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Reproducing Kernels for Harmonic Bergman Spaces of the Unit Ball

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Abstract. We study reproducing kernels for harmonic Bergman spaces of the unit ball in $\mathbf{R}^n$. We establish some new properties for the reproducing kernels and give some applications of these properties.

1. Introduction

Let B denote the open unit ball in $\mathbf{R}^n$ for $n \geqslant 2$ and V be the Lebesgue volume measure on $\mathbf{R}^n$. The harmonic Bergman space $b^2_\alpha(B)$, with $\alpha > -1$, is the set of all complex-valued harmonic functions u on B with

$$\|u\|_{l_{2,\alpha}} = \left(\int_B |u(x)|^2 \, (1 - |x|^2)^\alpha \, dV(x) \right)^{\frac{1}{2}} < \infty.$$

Point evaluation is a bounded linear functional on $b^2_\alpha(B)$. Hence for every $x \in B$, there exists a unique $R_\alpha(x, \cdot) \in b^2_\alpha(B)$ such that

$$u(x) = \int_B u(y) R_\alpha(x, y)(1 - |y|^2)^\alpha \, dV(y)$$

for all $u \in b^2_\alpha(B)$. The functions $R_\alpha(x, \cdot)$ are called reproducing kernels for $b^2_\alpha(B)$. We will see that each R_α is real valued for $\alpha > -1$ in Section 3.

The purpose of this paper is to study these reproducing kernels. These reproducing kernels have been studied by different authors in [1], [3], [4], and [8]. While reproducing kernels for (analytic) Bergman spaces of the unit ball in $\mathbf{C}^n$ have simple formulas in closed form, those for harmonic Bergman spaces are much more complicated, and it appears to be impossible to find formulas in closed form for R_α in general, except when $n = 2$. In Section 2, we point out how harmonic reproducing kernels behave differently from analytic ones on the unit disk. In Section 3, we give a representation for R_α in terms of zonal harmonics in higher dimensions and establish some properties for R_α. We use an estimate on R_α given recently in [8] to prove the last property for R_α. In the last two sections, we give some applications of these properties.

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2. Reproducing Kernels on the Unit Disk

We consider R_α when $n = 2$ in this section. Let D denote the open unit disk in the complex plane $\mathbf{C}$ and A be the Lebesgue area measure on D. For $\alpha > -1$, the analytic Bergman space $A^2_\alpha(D)$ is the set of all analytic functions in $L^2(D, (1-|z|^2)^\alpha dA(z))$. Let K_α be the reproducing kernel for $A^2_\alpha(D)$, i.e.,

$$f(z) = \int_D f(w)\bar{K}_\alpha(z,w)(1-|w|^2)^\alpha dA(w), \quad z \in D,$$

for all $f \in A^2_\alpha(D)$. We know that

$$\bar{K}_\alpha(z,w) = \frac{\alpha+1}{\pi}\frac{1}{(1-z\bar{w})^{2+\alpha}}, \quad z,w \in D.$$

The reproducing kernels for $b^2_\alpha(D)$ are closely related to $\bar{K}_\alpha(z,w)$. We have (see page 357 of [11])

$$R_\alpha(z,w) = \frac{\alpha+1}{\pi}\left(2\mathrm{Re}\frac{1}{(1-z\bar{w})^{2+\alpha}} - 1\right), \quad z,w \in D.$$

For $z \in D, r \in (0,1)$, let $D_r(z) = \{w \in \mathbf{C} : |w-z| < r(1-|z|)\}$. An important property for $K_\alpha(z,w)$ is that

$$|K_\alpha(z,w)| \approx 1/(1-|z|)^{2+\alpha}, \quad w \in D_r(z).$$

Here the notation "$\approx$" indicates that the quotient of two positive quantities is bounded above and below by constants when the variable varies. For the unit disk, one usually uses the pseudo-hyperbolic disk instead of $D_r(z)$ because of its connection with Möbius transformations; see [2] for example. However we will use the obvious extension of $D_r(z)$ for higher dimensions in the next section.

We find that $R_\alpha(z,w)$ behaves quite differently from $K_\alpha(z,w)$.

