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Mirror Property for Nonsingular Mixed Configurations of Lines and Points in $\mathbb{R}^3$*

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Abstract. Let $M(h, k)$ denote the topological space whose points are the sets composed of h lines and k points of $\mathbb{R}^3$ in general position. We say that $f \in M(h, k)$ is mirror if a path exists in $M(h, k)$ that joins f and its mirror image after reflection in any plane of $\mathbb{R}^3$. In this paper we are principally concerned about the following problem (an affine version of a problem proposed by Viro for the projective space): Given $h, k \geq 0$, does some $f \in M(h, k)$ exist such that f is mirror? This question is solved for all cases except when $h \equiv 1 \pmod 4$ with $h \geq 5$ and $k = 2$ or 3.

Introduction

Any collection of h lines and k points of $\mathbb{R}^3$ in *general position* (that is, none of the points lies on any of the lines, no line of $\mathbb{R}^3$ contains three of the points, and no plane of $\mathbb{R}^3$ contains four of the points or one of the lines and two of the points or two of the lines) is said to be a *nonsingular configuration of h lines and k points in* $\mathbb{R}^3$.

Given $h, k \geq 0$, $M(h, k)$ denotes the topological space whose points are the nonsingular configurations of h lines and k points in $\mathbb{R}^3$. The topology induced by $\mathbb{R}^3$ in $M(h, k)$ is the obvious one. Each connected component of $M(h, k)$ is said to be a *chamber of* $M(h, k)$ (we point out that in the literature the term *camera* appears instead of *chamber*).

By a *rigid isotopy* we understand a continuous function $\gamma: [0, 1] \to M(h, k)$. We say that $f, g \in M(h, k)$ are *rigidly isotopic* if a rigid isotopy exists such that $\gamma(0) = f$

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and $\gamma(1) = g$. Note that $f, g \in M(h, k)$ are rigidly isotopic if and only if they lie in the same chamber of $M(h, k)$.

$f \in M(h, k)$ is said to be *mirror* or to have the *mirror property* if f is rigidly isotopic to its mirror image after reflection in any plane. Otherwise f is said to be *nonmirror*.

The study of configurations in general position in projective space was introduced by Viro in [3] under the influence of Kharlamov's account of their connection with the problem of classifying nonsingular real projective algebraic surfaces of degree 4 up to rigid isotopy (by which is meant an isotopy consisting of nonsingular algebraic surfaces of degree 4). In some cases the nonexistence in the projective space of mirror configurations of some determined number of lines and points can be used to prove in an elementary way that certain of these surfaces are not amphicheiral (see [2]). Questions related to projective configurations in general position can be founded in [4].

In a previous work [1] the question of the mirror property for $M(1, k)$ with $k \geq 0$ was solved. The methods employed in that work are completely different to those employed here. Therefore, the present work can be read independently.

1. Mirror Permutations

Let $f \in M(h, k)$ be a mirror configuration. Given any reflection σ in a plane of $\mathbb{R}^3$, consider a rigid isotopy γ: $[0, 1] \to M(h, k)$ such that $\gamma(0) = \sigma(f)$ and $\gamma(1) = f$. The composition of σ with γ acts over f permuting its h lines and its k points. Any permutation of the lines and the points of f obtained in this way are called a *mirror permutation of* f.

Given $f \in M(h, k)$, any $f' \in M(h', k')$ such that the lines of f' are a subset of the lines of f and the points of f' are a subset of the points of f is said to be a *subconfiguration of* f (clearly, $h' \leq h$ and $k' \leq k$).

Let $f \in M(h, k)$ be a mirror configuration, let φ be a mirror permutation of f, and let $f' \in M(h', k')$ be a subconfiguration of f. If $\varphi(f') = f'$, then it is easy to see that f' is mirror and $\varphi' = \varphi|_{f'}$ is a mirror permutation of f'. In particular the configurations of $M(h, 0)$ and $M(0, k)$ composed, respectively, of all the lines and all the points of f are mirror.

