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Finite Sphere Packing and Sphere Covering*

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Abstract. A basic problem of finite packing and covering is to determine, for a given number of k unit balls in Euclidean d-space E^d, (1) the minimal volume of all convex bodies into which the k balls can be packed and (2) the maximal volume of all convex bodies which can be covered by the k balls. In the sausage conjectures by L. Fejes Tóth and J. M. Wills it is conjectured that, for all $d \geq 5$, linear arrangements of the k balls are best possible. In the paper several partial results are given to support both conjectures. Furthermore, some relations between finite and infinite (space) packing and covering are investigated.

1. Introduction

Let $\mathcal{K}^d$ denote the set of convex bodies in Euclidean d-space E^d, i.e., the set of all compact convex subsets of E^d with nonempty interior. Further, let B^d be the unit ball in E^d. A basic problem of the *theory of finite packing* is to determine for a given positive integer k the *minimum of the volume of all convex bodies into which k translates of B^d can be packed.*

In the plane this problem and several ramifications have been extensively studied. In particular, Fejes Tóth [5] (see also [6]) showed that finite packings with B^2 cannot be denser than an optimal packing of disks in the whole plane, and Wills [25] conjectured that this also holds (except for some small k) in E^3 and E^4. Further, according to Groemer [11] and Wegner [22] extremal finite

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packings of B^2 are essentially hexagonal parts of the densest lattice packing of disks in E^2. This result has no direct anology in E^3 and E^4 (except perhaps for large k). In the optimal arrangement of three balls in E^3 the centers are not, as one might possibly expect, the vertices of a regular triangle, but rather they lie on a straight line. Such linear arrangements are called *sausages*, the name originating from the shape of the convex hull of the balls. In E^3 sausages seem to be best possible up to 56 balls (see Wills [23], [24]). Then the extremal configurations change drastically and clusters of balls yield greater densities. In E^4 the same phenomenon occurs, but much later, perhaps between 50,000 and 100,000 balls (see [23] and [24]), which justifies the name *sausage catastrophe*.

In E^5 the situation changes completely. In E^d, $d \geq 5$, sausages seem to be best possible for any number of k unit balls. This is Fejes Tóth's [7] *sausage conjecture* for finite packings.

Correspondingly, a basic problem of the *theory of finite covering* is to determine, for a given positive integer k, the *maximum of the volume of all convex bodies which can be covered by k translates of* B^d. In this context there seems to be quite a strong analogy between finite packings and finite coverings with the unit ball. In particular, as a counterpart to Fejes Tóth's sausage conjecture, Wills [24] conjectured that sausage arrangements are also extremal, for $d \geq 5$, in the case of finite coverings of B^d.

In spite of their importance it has not yet been possible to prove either of these two sausage conjectures completely in any dimension. But there are a number of results which support the conjectures. For a survey of the results known so far, refer to Gritzmann and Wills [10].

In the present paper we give further evidence to support both conjectures. Furthermore, we point out some relations to packing and covering with B^d with respect to the whole space. Finally, we present a simple result concerning sausage catastrophes and outline some possible applications in chemistry.

In Section 2 we give some basic notation. In Section 3 we introduce a uniform concept of packing and covering densities which, in particular, makes the occurrence of sausage and sausage catastrophe phenomena much more lucid. Section 4 contains the statement of our results, the proofs of which are the content of Sections 5, 7, 9, and 10. In Sections 6 and 8 we give upper and lower estimates for the volume of parallel bodies of the regular simplex which are essential for the proofs of Theorems 3 and 4, and which may be of interest themselves.

2. Basic Notation and the Sausage Conjectures

Let $V = V^d$ denote the usual d-volume of convex bodies and, in particular, set

$$\omega_d = V(B^d) = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.$$

If K is an arbitrary convex body and if r runs through all nonnegative real numbers then, by Steiner's formula, $V(K + rB^d)$ is a polynomial in r of degree

d which can be written in the form

$$V(K + rB^d) = \sum_{i=0}^{d} \omega_{d-i} V_i(K) r^{d-i}$$

(see, e.g., Hadwiger [12]). The coefficients $V_0(K), \ldots, V_d(K)$ are the *intrinsic volumes* of K (see McMullen [16]). Let us remark that the intrinsic volumes are just a renormalization of the quermass integrals. Therefore, regarded as functionals on $\mathcal{K}^d$ they have the well-known properties of the quermass integrals, but, moreover, they are independent of the dimension of the space in which K is embedded. Of particular service is the following description of the intrinsic volumes V_i of polytopes. Let P be a polytope and let $F_i(P)$ denote the set of all i-dimensional faces f of P. Furthermore, let $C(f)$ be the cone with vertex 0 of outer normals of P taken at any relatively interior point of f and let $\alpha(f)$ be the fraction of the linear hull of $C(f)$ taken up by $C(f)$. Clearly, $0 < \alpha(f) \leq 1$. But, moreover,

$$V_i(P) = \sum_{f \in F_i(P)} V^i(f)\alpha(f).$$

Let us stress the fact that the sets $f + C(f)$ dissect E^d, and so, in particular, they dissect $P + B^d$.

Let $k \in \mathbb{N}$ and let

$$\mathcal{F}_k = \{(2i, 0, \ldots, 0) + B^d \mid i = 1, \ldots, k\}.$$

Furthermore, let $\mathcal{W}_k$ be a set of k translates of B^d, the centers of which are equally spaced on a straight line at such a distance that, for all such configurations, $\mathcal{W}_k$ covers a convex body of greatest volume.

Then the sausage conjectures can be expressed as follows:

Sausage conjecture for finite packings
For $d \geq 5$ $V(\text{conv}(\mathcal{F}_k))$ is the minimum of the volume of all convex bodies into which k translates of B^d can be packed.

Sausage conjecture for finite coverings
For $d \geq 5$ $\mathcal{W}_k$ contains a convex body of maximal volume of all such convex bodies that can be covered by k translates of B^d.

We remark that there is an extremely useful alternative statement of the sausage conjecture for finite packings in terms of parallel bodies.

Let C_k denote the convex hull of the centres of k nonoverlapping translates of B^d and let S_k be a segment of length $2(k-1)$. Then the conjecture is that

$$V(S_k + B^d) \leq V(C_k + B^d) \qquad \text{for} \quad d \geq 5.$$

Clearly, this way of stating the conjecture is well suited for methods involving intrinsic volumes.