Proposition 1. *For each $r \in (0,1)$, there exist $z \in D$ and $\alpha > -1$ such that $R_\alpha(z,w) = 0$ for some $w \in D_r(z)$.*

Proof. For $z, w \in D$, we have

$$R_\alpha(z,w) = \frac{\alpha+1}{\pi}\left(2\mathrm{Re}\frac{(1-z\bar{w})^{2+\alpha}}{|1-z\bar{w}|^{4+2\alpha}} - 1\right).$$

Let $z = t \in (0,1)$ and $\dfrac{1}{t} - w = se^{i\theta}$, where $s > 0$. It is easy to see that for $w \in D_r(t)$,

$$-\frac{rt}{1+t} = -\frac{r(1-t)}{\frac{1}{t}-t} < \sin\theta < \frac{r(1-t)}{\frac{1}{t}-t} = \frac{rt}{1+t},$$

and the range of θ is $\left(-\arcsin\dfrac{rt}{1+t}, \arcsin\dfrac{rt}{1+t}\right)$ when w ranges over $D_r(t)$. Since $|1-t\bar{w}| \leqslant (1-t)(1+t+rt)$ for $w \in D_r(t)$, we can choose t close enough to 1 such that $|1-t\bar{w}|^{2+\alpha} \leqslant \frac{1}{2}$. If we choose α large enough, then the range of

$\cos(2+\alpha)\theta$ is $[-1, 1]$ when w ranges over $D_r(t)$. Hence the conclusion follows from

$$R_\alpha(t,w) = \frac{\alpha+1}{\pi}\left(\frac{2\cos(2+\alpha)\theta - |1-t\bar{w}|^{2+\alpha}}{|1-t\bar{w}|^{2+\alpha}}\right). \qquad \square$$

It is not difficult to see from the proof above that we still have

$$R_\alpha(z,w) \approx 1/(1-|z|)^{2+\alpha}, \quad w \in D_r(z),$$

provided that r is small enough (depending only on α). In the next section we will prove this property for $R_\alpha(x,y)$ in higher dimensions.

3. Some Properties of the Reproducing Kernels

In order to give a description of R_α, we need to introduce zonal harmonics first. Let $\mathscr{H}_m(\mathbf{R}^n)$ denote the space of all homogeneous harmonic polynomials on $\mathbf{R}^n$ of degree m. A spherical harmonic of degree m is the restriction to S, the unit sphere, of an element of $\mathscr{H}_m(\mathbf{R}^n)$. The collection of all spherical harmonics of degree m is denoted by $\mathscr{H}_m(S)$. For every $\eta \in S$, there exists a unique $Z_m(\eta,\cdot) \in \mathscr{H}_m(S)$ such that

$$p(\eta) = \int_S p(\zeta) Z_m(\eta,\zeta)\, d\sigma(\zeta)$$

for all $p \in \mathscr{H}_m(S)$, where σ is the normalized surface-area measure on S. The spherical harmonic $Z_m(\eta,\cdot)$ is called the zonal harmonic of degree m. One can extend the zonal harmonic to a function on $\mathbf{R}^n \times \mathbf{R}^n$ by making Z_m homogeneous of degree m in the second variable as well as in the first. Let h_m denote the dimension (over $\mathbf{C}$) of the vector space $\mathscr{H}_m(S)$. One can compute h_m explicitly (see Ex 5.5 of [1]):

$$h_m = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2},$$

for $m > 0$. Also, $h_0 = 1$.

The following lemma states some properties of zonal harmonics that we will need. For more information of zonal harmonics, see Chapter 5 of [1].

Lemma 2. *Let m be a non-negative integer.*
(i) *If $\zeta, \eta \in S$, then $Z_m(\zeta,\zeta) = Z_m(\eta,\eta) = h_m$;*
(ii) *If $\zeta \in S$, then $\max_{\eta\in S}|Z_m(\zeta,\eta)| = Z_m(\zeta,\zeta) = h_m$.*

Now we can state the following representation for R_α.

Proposition 3. *Let $\alpha > -1$. If $x, y \in B$, then*

$$R_\alpha(x,y) = \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} Z_m(x,y).$$

The series converges absolutely and uniformly on $K \times B$ for every compact $K \subset B$.

Proof. This can be proved using the same argument as for the proof of Theorem 8.9 of [1]. $\square$

Since Z_m is real valued for each m (see Theorem 5.24 of [1]), we see that R_α is real valued.

All constants that depend only on α, n or other parameters and do not depend on the variable in B will be denoted by a single letter "C". Now we can give an estimate for R_α.