Let $f \in M(h, k)$ be a mirror configuration. If φ is a mirror permutation of f, then, $\forall n \in \mathbb{N}$, φ^{2n-1} is also a mirror permutation of f since it is possible to obtain φ^{2n-1} from the composition of a reflection with a rigid isotopy.

Mirror permutations play a very important role in this work.

2. Mirror Property in $M(h, 0)$ and $M(0, k)$

The following theorems are affine versions of Viro's projective results:

Theorem 1. *$M(h, 0)$ contains mirror configurations if and only if $h \not\equiv 3 \pmod 4$. Moreover, for any $h \geq 3$, $M(h, 0)$ contains nonmirror configurations.*

Theorem 2. *Any configuration of* $M(0, k)$, *where* $k \equiv 3 \pmod 4$ *with* $k \geq 7$ *or* $k \equiv 6 \pmod 8$, *is nonmirror.*

Note. (i) In [3] Viro introduced into the subject the *linking coefficient* associated to any configuration of three lines in general position in the projective space and he employed it, with elementary geometrical and combinatorial arguments, to prove the projective version of the theorems above.

(ii) The inclusion $\mathbb{R}^3 \subset \mathbb{R}P^3$ implies that if, for some $h, k \geq 0$, $M(h, k)$ contains mirror configurations, then an equivalent result is valid for the projective case. Therefore, the projective version of Theorem 2 implies its affine version. On the other hand, the constructions made in [3] of mirror configurations of h lines in the projective space when $h \not\equiv 3 \pmod 4$ are easily adaptable to the affine case, and it follows as before that the projective version of Theorem 1 implies its affine version.

In what follows the study of the mirror property for nonsingular configurations in $\mathbb{R}^3$ consisting only of points is completed. The following construction is very useful:

Let $f \in M(0, k)$. Consider the convex hull of the k points of f. We obtain one polyhedron $P_1(f)$ such that all its vertices are points of f. Since the k points are in general position, no face of $P_1(f)$ contains more that three points of f. Then the faces of $P_1(f)$ are triangles. Now remove the vertices of $P_1(f)$ from f and consider the convex hull of the remaining points of f. We obtain again another polyhedron $P_2(f)$ with triangular faces and whose vertices are points of f. Repeating this argument until all the points of f are removed we obtain a sequence of polyhedrons $P_1(f), P_2(f), \ldots, P_n(f)$ such that $P_{i+1}(f) \subset \mathring{P}_i(f)$ for $i = 1, 2, \ldots, n-1$, and such that if $P_n(f)$ has less than four vertices, then $P_n(f)$ will be a triangle, a segment, or a single point. The sequence of nested polyhedrons $P_1(f) \supset P_2(f) \supset \cdots \supset P_n(f)$ is called the *polyhedral structure of* f.

Two convex polyhedrons P and Q are *combinatorially equivalent* if some bijection between the vertices of P and Q that preserves adjacency exists. Given $f \in M(0, k)$, we say that two points of f are *i-adjacent in* f if they are joined by means of one edge of $P_i(f)$. We say that $f, g \in M(0, k)$ have *equivalent polyhedral structures* if some bijection ψ from the points of f to the points of g exists with the following property: p and q are i-adjacent in f if and only if $\psi(p)$ and $\psi(q)$ are i-adjacent in g (this means that the polyhedral structures of f and g contain the same number of polyhedrons with $P_1(f)$ combinatorially equivalent to $P_1(g)$, $P_2(f)$ combinatorially equivalent to $P_2(g)$, and so on). This defines an equivalence relation on $M(0, k)$. Each equivalence class is called a *polyhedral class of* $M(0, k)$.

Let $f \in M(0, k)$ and $\gamma: [0, 1] \to M(0, k)$ be a rigid isotopy with $\gamma(0) = f$. Since any rigid isotopy preserves the general position, then $\gamma(0) = f$ and $\gamma(1)$ have equivalent polyhedral structures (this implies that each chamber of $M(0, k)$ is contained in one polyhedral class of $M(0, k)$). The bijection that sends any point of $\gamma(0) = f$ to its image in $\gamma(1)$ satisfies the definition of equivalent polyhedral structures. This is clear since during the rigid isotopy two polyhedrons of the

polyhedral structure cannot intersect each other, and the equivalence class of each polyhedron remains invariant.