If A is a discrete set of points in E^d and $a \in A$ then the set of all points of E^d which are not farther away from a than from any other point of A is called the *Voronoi polyhedron* of a. Departing from the usual notation, for a fixed convex body K we call the intersection of all Voronoi polyhedra of points of $A \cap K$ with K *Dirichlet cells* with respect to K.

The rest of the notation used below is quite standard (see, for example, Hadwiger [12] or Rogers [20]).

3. Packing and Covering Densities

Let $A=\{a_1, a_2, \dots\}$ be a finite or infinite discrete set of points in E^d. We first introduce the so-called *π-density* and *γ-density* of the system $\mathcal{B}_A=[a_1+B^d, a_2+B^d, \dots]$ of translates of B^d. (The restriction to the case of the unit ball is merely a matter of simplicity and clarity. In fact, with some obvious changes most of the things discussed in this section hold for arbitrary convex bodies.) We use C^d to denote the unit cube with its edges parallel to the coordinate axis and with center 0. We write, for $\lambda \in \mathbb{R}$, $\lambda > 0$,

$$\pi(\mathcal{B}_A, \lambda) = \max[\omega_d \operatorname{card}(A \cap \lambda C^d)/V(\lambda C^d \cap C) \mid C \in \mathcal{K}^d \wedge (A \cap \lambda C^d) + B^d \subset C],$$

$$\gamma(\mathcal{B}_A, \lambda) = \min[\omega_d \operatorname{card}(A \cap \lambda C^d)/V(\lambda C^d \cap C) \mid C \in \mathcal{K}^d \wedge C \subset (A \cap \lambda C^d) + B^d].$$

Then the π-density and the γ-density of $\mathcal{B}_A$ is defined as follows:

$$\pi(\mathcal{B}_A) = \limsup_{\lambda \to \infty}(\mathcal{B}_A, \lambda)$$

$$\gamma(\mathcal{B}_A) = \liminf_{\lambda \to \infty} \gamma(\mathcal{B}_A, \lambda).$$

The system $\mathcal{B}_A$ is called a *packing* if each pair of translates a_i+B^d, a_j+B^d $(i \neq j)$ are nonoverlapping, i.e., if they do not have interior points in common. Clearly, if $\mathcal{B}_A$ is a packing then $\pi(\mathcal{B}_A) \leq 1$. Furthermore, $\gamma(\mathcal{B}_A) \geq 1$ in general.

We use the π- and γ-densities to define packing and covering densities for all the problems considered below, including the classical densities for packings and coverings with respect to the whole space (see, e.g., [20]), the densities of finite packings and coverings (see e.g., [10]), and also some intermediate types.

Let us start by defining certain types of packings and coverings. Let $1 \leq n \leq d$. Then $\mathcal{B}_A$ is called an *n-dimensional packing* of B^d if $\mathcal{B}_A$ is a packing and A is contained in some n-dimensional affine subspace E of E^d, usually identified with E^n such that $\operatorname{conv}(A)=E$. $\mathcal{B}_A$ is called an *n-dimensional covering* with B^d if $A \subset E^n$ such that $E^n \subset A+B^d$. Obviously, d-dimensional packings and coverings are the classical packings and coverings with respect to the whole space E^d. If A is finite and $\mathcal{B}_A$ is a packing, then $\mathcal{B}_A$ is called a *finite packing*. If A is finite, $C \in \mathcal{K}^d$, and $C \subset A+B^d$, then $\mathcal{B}_A$ is called a *finite covering* of C. If, further, $k \in \mathbb{N}$ and $\mathcal{B}_A$ is a finite packing or covering with $\operatorname{card}(A)=k$, then $\mathcal{B}_A$ is called a *k-packing*, *k-covering*, respectively.

Now we associate with B^d the following densities:

$$\delta_k(B^d)=\sup[\pi(\mathcal{B}_A)\,|\,\mathcal{B}_A \text{ is a } k\text{-packing of } B^d],$$

$$\vartheta_k(B^d)=\inf[\gamma(\mathcal{B}_A)\,|\,\mathcal{B}_A \text{ is a } k\text{-covering with } B^d],$$

$$\delta^n(B^d)=\sup[\pi(\mathcal{B}_A)\,|\,\mathcal{B}_A \text{ is an } n\text{-dimensional packing of } B^d],$$

$$\vartheta^n(B^d)=\inf[\gamma(\mathcal{B}_A)\,|\,\mathcal{B}_A \text{ is an } n\text{-dimensional covering with } B^d].$$

As usual, we further set $\delta(B^d)=\delta^d(B^d)$, $\vartheta(B^d)=\vartheta^d(B^d)$. The reason for introducing the *subdensities* δ^n, ϑ^n is precisely that the sausage and sausage-catastrophe phenomena become much more evident by means of a study of these densities.

To start with let us consider the packing problem.

Since the minimal convex set which contains a given set is its convex hull it is easy to see how the densities of n-dimensional packings of B^d are related. In fact, we have

$$\delta^n(B^d)=\frac{\omega_d}{\omega_n\omega_{d-n}}\,\delta(B^n).$$

Since the gamma function $\Gamma(x)$ is strictly convex the ratio $\omega_d(\omega_n\omega_{d-n})^{-1}$ is also strictly convex (considered as a discrete function of n, $1\le n\le d$). To give an impression of the size of this ratio let us remark that

$$\frac{\omega_d}{\omega_n\omega_{d-n}}=\binom{d/2}{n/2}^{-1}\quad\text{for even } d, n,$$

and

$$\sqrt{\frac{\pi}{2(d+1)}}<\frac{\omega_d}{\omega_1\omega_{d-1}}<\sqrt{\frac{\pi}{2d}}.$$

So, in particular, we have

$$\sqrt{\frac{\pi}{2(d+1)}}<\delta^1(B^d).$$

Now Rogers [19] has shown

$$\delta(B^d)\le\sigma_d,$$

where σ_d denotes the ratio of the sum of the volumes of the intersection of $d+1$ unit balls centered at the vertices of a regular simplex of side 2 to the volume of the simplex. Since this bound is less than Blichfeldt's bound [3]

$$\sigma_d<\frac{d+2}{2}\left(\frac{1}{2}\right)^{d/2},$$

which, for $d \geq 7$, is less than $\delta^1(B^d)$ we have, for $d \geq 7$,

$$\delta(B^d) \leq \sigma_d < \delta^1(B^d).$$

Accordingly, since the sequence

$$\frac{\omega_d}{\omega_n \omega_{d-1}} \frac{n+2}{2} \left(\frac{1}{2}\right)^{n/2}$$

is convex in n for fixed d, we have

$$\delta^n(B^d) \leq \delta^1(B^d) \qquad \text{for} \quad 1 \leq n \leq d, \quad d \geq 7.$$

But using the exact values of σ_d (see, e.g., Leech [15]) it is easy to verify that this inequality is valid for $d \geq 5$.

