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RESEARCH ARTICLE

RIGHT PP MONOIDS WITH CENTRAL IDEMPOTENTS

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For a monoid S, a (right, unitary) S-system is a set M in which for all s,t ε S and all m ε M the product ms ε M is defined and m(st) = (ms)t, m1 = m. An S-system P is projective when for every epimorphism g : M $\rightarrow$ N and every homomorphism h : P $\rightarrow$ N there is a homomorphism k : P $\rightarrow$ M such that gk = h where M,N are S-systems. A monoid S is called right PP when every principal right ideal of S is projective. It is right [semi-] hereditary when all [finitely generated] right ideals of S are projective.

Commutative PP monoids have been investigated by Kilp in [6] where it is proved that such monoids are semilattices of cancellative monoids. The present note provides a generalization of this result to the case of right PP monoids with central idempotents. The main result is that such monoids are semilattices of left cancellative monoids. From this we can deduce results giving the structure of right semi-hereditary and right hereditary monoids with central idempotents generalizing the commutative results obtained by Dorofeeva in [2]. An easy specialization yields the theorem of Feller [4] on the structure of right hereditary principal right ideal monoids with central idempotents.

We begin by recalling a definition and two basic results. Let e be an idempotent in a monoid S. An element a in S is *left* e-*cancellable* if ae = a and for all $s,t \in S$, as = at implies es = et. From [6], we have

RESULT 1. *A monoid* S *is a right* PP *monoid if and only if every element of* S *is a left* e-*cancellable element for some idempotent* e *in* S.

From [2], we have

RESULT 2. *A monoid* S *is right semi-hereditary [hereditary] if and only if*

(i) S *is right* PP

(ii) *incomparable principal right ideals are disjoint and*

[(iii) S *satisfies the maximum condition for principal right ideals*].

We also need the following lemma whose easy proof is left to the reader.

LEMMA 1. *If the idempotents of a right* PP *monoid* S *commute, then for each element* $a \in S$ *there is a unique idempotent* e *such that* a *is left* e-*cancellable*.

From now on S will denote a right PP monoid with central idempotents. For each idempotent e in S, put

$$T_e = \{a \in S : a \text{ is left e-cancellable}\}.$$

In view of Lemma 1, $T_e \cap T_f = \emptyset$ when $e \neq f$, and from

Result 1 we see that $S = \bigcup_{e \in E} T_e$ where E is the set of idempotents of S.

Now let $e,f \in E$ and $a \in T_e$, $b \in T_f$. Then $abef = aebf = ab$ and if $abs = abt$ $(s,t \in S)$, then $ebs = ebt$ so that $bes = bet$ and thus $fes = fet$. Hence $ab \in T_{ef}$ and similarly $ba \in T_{ef}$. In particular, for each $e \in E$, T_e is a subsemigroup with identity e. In fact, T_e is left cancellative since $ab = ac$ for $a,b,c \in T_e$ implies $eb = ec$, that is, $b = c$.

Furthermore, if we define $\phi_{e,f} : T_e \to T_f$ for $f \leqslant e$ by $\phi_{e,f}(a) = fa$, then this is a monoid homomorphism and if $h \leqslant f \leqslant e$, then clearly $\phi_{f,h}\phi_{e,f} = \phi_{e,h}$ and $\phi_{e,e}$ is the identity mapping of T_e.

Finally, if $a \in T_e$, $b \in T_f$, then $ab = abef = aefbef = \phi_{e,ef}(a)\phi_{f,ef}(b)$.

Thus we have proved the converse part of

THEOREM 1. *Let Y be a semilattice with identity, and to each element α of Y assign a left cancellative monoid T_α such that T_α and T_β are disjoint if $\alpha \neq \beta$ in Y. To each pair of elements α,β of Y such that $\alpha > \beta$ assign a monoid homomorphism $\phi_{\alpha,\beta} : T_\alpha \to T_\beta$ such that if $\alpha > \beta > \gamma$, then*

$$\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma} .$$

Let $\phi_{\alpha,\alpha}$ be the identity automorphism of T_α. Let $S = \bigcup_{\alpha \in Y} T_\alpha$ and define the product of elements a_α, b_β of S $(a_\alpha \in T_\alpha, b_\beta \in T_\beta)$ by

$$a_\alpha b_\beta = \phi_{\alpha,\gamma}(a_\alpha)\phi_{\beta,\gamma}(b_\beta)$$

where $\gamma = \alpha\beta$ in Y.

Then S is a right PP monoid with central idempotents. Conversely, every such monoid can be constructed in this manner.

Proof. The converse part has been established using the semilattice E of idempotents of S for Y.