On the other hand, let σ be any reflection in a plane of $\mathbb{R}^3$. Then f and $\sigma(f)$ have equivalent polyhedral structures (this implies that the chambers of $M(0, k)$ in which f and $\sigma(f)$ lie are contained in the same polyhedral class of $M(0, k)$). Clearly, the bijection that sends the points of f to their images in $\sigma(f)$ satisfies the definition of an equivalent polyhedral structure.

As a consequence of the observations in the previous paragraphs we have

Lemma 3. *Let $f \in M(0, k)$ be a mirror configuration, let φ be a mirror permutation of f, and let $P_1(f) \supset P_2(f) \supset \cdots \supset P_n(f)$ be the polyhedral structure of f. If $p \in f$ is a vertex of $P_i(f)$, then $\varphi(p)$ is also a vertex of $P_i(f)$ and the number of edges landing on p and $\varphi(p)$ is the same since φ preserves i-adjacency in f.*

Let $f = \{p_1, p_2, \ldots, p_k\} \in M(0, k)$ be mirror and let φ be a mirror permutation of f. As φ acts over f permuting its points, then φ divides $\{p_1, p_2, \ldots, p_k\}$ in cycles. Any subconfiguration of f composed of all the points of one of these cycles is called a *φ-cycle* of f. Let $f' \in M(0, k')$ be a φ-cycle of f. Clearly, f' is mirror and $\varphi' = \varphi|_{f'}$ is a mirror permutation of f' that cyclically permutes its k' points.

Theorem 4. *If $f \in M(0, k)$ is mirror and φ is a mirror permutation of f, then any φ-cycle has at most four points.*

Proof. Let $f' \in M(0, k')$ be a φ-cycle of f and $\varphi' = \varphi|_{f'}$. From Lemma 3 we conclude that the polyhedral structure of f' consists only of $P_1(f')$ and that the number of edges landing on each one of its k' vertices is the same. Suppose that $k' > 4$. As all the faces of $P_1(f')$ are triangles, it would follow from Euler's formula that $P_1(f')$ is an octahedron or an icosahedron.

(1) $P_1(f')$ cannot be an octahedron. Otherwise $f' \in M(0, 6)$ will be mirror, which is not possible (see Theorem 2).

(2) $P_1(f')$ cannot be an icosahedron. Suppose the opposite. Define one bijection ψ from the vertices of the icosahedron $P_1(f')$ over the vertices of a regular icosahedron I with its vertices in the unit sphere $S^2 \subset \mathbb{R}^3$ in such a way that adjacent vertices in $P_1(f')$ go to adjacent vertices in I. As φ' preserves adjacency in $P_1(f')$, then it induces (by means of ψ) the action over I of some element of SO(3) that leaves invariant I. Now, as the order of any element of the group of transformations of SO(3) that leave invariant I is at most 10 we would conclude (via ψ) that φ' does not cyclically permute the 12 points of f', which is a contradiction. □

In the next theorem the question of the mirror property for $M(0, k)$ is completely solved.

Theorem 5

(i) *If $k \leq 5$ and $f \in M(0, k)$, then f is mirror.*
(ii) *If $k \equiv 2$ or 3 (mod 4) with $k \geq 6$ and $f \in M(0, k)$, then f is nonmirror.*
(iii) *If $k \equiv 0$ or 1 (mod 4) with $k \geq 8$, then $M(0, k)$ contains mirror and nonmirror configurations.*

Proof. (i) If $k \leq 4$ it is easy to see that $M(0, k)$ has only one chamber. Therefore $M(0, k)$ only contains mirror configurations if $k \leq 4$.