Figure 1 illustrates the behavior of the subdensities in dimension 24. For the calculations we used the densest known packings of B^d. Further, Figure 2 shows the subdensities for $d \leq 10$ again with the uncertainty of having used the best-known lattice packings for $d = 9, 10$ since in these dimensions the exact quantities have not been determined as yet. Roughly speaking, the figures show that, for small n, $\delta(B^d) < \delta^n(B^d)$ with the extreme case of one-dimensional arrangements.

So the sausage conjecture for finite packings may be regarded as a generalization of this fact to the case of finite packings of B^d.

Now let us turn to coverings. In contrast to packings, in order to determine the subdensities $\vartheta^n(B^d)$ we have to solve a simple extremum problem to calculate the extra dilatation factor to be used to maximize the contained convex body asymptotically. Doing so, we obtain

$$\vartheta^n(B^d) = \frac{d^{d/2}}{(d-n)^{(d-n)/2} n^{n/2}} \frac{\omega_d}{\omega_n \omega_{d-n}} \vartheta(B^n),$$

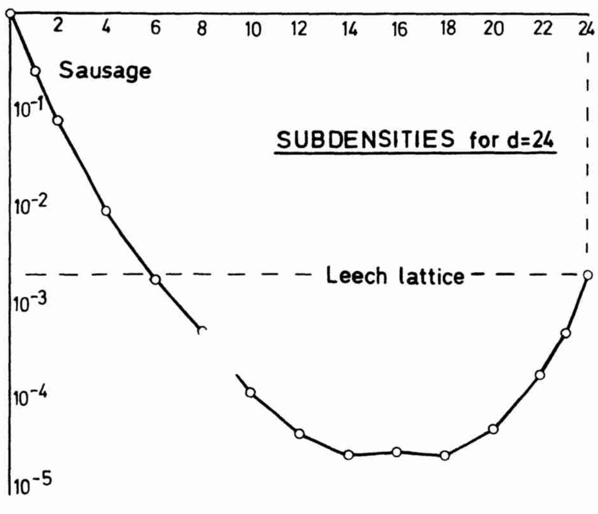

Fig. 1

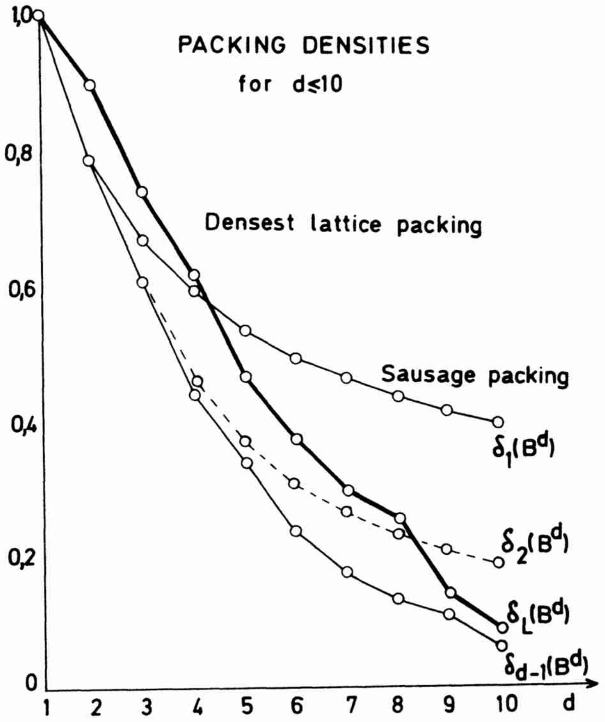

Fig. 2

which is, for large d, $d-n$, by Stirling's formula,

$$\vartheta^n(B^d) \sim \sqrt{\pi}\sqrt{\frac{n(d-n)}{d}}\,\vartheta(B^n).$$

Now in view of Coxeter *et al.*, bound [4]

$$\frac{d}{e\sqrt{e}} \sim \tau_d \leq \vartheta(B^d)$$

(where τ_d denotes the ratio of the sum of the volumes of the intersection of a regular simplex of side $[2(d+1)d^{-1}]^{1/2}$ with $d+1$ unit balls centered at its vertices to the volume of the simplex) it is easy to see that, for coverings with B^d, subdensities behave like packings of B^d, thus motivating the sausage conjecture for finite coverings.

To close Section 3 let us give one further definition for packings. Let k_d denote the largest number k such that

$$\delta_k(B^d) = \pi(\mathcal{F}_k).$$

If such a number does not exist we set $k_d = \infty$. k_d is called the *sausage-catastrophe* *number* of finite sphere packings. (Clearly, the sausage conjecture for packings states that $k_d=\infty$ for $d\geq 5$ in this notation.) For reasons that will become clear in the next section, for a d-dimensional packing lattice $\mathcal{G}$ of B^d let $k(\mathcal{G})$ denote the largest number k such that

$$\pi(\mathcal{B}_A)\leq \pi(\mathcal{F}_k)$$

whenever $A\subset \mathcal{G}$, $\operatorname{card}(A)=k$.

Let us finally remark that such notion also makes sense for finite coverings (see [24]).

4. Results

We show that the sausage conjectures hold for several types of arrangements. Let us start with a counterpart to the results of Betke *et al.* [2] and Betke and Gritzmann [1] who showed that Fejes Tóth's sausage conjecture holds if the dimension of C_k is sufficiently small compared with d, i.e., it holds for "flat" arrangements. Our first theorem gives a corresponding result for "large" arrangements.

Theorem 1. *Let $A\subset E^d$, $\operatorname{card}(A)=k$ be such that $\mathcal{B}_A$ is a packing. Further, let $C_k=\operatorname{conv}(A)$, $\sigma\in\mathbb{R}$, $\sigma\leq\pi(\mathcal{B}_A)$. Then*

$$\sum_{i=0}^{d}(\sigma-\sigma_d 2^{d-i})\omega_{d-i}V_i(C_k)\leq 0.$$

Setting, in particular, $\sigma=\pi(\mathcal{F}_k)$ Theorem 1 becomes a sausage criterion showing that a certain class of finite packings of B^d yields a lower density than the sausage arrangements. In particular, Theorem 1 implies that, for counter-examples to the sausage conjecture, the insphere radius of C_k must be small compared with the circumsphere radius.