For the direct part, the proof of associativity is routine and is omitted. Let e_α be the identity of T_α. Clearly S has no idempotents other than the e_α's. Let a be an element of S, say $a \in T_\beta$, and let $\gamma = \alpha\beta$. Since $\phi_{\alpha,\gamma}$ is a monoid homomorphism, we have

$$e_\alpha a = \phi_{\alpha,\gamma}(e_\alpha)\phi_{\beta,\gamma}(a) = e_\gamma\phi_{\beta,\gamma}(a) = \phi_{\beta,\gamma}(a)e_\gamma$$

$$= \phi_{\beta,\gamma}(a)\phi_{\alpha,\gamma}(e_\alpha) = ae_\alpha$$

and so the idempotents of S are central.

Certainly $ae_\beta = a$. Now let $s,t \in S$, say $s \in T_\alpha, t \in T_\gamma$ and suppose that as = at. Then we must have $\beta\alpha = \beta\gamma = \delta$, say, and $\phi_{\beta,\delta}(a)\phi_{\alpha,\delta}(s) = \phi_{\beta,\delta}(a)\phi_{\gamma,\delta}(t)$. Since T_δ is left cancellative, this gives $\phi_{\alpha,\delta}(s) = \phi_{\gamma,\delta}(t)$ and it follows that $e_\beta s = e_\beta t$. Thus a is left e_β-cancellable and S is a right PP monoid with central idempotents.

We note that the semilattice Y is isomorphic to the semilattice of idempotents of S.

Remark. Clearly by omitting the condition that the semilattice Y has an identity we obtain a stucture theorem for semigroups in which the idempotents are central and each element is left e-cancellable for some idempotent e. If a is a regular element in such a semigroup, then clearly a is left a'a-cancellable where a' is an inverse of a. Putting a'a = e, we have $a \in T_e$ and from the earlier discussion aa' is also a member of T_e so that aa' = e. But a' is left aa'-cancellable and so $a' \in T_e$. Hence a has a unique inverse and is a unit of T_e. Thus if S is a regular semigroup (and so _a fortiori_ each element is left e-cancellable for some idempotent e) whose idempotents are central, then each T_e is a group and we re-capture the well-known theorem on the structure of such semigroups [1, chapter 4].

Returning to the case of a right PP monoid S with its set E of idempotents in the centre of S, we make the following observation:

(A) If $a,b \in S$ and $ab \in T_e$ $(e \in E)$, then $a \in T_f$, $b \in T_h$ for some $f,h \in E$ with $fh = e$.

Now assume that S is right semi-hereditary. If $e,f \in E$, then $ef \in eS \cap fS$ and consequently, by Result 2, eS and fS are comparable. It follows that E is a chain.

If the chain E has a zero z, then T_z is an ideal of S and so $aT_z = aS$ for $a \in T_z$. It follows that incomparable principal right ideals of T_z are disjoint. Since T_z is also left cancellative, Result 2 gives that it is right semi-hereditary.

If e is a non-zero element of E, let $f \in E$ with $f < e$. For an element t of T_e we have $tf = ft \in fS \cap tS$ and consequently, by Result 2, fS and tS are comparable. Clearly $tf \in T_f$ so that $tf \neq t$ and hence $fS \subseteq tS$, say $f = ta$ $(a \in S)$. Then $f = ft.fa$ and it follows from (A) that $ft, fa \in T_f$. As T_f is left cancellative, ft is a unit of T_f. Also if $t_1, t_2 \in T_e$, then $f \in t_1S \cap t_2S$ and so $t_1S \subsetneq t_2S$, say. Thus $t_1 = t_2b = t_2eb$ ($b \in S$) and by (A), we have $eb \in T_e$ giving $t_1T_e \subseteq t_2T_e$. The principal right ideals of T_e are therefore linearly ordered.

Now assume that S is right hereditary. From Result 2 we have that S satisfies the maximum condition for principal right ideals and so clearly E must be an inversely well-ordered chain. From the preceding two paragraphs it is clear that for any $e \in E$ and $t_1, t_2 \in T_e$ we have $t_1T_e \subseteq t_2T_e$ if and only if $t_1S \subseteq t_2S$, so that each T_e satisfies the maximum condition for principal right ideals. Hence each T_e is certainly right hereditary. Furthermore, if e is a non-zero element of E, then since the principal right ideals are also linearly ordered, it follows that T_e is a principal right ideal monoid.