There are two possibilities for the polyhedral structure of any $f \in M(0, 5)$: $P_1(f)$ is a bipyramide with triangular basis, or $P_1(f)$ is a tetrahedron and $P_2(f)$ is one point. It is easy to see that when two configurations of $M(0, 5)$ have equivalent polyhedral structures, then they are rigidly isotopic. As any reflection preserves the equivalence class of the polyhedral structure, then $M(0, 5)$ only contains mirror configurations.

(ii) Let $f \in M(0, k)$ with $k \geq 6$ be mirror and let φ be a mirror permutation of f. By Theorem 4, each φ-cycle of f has at most four points. Suppose that all φ-cycles have less than four points. If this was the case, we could consider some appropriate set of φ-cycles such that with their points we would obtain a mirror subconfiguration in $M(0, 6)$ or $M(0, 7)$, but this is not possible (see Theorem 2). Therefore some φ-cycle has four points. Then we conclude, arguing in a similar way, that there are no φ-cycles of two or three points, nor two φ-cycles of one point. Then all φ-cycles have four points except perhaps one of one point, and therefore $k \equiv 0$ or 1 (mod 4).

(iii) Let $k \equiv 0$ or 1 (mod 4) with $k \geq 8$.

$M(0, k)$ contains nonmirror configurations. If $f \in M(0, k)$ is such that its polyhedral structure contains one polyhedron $P_i(f)$ with six or seven vertices, then f is nonmirror. Suppose the opposite. Let φ be a mirror permutation of f and let $f' \in M(0, 6)$ or $M(0, 7)$ be the subconfiguration of f composed of the vertices of $P_i(f)$. By Lemma 3, $\varphi(f') = f'$ and therefore f' would be mirror which is not possible.

$M(0, k)$ also contains mirror configurations. If $a, b, c \in \mathbb{R}$ with $c \neq 0$ and $(a, b) \neq (0, 0)$ we denote the set $\{(a, b, c), (-a, -b, c), (b, -a, -c), (-b, a, -c)\} \in M(0, 4)$ by $S(a, b, c)$. Let φ be the composition of the reflection in the plane OXY followed by the rigid isotopy given by a rotation of 90° with axis OZ. Clearly, $S(a, b, c)$ is mirror since $\varphi(S(a, b, c)) = S(a, b, c)$ (note that its four points are cyclically permuted by φ). Given $d \geq 2$ consider one set $S = \bigcup_{i=1}^{d} S(a_i, b_i, c_i)$ such that no plane of $\mathbb{R}^3$ contains four points of S. Then $S \in M(0, 4d)$ is mirror since $\varphi(S) = S$. Moreover, we can choose S such that no plane of $\mathbb{R}^3$ contains four points of $S' = S \cup \{(0, 0, 0)\}$. Then $S' \in M(0, 4d + 1)$ is mirror since $\varphi(S') = S'$. □

3. Mirror Property in $M(h, k)$

In this section, using essentially the results of Section 2, we determine in a large number of cases when $M(h, k)$ contains mirror configurations.

Theorem 6. *$M(h, k)$ does not contain mirror configurations when*:

(i) $h \equiv 3 \pmod 4$.
(ii) $h \equiv 2 \pmod 4$ *and* $k \equiv 2$ *or* $3 \pmod 4$.
(iii) $h \equiv 1 \pmod 4$ *and* $k \equiv 4, 5, 6,$ *or* $7 \pmod 8$.
(iv) $h \equiv 0 \pmod 4$ *and* $k \equiv 6$ *or* $7 \pmod 8$.

Proof. (i) It is an immediate consequence of Theorem 1.

(ii) Let $f \in M(h, k)$ with $h \equiv 2 \pmod 4$ and $k \equiv 2$ or $3 \pmod 4$. Define $R(f) = \{f_1, f_2, \ldots, f_m\}$ to be the set composed of the

$$m = \binom{k}{2}$$

configurations of $M(h + 1, 0)$ that consist of the h lines of f and one line determined by one pair of points of f. By Theorem 1, $R(f)$ only contains nonmirror configurations since $h + 1 \equiv 3 \pmod 4$.