Corollary. *Let $A\subset E^d$, $\operatorname{card}(A)=k$ be such that $\mathcal{B}_A$ is a packing. Further, let $C_k=\operatorname{conv}(A)$ and let r, R denote the insphere radius and circumsphere radius of C_k, respectively. If, moreover, $\pi(\mathcal{F}_k)\leq\pi(\mathcal{B}_A)$, then*

$$1+r<\sqrt{2}\left(1+\frac{R}{2}\right)[(2\pi)^{-1/2}(d+2)(d+1)^{1/2}]^{1/d}.$$

Let us point out that for large d this inequality is asymptotically of the form

$$r\leq\frac{R}{\sqrt{2}}+\sqrt{2}-1.$$

(For coverings a result of the same spirit was proved by Gritzmann and Wills [9].)

Our second theorem is closely related to Theorem 1 in the sense that it again deals with "large" arrangements of closely packed balls, the so-called *kissing number configurations.* By a kissing number configuration we understand a packing $\mathscr{B}_A$ of B^d such that there is an element $a_0 \in A$ so that $\|a_0 - a\| = 2$ for all $a \in A \setminus \{a_0\}$ and, further, if there is an $x \in \mathbb{R}^d$ with $\|a_0 - x\| = 2$ and $\|a - x\| \geq 2$ for all $a \in A \setminus \{a_0\}$ then $x \in A$. So, intuitively speaking, a kissing number configuration is a maximal packing of B^d such that each ball touches a given one. In a sense, kissing number configurations may be regarded as building blocks of densest d-dimensional packings of B^d. However, Theorem 2 shows that, at least for large d, they do not serve as counterexamples for Fejes Tóth's sausage conjecture.

Theorem 2. *Let $d \geq 12$ and let $\mathscr{B}_A$ be a kissing number configuration in E^d of cardinality k. Then*

$$\pi(\mathscr{B}_A) \leq \pi(\mathscr{F}_k).$$

Let us remark that this result is not true for arbitrary dimensions. In fact, in the plane the kissing number configuration of seven disks has greater π-density than the respective sausage arrangement. But, as a special case of the sausage conjecture for packings, Theorem 2 should hold for $d \geq 5$.

Now we turn to special arrangements which seem to play a crucial role in the understanding of Euclidean d-space manifesting itself not only in packing and covering problems but also in different problems such as determining the lattice point enumerator of convex bodies. These are configurations related to the regular simplex T^d.

Theorem 3.

(a) *Let $d \geq 5$, $n \in \mathbb{N}$, $i \leq d$. Further, let A be the set of vertices of a regular i-dimensional simplex of side 2 in E^d. Then*

$$\pi(\mathscr{B}_A) < \pi(\mathscr{F}_{i+1}).$$

(b) *Let $s \in \mathbb{R}$, $s \geq 2$. Further, let A_s be the set of vertices of a regular d-dimensional simplex of side s. Then, as $d \to \infty$, we have*

$$\pi(\mathscr{B}_{A_s}) \geq 2^{-\sqrt{d}\log_2 d} \quad \textit{for} \quad s \leq 2\sqrt{\pi},$$

$$\pi(\mathscr{B}_{A_s}) \geq 2^{-d/2} \quad \textit{for} \quad s \leq \frac{\sqrt{d}}{13}.$$

Part (a) of the theorem shows that the π-density of the simplex configuration is less than the respective sausage density, at least for $d \geq 5$. In fact, the same result is true even for $d \geq 3$ (and fails for $d = 2$). The reason for the assumption $d \geq 5$ is that we prefer a very short proof in the spirit of Section 3 rather than a longer and much more technical one.

Part (b) of the theorem proves that, on the other hand, the π-density of the simplex configuration is much greater than the density $\delta(B^d)$ of d-dimensional sphere packing. In fact, by Kabatjanski and Levenstein [14],

$$\delta(B^d) \leq 2^{-0.599d + o(d)}$$

Surprisingly we even obtain a finite packing of greater density than $\delta(B^d)$ if we arrange $d+1$ balls such that their centres form a regular simplex of side $\sqrt{d}/13$. To demonstrate what this means, in dimension, say, $d = 169{,}676{,}676$, the distance of the balls may be 1000.

Let us now turn to asymptotic results for finite sphere coverings. As a counterpart to part (a) of Theorem 3 we show

Theorem 4. *Let $d \geq 3$, $s_0 = (2/(d-1))^{1/2}$. Further, let A_0 be the set of vertices of a regular d-dimensional simplex of side s_0. Then we have*

$$\gamma(\mathcal{B}_A) < 4\sqrt{5\pi e}\,\frac{d+1}{(\ln d)^{1/4}}.$$

Compared with Coxeter *et al.*'s [4] asymptotic bound $d/e\sqrt{e}$ the theorem shows that at least for high dimensions the γ-density of a regular simplex of suitable side is less than $\vartheta(B^d)$. For coverings, this is, indeed, a result similar to Theorem 3(b) for packings. It would, of course, be nice to carry over part (a) as well. The problem is that contrary to finite packings, where the least convex body that contains all balls of a given arrangement is simply the convex hull, convex bodies of greatest volume covered by a given arrangement are not characterized. So, for finite coverings we have one more optimization process (see Section 3).

Let us further point out that for a certain ramification of the finite covering problem considered in [9], the so-called cocoverings, the density of the simplex arrangement is much worse than $\vartheta(B^d)$.

Now let us finally turn to the phenomenon of sausage catastrophes for packings of B^3. This problem is of particular interest because of its connections to chemistry.

As it is well known, the study of densely arranged configurations of three-dimensional balls gives some insight into the behavior of solids and liquids. For example, the molecular properties of many crystals which, of course, bear a lattice structure can be described, at least approximately, as the effect of forces on a huge number of closely packed balls. In view of this close connection it is not very surprising that there is also a phenomenon in physical chemistry that corresponds to our sausage catastrophes in E^3.

In fact, such phenomena of one-dimensional growth of atomic structures up to a critical limit were observed by chemists and engineers almost 40 years ago. Such metal threads, the so-called *Whiskers*, caused short circuits in condensers. In particular, the case of *iron whiskers* is of some interest because of their extremal stretching properties. Since the iron atoms are usually arranged in the *space-centered cubical lattice* our last result (which should serve as an illustrating example) deals with the sausage catastrophe for the space-centered cubical lattice.