We have thus demonstrated the converse part of

THEOREM 2. *Let Y be a chain with a maximum element [an inversely well-ordered chain]. To each element α of Y assign a left cancellative monoid T_α such that T_α and T_β are disjoint if $\alpha \neq \beta$ in Y. Further, if Y has a zero ζ, let T_ζ be right semi-hereditary [right hereditary] and if α is a non-zero element of Y let T_α have its principal right ideals linearly ordered [be a principal right ideal monoid]. To each pair of elements α, β of Y such that $\alpha > \beta$ assign a monoid homomorphism $\phi_{\alpha,\beta} : T_\alpha \to T_\beta$ such that the image of T_α under $\phi_{\alpha,\beta}$ is contained in the group of units of T_β and if $\alpha > \beta > \gamma$, then $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$. Let $\phi_{\alpha,\alpha}$ be the identity automorphism of T_α. Let $S = \bigcup_{\alpha \in Y} T_\alpha$ and define the* *product of elements* a_α, b_β *of* S $(a_\alpha \in T_\alpha,\ b_\beta \in T_\beta)$ *by*

$$a_\alpha b_\beta = \phi_{\alpha,\gamma}(a_\alpha)\phi_{\beta,\gamma}(b_\beta)$$

where $\gamma = \alpha\beta$ *in* Y.

Then S *is a right semi-hereditary [right hereditary] monoid with central idempotents. Conversely, every such monoid can be constructed in this manner.*

Proof. We have to prove the direct part. By Theorem 1, S is a right PP monoid with central idempotents. Let a,b be elements of S with $a \in T_\alpha$, $b \in T_\beta$. If Y has a zero ζ and $\alpha = \beta = \zeta$, then T_ζ is an ideal of S and aS = aT_ζ, bS = bT_ζ. Since T_ζ is right semi-hereditary Result 2 gives that aS, bS are comparable or disjoint. If $\alpha = \beta$ and α is a non-zero element of Y, then aT_α, bT_α are comparable and so clearly aS, bS are comparable. If $\alpha \neq \beta$, say $\alpha > \beta$, then $\phi_{\alpha,\beta}(a)$ is a unit of T_β, so that $b = \phi_{\alpha,\beta}(a)c$ where $c = \phi_{\alpha,\beta}(a)^{-1}b \in T_\beta$. Hence b = ac and bS $\subseteq$ aS. Thus aS, bS are comparable or disjoint and by Result 2, S is right semi-hereditary.

To prove the bracketed statement we have now, by Result 2, simply to show that S satisfies the maximum condition for principal right ideals. Let $P = \{I_\lambda : \lambda \in \Lambda\}$ be a collection of principal right ideals of S. Let α be the largest member of Y such that $I_\lambda \cap T_\alpha$ is non-empty for at least one $\lambda \in \Lambda$. If Y has a zero ζ and $\alpha = \zeta$, then $I_\lambda \subseteq T_\zeta$ for all $\lambda \in \Lambda$ so that each I_λ is a principal right ideal of T_ζ. Now T_ζ is a right hereditary and consequently, by Result 2, P has a maximal member. If α is a non-zero element of Y, then since T_α is a principal right ideal monoid its right ideals are inversely well-ordered. Hence there is a maximal one among the right ideals $T_\alpha \cap I_\lambda$ of T_α. Let this be $T_\alpha \cap aS$. Clearly $a \in T_\alpha$ and since for $\alpha > \beta$, $\phi_{\alpha,\beta}(a)$ is a unit of T_β, it follows that $T_\beta \subseteq aS$ for all $\beta < \alpha$. Clearly then, aS is maximal in the set P and so S does have the desired property.

The structure of right hereditary principal right ideal monoids with central idempotents has been considered by Feller [4]. His Theorem, which reads as the bracketed version of Theorem 2 with the exception that T_α is a principal right ideal monoid for all α in Y, is now easily obtained from Theorem 2.

The results presented here lead one to ask about the structure of various types of left cancellative monoids. In general it seems that there is little one can say. In the commutative case many equivalent characterisations of semi-hereditary and hereditary cancellative monoids have been given in [3]. However, several of these characterisations have no analogue in the general left cancellative case and for some of the others the non-commutative analogues are not equivalent. We do have that for a monoid T the following conditions are obviously equivalent:

(i) *The set of all principal right ideals of* T *is linearly ordered*,

(ii) *The set of all right ideals of* T *is linearly ordered*,

(iii) *Every finitely generated right ideal of* T *is principal*.

But while a left cancellative monoid satisfying these conditions is right semi-hereditary, even a right and left hereditary cancellative monoid need not satisfy them as is evidenced by a non-commutative free monoid. This also shows that the extra possibilities allowed for in Theorem 2 when Y has a zero are genuine. The one non-commutative situation where there is a definitive result is the case of left cancellative principal right ideal monoids, or equivalently left cancellative monoids whose principal right ideals are inversely well-ordered by inclusion. The structure of such a monoid (actually one satisfying the left-right dual condition) has been determined by Hogan [5] in terms of ordinal arithmetic and a group. We refer the reader to the cited paper for details.

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