Suppose that f is mirror. If φ is a mirror permutation of f, then we can extend the action of φ to any $f_i \in R(f)$ in the following way: the image of the line of f_i determined by the pair of points of f is the line determined by the images of both points. Therefore $\varphi(f_i) \in R(f)$ and φ defines a permutation in $R(f)$ that divides $\{f_1, f_2, \ldots, f_m\}$ in cycles. The length of each cycle is an even number. Otherwise, for some $f_i \in R(f)$ and some $n \in \mathbb{N}$, $\varphi^{2n-1}(f_i) = f_i$, and it is easy to see that this implies that f_i is mirror, which is not true. We conclude that $\# R(f)$ is an even number. However, this give us a contradiction since

$$\# R(f) = \binom{k}{2}$$

is an odd number when $k \equiv 2$ or $3 \pmod 4$. Therefore f is nonmirror.

(iii) Let $f \in M(h, k)$ with $h \equiv 1 \pmod 4$ and $k \equiv 4, 5, 6,$ or $7 \pmod 8$. Define $R(f)$ to be the set composed of the

$$\binom{k}{2} \cdot \binom{k-2}{2} \Big/ 2$$

nonmirror configurations of $M(h + 2, 0)$ that consist of the h lines of f and two lines determined by two disjoint pairs of points of f. If we argue as in (ii), we conclude that f is nonmirror.

(iv) Let $f \in M(h, k)$ with $h \equiv 0 \pmod 4$ and $k \equiv 6$ or $7 \pmod 8$. Define $R(f)$ to be the set composed of the

$$\binom{k}{2} \cdot \binom{k-2}{2} \cdot \binom{k-4}{2} \Big/ 6$$

nonmirror configurations of $M(h+3, 0)$ that consist of the h lines of f and three lines determined by three disjoint pairs of points of f. Again we conclude, arguing in the same way, that f is nonmirror. □

Note. The argument in the proof of Theorem 6 is a modification of Viro's argument in order to prove Theorem 2 in the projective case (see [3]).

Now we complete the study of the mirror property for $M(h, k)$ when $k \geq 4$:

Theorem 7.

(i) *If* $k \equiv 2$ *or* $3 \pmod 4$ *with* $k \geq 6$, *then* $M(h, k)$ *does not contain mirror configurations.*
(ii) *If* $h \equiv 1 \pmod 4$ *and* $k \equiv 0$ *or* $1 \pmod 4$ *with* $k \geq 8$, *then* $M(h, k)$ *does not contain mirror configurations.*
(iii) *If* $h \equiv 0$ *or* $2 \pmod 4$ *and* $k \equiv 0$ *or* $1 \pmod 4$, *then* $M(h, k)$ *contains mirror configurations.*

Proof. (i) It is an immediate consequence of Theorem 5(ii).

(ii) Let $h \equiv 1 \pmod 4$ and $k \equiv 0$ or $1 \pmod 4$ with $k \geq 8$. Suppose that $f \in M(h, k)$ is mirror and φ is a mirror permutation of f. Let $f' \in M(0, k)$ be the subconfiguration of f composed of the k points of f. Then $\varphi' = \varphi|_{f'}$ is a mirror permutation of f' that divides f' in φ'-cycles of four points except one φ'-cycle of one point when $k \equiv 1 \pmod 4$ (see the proof of (ii) of Theorem 5). Consider one subconfiguration $g \in M(h, 4)$ of f composed of the h lines of f and the four points of one φ'-cycle. Then $\varphi(g) = g$ and therefore $g \in M(h, 4)$ with $h \equiv 1 \pmod 4$ would be mirror, but this is not possible by Theorem 6.