Theorem 5. *Let* $u = 2/\sqrt{3}$ *and let* $\mathcal{G}$ *be the lattice generated by* $(2u, 0, 0)$, $(0, 2u, 0)$, (u, u, u). *Then*

$$k(\mathcal{G}) \le 23{,}968.$$

5. Proof of Theorem 1, Its Corollary, and Theorem 2

According to an estimate of Rogers [19] we have, for every Voronoi polyhedron of a d-dimensional packing of B^d,

$$V(P) \ge \frac{\omega_d}{\sigma_d}.$$

The following lemma shows that in some sense we can also make use of this inequality in the case of finite packings of B^d.

Lemma 1. *Let* $A \subset E^d$, $\mathrm{card}(A) = k$ *such that* $\mathcal{B}_A$ *is a packing and* $C_k = \mathrm{conv}(A)$. *Then there is a finite set* A' *with* $A \subset A'$, $\mathcal{B}_{A'}$ *being a packing such that the following property holds. Let* $a \in A$ *and let* P_a *be the Voronoi polyhedron of* a *with respect to* A'. *Then*

$$P_a \subset C_k + 2B^d.$$

Proof. Let $C = C_k + B^d$. Now we successively add points of $\mathrm{bd}(C)$ to A such that the distance of any two points is at least 2. After finitely many steps we obtain a maximal set A'. Clearly, $\mathcal{B}_{A'}$ is a packing. Then $\mathrm{int}(P_a) \cap \mathrm{bd}(C) = \varnothing$, which yields the assertion. □

Proof of Theorem 1. Let $A = \{a_1, \ldots, a_k\}$ and let $D_1, \ldots, D_k$ denote the Dirichlet cells of $a_1, \ldots, a_k$ with respect to $C_k + 2B^d$. Further, let $P_1, \ldots, P_k$ be respective Voronoi polyhedra according to Lemma 1. Then we can apply Rogers's estimate to $P_1, \ldots, P_k$. Thus

$$\begin{aligned} \sigma V(C_k + B^d) \le k\omega_d &= k\omega_d \left(\sum_{i=1}^{k} V(D_i) \right)^{-1} V(C_k + 2B^d) \\ &\le k\omega_d \left(\sum_{i=1}^{k} V(P_i) \right)^{-1} V(C_k + 2B^d) \\ &\le \sigma_d V(C_k + 2B^d). \end{aligned}$$

So, by Steiner's formula, we have

$$\sigma \sum_{i=1}^{d} \omega_{d-i} V_i(C_k) \le \sigma_d \sum_{i=1}^{d} \omega_{d-i} 2^{d-i} V_i(C_k)$$

which completes the proof of Theorem 1. □

Proof of the Corollary. Using the fact that the intrinsic volumes V_i are homogeneous of degree i and monotonous and applying the estimates

$$\pi(\mathscr{F}_k) > \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{d+1}}, \qquad \sigma_d \leq \frac{d+2}{2} \left(\frac{1}{\sqrt{2}}\right)^d,$$

we have

$$\sum_{i=0}^{d} \pi(\mathscr{F}_k) \omega_{d-i} V_i(C_k) > \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{d+1}} \sum_{i=0}^{d} \binom{d}{i} \omega_d r^i$$

$$= \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{d+1}} \omega_d (1+r)^d,$$

$$\sum_{i=0}^{d} \sigma_d 2^{d-i} \omega_{d-i} V_i(C_k) \leq \frac{d+2}{2} 2^{d/2} \omega_d \left(1 + \frac{R}{2}\right)^d.$$

Thus, by Theorem 1

$$\sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{d+1}} (1+r)^d < \frac{d+2}{2} 2^{d/2} \left(1 + \frac{R}{2}\right)^d,$$

which yields the assertion. □

Proof of Theorem 2. Let $\mathscr{B}_A$ be a kissing number configuration of cardinality k in E^d. Assume that $0 = a_0 \in A$ and all other balls of $\mathscr{B}_A$ touch B^d. Let r be the maximum radius such that $rB^d \subset C_k = \text{conv}(A)$. We consider a point $x \in rB^d \cap \text{bd}(C_k)$ and a supporting hyperplane H of C_k through x. Since we cannot add another ball to $\mathscr{B}_A$ without violating the packing or kissing number property, the radius of $2B^d \cap H$ is at most $\sqrt{3}$ and we have $r \geq 1$. It follows that

$$2^d \omega_d \leq V(C_k + B^d).$$

Let M_d denote the maximum number of nonoverlapping equal spheres in E^d that can touch a sphere of the same size. In other words, $M_d + 1$ is the maximum cardinality of a kissing number configuration in E^d. Since

$$V(S_k + B^d) \leq V(S_{M_d} + B^d) < 2M_d \omega_{d-1},$$

Theorem 2 will be proved by showing that, for $d \geq 12$,

$$M_d \leq 2^{d-1} \frac{\omega_d}{\omega_{d-1}}.$$

A result of Rankin [18] states that

$$M_d \le \frac{2^{-3/2}\omega_{d-2}}{\omega_{d-3}\int_0^{\pi/4} (\sin t)^{d-2}(\cos t - \cos^{\pi/4})\, dt}.$$

To estimate the upper bound, observe that iterative integration by parts yields, for arbitrary nonnegative integer m,

$$\begin{aligned}\int_0^{\pi/4} \sin^{d-2} t\, dt &= \frac{1}{(d-1)2^{d/2}} \sum_{i=0}^{m} \left(\prod_{j=0}^{m-1} \frac{d+2j}{d+2j+1}\right)\frac{1}{2^i} \\ &\quad + \prod_{j=0}^{m}\left(\frac{d+2j}{d+2j-1}\right)\int_0^{\pi/4} \sin^{d+2mt} t\, dt \\ &\ge \frac{1}{(d-1)2^{d/2}} \sum_{i=0}^{m} \left(\frac{d}{d+1}\right)^i \frac{1}{2^i},\end{aligned}$$

thus

$$\int_0^{\pi/4} \sin^{d-2} t\, dt \ge \frac{d+1}{(d-1)(d+2)} \frac{1}{2^{(d-2)/2}}.$$

Using this inequality we deduce

$$\int_0^{\pi/4} \sin^{d-2} t(\cos t - \cos \pi/4)\, dt \ge \frac{1}{(d-1)(d+2)} \frac{1}{2^{(d-1)/2}}$$

and so, by means of Rankin's estimate,

$$M_d \le d(d+2)2^{(d-4)/2} \frac{\omega_d}{\omega_{d-1}}.$$

Thus we have

$$M_d \le 2^{d-1} \frac{\omega_d}{\omega_{d-1}},$$

at least for $d \ge 14$. But applying the estimate

$$\sqrt{\frac{2\pi}{d+1}} < \frac{\omega_d}{\omega_{d-1}}$$

and using the upper bounds

$$M_{12} \le 1416, \qquad M_{13} \le 2233,$$

which are due to Odlyzko and Sloane [17], it is obvious that the inequality holds for $d \geq 12$ which completes the proof of Theorem 2. □

6. Upper Bounds for the Volume of Parallel Bodies of the Regular Simplex

For the proof of Theorem 3 we shall need some uppcr estimates of the volume of parallel bodies of the regular simplex. Let T^d be the regular d-dimensional simplex of side 1 and $s \in \mathbb{R}$, $s \geq 0$. Then, by Steiner's formula,

$$V(sT^d + B^d) = \sum_{i=0}^{d} \omega_{d-i} s^i V_i(T^d) = \sum_{i=0}^{d} \omega_{d-i} s^i \sum_{f \in F_i(T^d)} \alpha(f) V^i(f).$$

Since T^d has $\binom{d+1}{i+1}$ i-dimensional faces, all being regular i-dimensional simplices of volume $\frac{(i+1)^{1/2}}{i!2^{i/2}}$, we have

$$V(sT^d + B^d) = \sum_{i=0}^{d} \binom{d+1}{i+1} \frac{(i+1)^{1/2}}{i!2^{i/2}} \omega_{d-i} \alpha_i s^i,$$

where α_i denotes the external angle $\alpha(T^i)$ of T^i considered as a face of T^d. In particular,

$$\alpha_0 = \frac{1}{d+1}, \qquad \alpha_d = 1.$$

Let

$$P_i(s) = \binom{d+1}{i+1} \frac{(i+1)^{1/2}}{i!2^{1/2}} \frac{\omega_{d-i}}{\omega_d} s^i.$$

Then

$$P_0(s) = (d+1)$$

and, by means of Stirling's formula,

$$P_d(s) = O\left(\frac{1}{d^{(d-1)/2}} \left(\frac{e}{2\pi}\right)^{d/2} s^d\right).$$

Now we also prove a respective estimate in the intermediate cases. Let

$$\kappa = \frac{1}{1 + 2\sqrt{\pi e}}.$$

Lemma 2. *Let $s \in \mathbb{R}$, $s \geq 0$, $t_0 = (es^2)^{1/4}[(4\pi d)^{1/4} + (es^2)^{1/4}]^{-1}$. Then*:

(a) $P_i(s) = O(d[1 + (es^2/4\pi d)^{1/4}]^{2d})$ *for* $1 \leq i \leq d-1$.
(b) $P_i(s) = O(d[2(s^2/d)^{\kappa/2}]^d)$ *for* $\kappa d \leq i \leq d-1$.

Proof. Setting $\lambda = i/d$, by Stirling's formula we have

$$P_i(s) = O\left(\frac{d^2}{i^{3/2}(d-i)} \frac{d^{3d/2} e^{i/2}}{i^{2i}(d-i)^{3(d-i)/2} 2^i \pi^{i/2}} s^i\right)$$
$$= O\left(\frac{1}{d^{1/2}\lambda^{3/2}(1-\lambda)} \left[\frac{e^{\lambda/2}}{2^\lambda \pi^{\lambda/2}} \frac{s^\lambda}{d^{\lambda/2}\lambda^{2\lambda}(1-\lambda)^{3(1-\lambda)/2}}\right]^d\right).$$

The first factor is maximal for $\lambda = 1/d$, thus

$$\frac{1}{d^{1/2}\lambda^{3/2}(1-\lambda)} \leq d.$$

Now, for $0 < t < 1$, let

$$y(t) = \frac{t}{2} \ln \frac{es^2}{4\pi d} - 2t \ln t - 2(1-t) \ln(1-t).$$

Then the second factor is bounded by $e^{y(\lambda)d}$. Furthermore, we have

$$y'(t) = \tfrac{1}{2} \ln \frac{es^2}{4\pi d} - 2 \ln t + 2 \ln(1-t),$$
$$y''(t) = -2\left(\frac{1}{t} + \frac{1}{1-t}\right) < 0.$$

Thus y is a concave functional which is maximal for t_0. So, with some calculation, we have

$$y(t) \leq y(t_0) = 2 \ln\left[1 + \left(\frac{es^2}{4\pi d}\right)^{1/4}\right].$$

Putting things together, we obtain the first asserted equality (a). Now, for $t_0 \leq t < 1$, $y(t)$ is decreasing. On the other hand, $t_0 = o(\kappa)$ and, with some further calculation,

$$y(\kappa) = \frac{2}{1 + 2\sqrt{\pi e}} \left[\ln\left(\frac{es^2}{4\pi d}\right)^{1/4} - 2\sqrt{\pi e} \ln(2\sqrt{\pi e})\right] + 2 \ln(1 + 2\sqrt{\pi e})$$
$$\leq \ln 2 + \frac{\kappa}{2} \ln \frac{s^2}{d},$$

thus

$$P_i(s) = O\left(d\left[2\left(\frac{s^2}{d}\right)^{\kappa/2}\right]^d\right)$$

for $\kappa d \leq i \leq d-1$, which also proves (b). □

Using Lemma 2 and the trivial bound $\alpha_i \leq 1$ we obtain an upper bound for $V(sT^d + B^d)$. But we can do even better if we take Hadwiger's [13] estimate

$$\alpha_i < \sqrt{i+1}\left(2\sqrt{\pi}\frac{i}{d-i}\right)^i\left(-\ln 2\sqrt{\pi}\frac{i}{d-i}\right)^{i/2}$$

for $1 \leq i \leq \kappa d$. As Hadwiger [13] further shows

$$P_i(s)\alpha_i \leq \frac{1}{(2d)^{i/2}}\binom{d+\sqrt{d}}{d}\frac{1}{d^{\sqrt{d/2}}}\frac{\omega_{d-i}}{\omega_d}s^i o(d^{\sqrt{d}}).$$

Using Stirling's formula again we obtain the following estimate for $P_i(s)\alpha_i$, $1 \leq i \leq \kappa d$, $s \leq 2\sqrt{\pi}$.