(iii) Let $h \equiv 0$ or $2 \pmod 4$ and $k \equiv 0 \pmod 4$. Construct some

$$S = \bigcup_{i=1}^{(h/2)+(k/4)} S(a_i, b_i, c_i) \in M(0, 2h+k)$$

as in the proof of (iii) of Theorem 5. For $j = 1, 2, \ldots, h/2$ let l_j be the line determined by (a_j, b_j, c_j) and $(-a_j, -b_j, c_j)$, and let l'_j be the line determined by $(-b_j, a_j, -c_j)$ and $(b_j, -a_j, -c_j)$. Then

$$f = \left\{ \bigcup_{j=1}^{h/2} \{l_j, l'_j\}; \bigcup_{i=(h/2)+1}^{(h/2)+(k/4)} S(a_i, b_i, c_i) \right\} \in M(h, k).$$

Let φ be defined as in the proof of (iii) of Theorem 5. Then $\varphi(f) = f$ and therefore f is a mirror configuration.

Moreover, if no plane contains four points of $S' = S \cup \{(0, 0, 0)\}$, then $f' = f \cup \{(0, 0, 0)\} \in M(h, k+1)$ and $\varphi(f') = f'$. Hence f' is mirror. □

In order to solve part of the remaining cases the following lemma is useful.

Lemma 8.

(i) *$M(h, 1)$ contains mirror configurations if and only if $M(h, 0)$ contains mirror configurations.*
(ii) *$M(h, 2)$ and $M(h, 3)$ contain mirror configurations if and only if a mirror configuration $f \in M(h+1, 0)$ exists such that some mirror permutation of f leaves invariant one line of f.*

Proof. (i) It is enough to point out that given $f \in M(h, 1)$, if we move the point of f freely in the complement of the h lines of f in $\mathbb{R}^3$ we always obtain configurations of $M(h, 1)$.

(ii) ($\Rightarrow$) Let $f = \{l_1, \ldots, l_h; p, q\} \in M(h, 2)$ be a mirror configuration and let φ be any mirror permutation of f. Let l_{h+1} denote the line determined by p and q and let $g = \{l_1, \ldots, l_h, l_{h+1}\} \in M(h+1, 0)$. Define φ_g to be the following permutation of the lines of g: $\varphi_g(l_i) = \varphi(l_i)$, $\forall i \in \{1, 2, \ldots, h\}$, and $\varphi_g(l_{h+1}) = l_{h+1}$. It is easy to see that φ_g is a mirror permutation of g, and it leaves invariant one line of $g \in M(h+1, 0)$.

If $f \in M(h, 3)$ is mirror and φ is a mirror permutation of f, then φ^3 is a mirror permutation of f that leaves invariant some point $r \in f$, and therefore also its complementary. Hence $f \backslash \{r\} \in M(h, 2)$ is mirror. Having obtained $f \backslash \{r\}$ and φ^3 we can use the same argument as in the preceding case.

($\Leftarrow$) Let $f = \{l_1, \ldots, l_{h+1}\} \in M(h+1, 0)$ be mirror and let φ be a mirror permutation of f such that $\varphi(l_{h+1}) = l_{h+1}$. We know that φ can be obtained from the composition of one reflection σ with one rigid isotopy γ: $[0, 1] \to M(h+1, 0)$ such that $\gamma(0) = \sigma(f)$ and $\gamma(1) = f$. Let p and q be two different points of l_{h+1} and let $g = \{l_1, \ldots, l_h; p, q\} \in M(h, 2)$. We can restrict the action of σ and γ over g in such a way that $\gamma(0) = \sigma(g)$ and $\gamma(1) = g$. Therefore $g \in M(h, 2)$ is mirror. Now consider a disk D with the following properties:

(1) D is perpendicular to the segment of extremes p and q by its middle point $r = (p+q)/2$.
(2) $\forall s \in D \backslash \{r\}$, $g_s = g \cup \{s\} \in M(h, 3)$ (this implies that g_s and $g_{s'}$ will be rigidly isotopic $\forall s, s' \in D \backslash \{r\}$).