Lemma 3. *Let $s \in \mathbb{R}$, $0 \leq s \leq 2\sqrt{\pi}$, $1 \leq i \leq \kappa d$. Then*

$$P_i(s)\alpha_i = o(d^{\sqrt{d}}).$$

Proof. Setting $\lambda = i/d$ we have

$$P_i(s)\alpha_i = \frac{1}{2^i e^{i/2}\pi^{i/2}}\left(1+\frac{1}{\sqrt{d}}\right)^{d+\sqrt{d}}\left(\frac{d}{d-i}\right)^{(d-i)/2} s^i o(d^{\sqrt{d}})$$
$$= \left[\frac{1}{(4\pi e)^{\lambda/2}}\left(\frac{1}{1-\lambda}\right)^{(1-\lambda)/2} s^\lambda\right]^d o(d^{\sqrt{d}}).$$

Now, for $0 \leq t \leq \kappa$, let

$$y(t) = \frac{t}{2}\ln\frac{s^2}{4\pi e} - \frac{1-t}{2}\ln(1-t).$$

Then, of course,

$$P_i(s)\alpha_i = e^{y(\lambda)d}o(d^{\sqrt{d}}).$$

Furthermore, since $y'(t) = \frac{1}{2}(\ln(s^2/4\pi) + \ln(1-t))$ the functional y is decreasing, thus

$$y(t) \leq y(0) = 0,$$

which yields the assertion. □

7. Proof of Theorem 3

Proof of Part (a). Let $T = \mathrm{conv}(A)$ and let T_n denote those parts of $T + B^d$ which are contained in the sum of an n-dimensional face of T and the associated $(d-n)$-dimensional cone of outer normals. Furthermore, let $K_n = T_n \cap \bigcup\{a + B^d \mid a \in A\}$. Then, by means of Section 3, we have

$$\begin{aligned} V(T + B^d) &= \sum_{n=0}^{i} V(T_n) \\ &= \omega_d + \sum_{n=1}^{i} V(K_n)\frac{\omega_{d-n}\omega_n}{\sigma_n \omega_d} > \omega_d + \frac{2\omega_{d-1}}{\omega_d}\sum_{n=1}^{i} V(K_n) \\ &= V(S_{i+1} + B^d), \end{aligned}$$

which proves part (a) of the theorem. □

Proof of Part (b). Using the notation of Section 6 we have

$$\pi^{-1}(\mathcal{B}_{A_i}) = \frac{V(sT + B^d)}{(d+1)\omega_d} = \frac{1}{d+1}\left(1 + \sum_{i=1}^{d} P_i(s)\alpha_i\right).$$

Now let $0 \le s \le 2\sqrt{\pi}$. Then, essentially by Lemmas 2(b) and 3, we obtain

$$\pi^{-1}(\mathcal{B}_{A_i}) = o(d^{\sqrt{d}}).$$

Furthermore, if $0 \le s \le \sqrt{d}/13$, then, by Lemma 2(a),

$$P_i(s) = O\left(d\left[1 + \left(\frac{e}{13^2 \times 4\pi}\right)^{1/4}\right]^{2d}\right) = o(2^{d/2})$$

for $1 \le i \le d-1$ and also

$$P_d(s) = o(2^{d/2}),$$

thus

$$\pi^{-1}(\mathcal{B}_{A_i}) = o(2^{d/2}),$$

which completes the proof of part (b). □

8. A Lower Bound for the Volume of Parallel Bodies of the Regular Simplex

For the proof of Theorem 4 we need a lower estimate for the volume of parallel bodies of the regular simplex. Let $s, r \in \mathbf{R}$, $s, r \ge 0$. As in Section 6 we have

$$V(sT^d + rB^d) = \sum_{i=0}^{d}\binom{d+1}{i+1}\frac{1}{i!}\sqrt{\frac{i+1}{2^i}}\, r^{d-i}s^i\omega_{d-i}\alpha_i.$$

As we have pointed out in Section 6, Hadwiger [13] gave upper estimates of the external angles α_i which, in particular, for $i=1$, $d\geq 3$, read as follows:

$$\alpha_1 < 2\sqrt{2\pi}\,\frac{1}{d-1}\sqrt{\ln(d-1)-\ln 2\sqrt{\pi}} = O\left(\frac{\sqrt{\ln d}}{d}\right).$$

We now deduce a lower bound of type $O((\ln d)^{1/4}/d^2)$.

Lemma 4. *Let $d\geq 3$. Then we have*

$$\frac{1}{2\sqrt{5}}\,\frac{(\ln d)^{1/4}}{d(d+1)} < \alpha_1.$$

Proof. Using results of Ruben [21] and Hadwiger [13] we have

$$\alpha_1 = \sqrt{2}\int_0^1 f(t)t^{d-1}\,dt,$$

where

$$f(t) = e^{-h^2(\sqrt{\pi}(t-1/2))}, \qquad 0\leq t\leq 1$$

(with the usual convention $e^{-\infty}=0$) and $h(z)$ is defined via

$$\int_0^{h(z)} e^{-t^2}\,dt = z.$$

Since

$$h'(z) = e^{h^2(z)}$$

we have

$$f''(t) = -2\pi\, e^{h^2(\sqrt{\pi}(t-1/2))} < 0,$$

thus f is a concave functional of t. Furthermore, f is symmetrical with respect to $\frac{1}{2}$, i.e.,

$$f(t)=f(1-t) \quad\text{and}\quad f(0)=0, \qquad f(\tfrac{1}{2})=1.$$

So, for every point t_0, $0<t_0<\frac{1}{2}$, we have

$$g_{t_0}(t)\leq f(t) \qquad \text{for}\quad 0\leq t\leq 1,$$

where

$$g_{t_0}(t) = \begin{cases} \dfrac{f(t_0)}{t_0}\,t & \text{for}\quad 0\leq t\leq t_0,\\[2ex] \dfrac{2(f(t_0)-1)}{1-2t_0}(\tfrac{1}{2}-t)+1 & \text{for}\quad t_0\leq t\leq \tfrac{1}{2},\end{cases}$$

and

$$g_{t_0}(t) = g_{t_0}(1-t).$$

In order to deduce a lower bound for $f(t_0)$ we have to give an upper estimate for $h(\sqrt{\pi}(\frac{1}{2}-t_0))$, which, in view of the inequality

$$\frac{\sqrt{\pi}}{2}-\frac{e^{-z^2}}{2z}\leq\int_0^z e^{-t^2}\,dt,$$

means finding solutions x_0 of

$$\sqrt{\pi}(\tfrac{1}{2}-t_0)\leq\frac{\sqrt{\pi}}{2}-\frac{e^{-x_0^2}}{2x_0}.$$