Note that it is possible to choose D so small that we can consider the action of σ and γ over $g \cup D$ in such a way that, as before, $\gamma(0) = \sigma(g)$ and $\gamma(1) = g$, and that properties (1) and (2) are always verified by the images of D with respect to the images of p and q. We conclude that if $s \in D \backslash \{r\}$, then $g \cup \{s\} \in M(h, 3)$ is mirror. □

As was shown graphically in [4], $M(h, 0)$ with $h \equiv 1 \pmod 4$ contains mirror configurations verifying that it is possible to construct a mirror permutation that leaves invariant exactly one line. By Lemma 8 we have

Theorem 9. *$M(h, k)$ contains mirror configurations when:*

(i) $h \equiv 1 \pmod 4$ *and* $k = 0$ *or* 1.
(ii) $h \equiv 0 \pmod 4$ *and* $k = 2$ *or* 3.

4. The Opposite Problem

Given $h, k \geq 0$, does $M(h, k)$ contain nonmirror configurations?

Theorem 10. *The only cases in which all their configurations are mirror are $M(0, k)$ when $k \leq 5$, $M(1, k)$ when $k \leq 3$, and $M(2, k)$ when $k \geq 1$.*

Proof. First we point out that $M(0, k)$ contains nonmirror configurations when $k \geq 6$ (see Theorem 5), and that $M(h, 0)$ contains nonmirror configurations when $h \geq 3$ (see Theorem 1). Then it is enough to study the cases $k \leq 5$ and $h \leq 2$.

Case (i): $h = 0$ and $k \leq 5$. We saw in Theorem 5 that $M(0, k)$ with $k \leq 5$ only contains mirror configurations.

Case (ii): $h = 1$ and $k \leq 5$. It is easy to see that $M(1, 0)$, $M(1, 1)$, and $M(1, 2)$ have only one chamber. Hence any $f \in M(1, 0)$, $M(1, 1)$, or $M(1, 2)$ is mirror.

Given $f \in M(1, 3)$, up to rigid isotopy we can consider that the line of f is perpendicular to the plane π that contains the three points of f. However, in this situation f coincides with its mirror image after reflection in π. We conclude that all the configurations of $M(1, 3)$ are mirror.

$M(1, 4)$ and $M(1, 5)$ contain nonmirror configurations (indeed all are nonmirror configurations, see Theorem 6).

Case (iii): $h = 2$ and $k \leq 5$. Is easy to see that $M(2, 0)$ and $M(2, 1)$ have only one chamber. Hence, any $f \in M(2, 0)$ or $M(2, 1)$ is mirror.

$M(2, 2)$ and $M(2, 3)$ contain nonmirror configurations (indeed, all are nonmirror, see Theorem 6).

$M(2, 4)$ and $M(2, 5)$ contain nonmirror configurations: If $f \in M(2, 4)$ has the property that only one line of f intersects the tetrahedron determined by the four points of f, then f is nonmirror. Suppose the opposite. Let φ be a mirror permutation of f. As the previous property is preserved under reflection and rigid isotopies, then φ leaves invariant each line of f. However, this implies that $M(1, 4)$ contains mirror configurations, which is not true.

The same kind of argument shows that $M(2, 5)$ contains nonmirror configurations. □

5. Summary

The following is a summary with all the results about the mirror property in $M(h, k)$:

(1) If $h \equiv 0 \pmod 4$, then $M(h, k)$ contains mirror configurations if and only if $k \equiv 0$ or $1 \pmod 4$, or $k = 2$ or 3.

(2) If $h \equiv 1 \pmod 4$ and $k = 0$ or 1, then $M(h, k)$ contains mirror configurations. If $h \equiv 1 \pmod 4$ and $k \geq 4$, then $M(h, k)$ does not contain mirror configurations. $M(1, 2)$ and $M(1, 3)$ contain mirror configurations. What does happen when $h \equiv 1 \pmod 4$ with $h \geq 5$ and $k = 2$ or 3?

(3) If $h \equiv 2 \pmod 4$, then $M(h, k)$ contains mirror configurations if and only if $k \equiv 0$ or $1 \pmod 4$.

(4) If $h \equiv 3 \pmod 4$, then $M(h, k)$ does not contain mirror configurations.

(5) The only cases in which all their configurations are mirror are $M(0, k)$ when $k \leq 5$, $M(1, k)$ when $k \leq 3$, and $M(2, k)$ when $k \leq 1$.

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