Setting

$$t_0=\frac{1}{2\sqrt{\pi}d},\qquad x_0=[\ln d-\tfrac{1}{4}\ln(\ln d)]^{1/2}$$

we have

$$e^{-x_0^2}=\frac{1}{d}(\ln d)^{1/4}\leq\frac{1}{d}[\ln d-\tfrac{1}{4}\ln(\ln d)]^{1/2}=2\sqrt{\pi}\,t_0x_0$$

and thus

$$h^2(\sqrt{\pi}(\tfrac{1}{2}-t_0))\leq\ln d-\tfrac{1}{4}\ln(\ln d).$$

So, with some calculation we obtain

$$\begin{aligned}\alpha_1&=\sqrt{2}\int_0^1 f(t)t^{d-1}\,dt\geq\sqrt{2}\frac{f(t_0)}{t_0}\int_{1-t_0}^1(1-t)t^{d-1}\,dt\\&=\sqrt{2}\frac{f(t_0)}{t_0}\left\{\frac{1}{d}[1-(1-t_0)^d]-\frac{1}{d+1}[1-(1-t_0)^{d+1}]\right\}\\&\geq 2\sqrt{2\pi}\left[1-\left(1+\frac{1}{2\sqrt{\pi}}\right)\left(1-\frac{1}{6\sqrt{\pi}}\right)^3\right]\frac{\ln d}{d(d+1)}\\&>\frac{1}{2\sqrt{5}}\frac{(\ln d)^{1/4}}{d(d+1)}.\end{aligned}$$

Let us point out that this estimate can easily be improved for small dimensions, but here we are mainly interested in its asymptotical behavior. □

Using Lemma 4 we at once obtain a suitable lower bound for $V(sT^d+B^d)$.

Corollary. *Let $d\geq 3$, r, $s\in\mathbb{R}$, r, $s\geq 0$. Then*

$$\omega_d r^d+\frac{1}{4\sqrt{5}}(\ln d)^{1/4}\omega_{d-1}r^{d-1}s\leq V(sT^d+rB^d).$$

9. Proof of Theorem 4

The proof is an application of the previous section and the following covering result.

Lemma 5. *Let* $r_0 = (1-1/d)^{1/2}$. *Then*

$$s_0 T^d + r_0 B^d \subset \bigcup_{a \in A_0} a + B^d.$$

Proof. As is well known (and easily calculated) the circumsphere radius of the n-dimensional regular simplex of side 1 is $(n/[2(n+1)])^{1/2}$. Thus, since $s_0 < 1$, in particular we have

$$S_0 T^d \subset \bigcup_{a \in A_0} (a + B^d)$$

and trivially

$$\bigcup_{a \in A_0} (a + r_0 B^d) \subset \bigcup_{a \in A_0} (a + B^d).$$

Now let $1 \leq n \leq d-1$ and let f be an n-dimensional face of $s_0 T^d$. Furthermore, let p be an outer normal of $s_0 T^d$ (taken at 0) with length r_0. Then it is sufficient to prove the following inclusion;

$$p + f \subset \bigcup_{a \in A_0 \cap f} (a + B^d).$$

Let $a \in A_0 \cap f$ and $E = \operatorname{aff}(p+f)$. Then $(a + B^d) \cap E$ is an n-dimensional ball of radius $(1 - r_0^2)^{1/2}$. Since $p + f$ is itself an n-dimensional simplex of side s_0 the assertion follows from

$$s_0 \left(\frac{n}{2(n+1)} \right)^{1/2} \leq (1 - r_0^2)^{1/2}. \qquad \square$$

Proof of Theorem 4. By Lemma 5 and the corollary to Lemma 4 we have

$$\gamma^{-1}(\mathcal{B}_{A_0}) \geq \frac{V(s_0 T^d + r_0 B^d)}{(d+1)\omega_d} \geq \frac{1}{d+1} r_0^d + \frac{1}{4\sqrt{5}} \frac{(\ln d)^{1/4}}{d+1} \frac{\omega_{d-1}}{\omega_d} r_0^{d-1} s_0.$$

So, using

$$\omega_{d-1} > \sqrt{\frac{d}{2\pi}}\, \omega_d,$$

we obtain, with some easy calculation,

$$\gamma(\mathcal{B}_{A_0}) < 4\sqrt{5\pi e}\, \frac{d+1}{(\ln d)^{1/4}},$$

which completes the proof of Theorem 4. $\square$

10. Proof of Theorem 5

Let Q be the octahedron

$$Q = 2\,\mathrm{conv}\{(u, u, 0), (u, -u, 0), (-u, u, 0)(-u, -u, 0), (0, 0, u), (0, 0, -u)\}$$

and let $s \in \mathbb{N}$. Let $G(s)$ denote the number of lattice points of $\mathcal{G}$ contained in sQ. Then

$$G(s) = (2s+1)^2 + 2\sum_{i=0}^{s-1}(2i+1)^2 + 2\sum_{i=0}^{s-1}(2i+2)^2$$
$$= \tfrac{16}{3}s^3 + 8s^2 + \tfrac{14}{3}s + 1.$$

After some easy calculations we obtain, for the intrinsic volumes of Q,

$$V_0(Q) = 1,$$

$$V_1(Q) = 4 \times \frac{8}{\sqrt{3}} \times \tfrac{1}{4} + 8 \times 4 \times \tfrac{1}{6} = \tfrac{8}{3}(2+\sqrt{3}),$$

$$V_2(Q) = 8\,\frac{16 \times \sqrt{2}}{3}\,\frac{1}{2} = \frac{64\sqrt{2}}{3},$$

$$V_3(Q) = \frac{512}{9\sqrt{3}}.$$

Thus

$$V(sQ + B^d) = \frac{4\pi}{3} + \pi\frac{8}{3}(2+\sqrt{3})s + 2 \times \frac{64\sqrt{2}}{3}s^2 + \frac{512}{9\sqrt{3}}s^3$$

and so

$$V(sQ + B^d) - V(S_{G(s)} + B^d) = \frac{32}{3}\left(\frac{16}{3\sqrt{3}} - \pi\right)s^3 + 16\left(\frac{8\sqrt{2}}{3} - \pi\right)s^2$$
$$+ \frac{4\pi}{3}(4 + 2\sqrt{3} - 7)s.$$

Therefore we have

$$G(s) > 0 \quad \text{for } s \leq 15, \qquad G(s) < 0 \quad \text{for } s \geq 16$$

with

$$G(16) = 23{,}909,$$

which proves the theorem. □

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