Proposition 4. *Let $\alpha > -1$. Then*
(i) $R_\alpha(x,x) \approx 1/(1-|x|)^{n+\alpha}$ *for* $x \in B$;
(ii) $\|R_\alpha(x,\cdot)\|^2_{2,\alpha} \approx 1/(1-|x|)^{n+\alpha}$ *for* $x \in B$;
(iii) $|R_\alpha(x,y)| \leqslant C/(1-|x|\,|y|)^{n+\alpha}$ *for* $x, y \in B$.

Proof. First we prove (i). By Proposition 3 and (i) of Lemma 2, we have

$$R_\alpha(x,x) = \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} h_m |x|^{2m}.$$

Since $h_m \approx (m+1)^{n-2}$, by Stirling's formula we see the coefficients in the series above are of order $m^{\alpha-1}$ as $m \to \infty$. This proves (i).

(ii) follows from $\|R_\alpha(x,\cdot)\|^2_{2,\alpha} = R_\alpha(x,x)$.

To show (iii), for $x, y \in B$, let $x = |x|\zeta, y = |y|\eta$. Then by (ii) of Lemma 2

$$\begin{aligned} |R_\alpha(x,y)| &\leqslant \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} (|x|\,|y|)^m |Z_m(\zeta,\eta)| \\ &\leqslant \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{n}{2}+\alpha+1)}{\Gamma(m+\frac{n}{2})} (|x|\,|y|)^m h_m \\ &\leqslant \frac{C}{(1-|x|\,|y|)^{n+\alpha}}. \end{aligned}$$

This finishes the proof. □

For $r \in (0,1), x \in B$, let $K_r(x) = \{y \in \mathbf{R}^n : |y-x| < r(1-|x|)\}$. The following fact will be used:

$$1 - |y| \approx 1 - |x|, \quad y \in K_r(x).$$

We have the following lower bound estimate for the reproducing kernels.

Proposition 5. *Let $\alpha > -1$ and $x \in B$. Then there exists $r = r(\alpha) \in (0,1)$ depending only on α such that $R_\alpha(x,y) \approx 1/(1-|x|)^{n+\alpha}$ for $y \in K_r(x)$.*

Proof. It follows from Proposition 4 (iii) that $R_\alpha(x,y) \leqslant C/(1-|x|)^{n+\alpha}$ *for* $y \in K_r(x)$.

To show the other direction, for $y \in K_r(x)$, by the mean value theorem we have

$$\begin{aligned} R_\alpha(x,y) &\leqslant R_\alpha(x,x) - \max_{u \in K_r(x)} |\nabla_u R_\alpha(x,u)|\,|y-x| \\ &\geqslant \frac{C}{(1-|x|)^{n+\alpha}} - \max_{u \in K_r(x)} |\nabla_u R_\alpha(x,u)|\,|y-x|. \end{aligned}$$

If $u \in K_{\frac{1}{2}}(x)$, then $1 - |u| > \frac{1}{2}(1 - |x|)$. Thus for $u \in K_{\frac{1}{2}}(x)$, Cauchy's estimates (2.4 of [1]) gives

$$|\nabla_u R_\alpha(x,u)| \geqslant \frac{C}{(1-|u|)} \max_{v \in K_{\frac{1}{2}}(u)} |R_\alpha(x,v)| \leqslant \frac{C}{(1-|x|)^{n+\alpha+1}}.$$

Thus if r is chosen small enough, for $y \in K_r(x)$, we get

$$R_\alpha(x,y) \geqslant \frac{C}{(1-|x|)^{n+\alpha}} - \frac{Cr}{(1-|x|)^{n+\alpha}} \geqslant \frac{C}{(1-|x|)^{n+\alpha}}.$$

This proves the proposition. □

When $\alpha = 0$, the proposition above was proved in [6] for any $r \in (0,1)$ using the explicit formula for $R_0(x,y)$ given in [1].

For $x, y \in B$, let $P(x,y)$ be the "extended Poisson kernel" for B. Then (see pages 156 and 157 of [1])

$$P(x,y) = \sum_{m=0}^{\infty} Z_m(x,y) = \frac{1 - |x|^2|y|^2}{(1 - 2x \cdot y + |x|^2|y|^2)^{\frac{n}{2}}}, \quad x, y \in B.$$

If α is a non-negative integer, then

$$\begin{aligned} R_\alpha(x,y) &= \frac{2}{nV(B)\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \left(m + \frac{n}{2} + \alpha\right) \cdots \left(m + \frac{n}{2}\right) Z_m(x,y) \\ &= \frac{2}{nV(B)\Gamma(\alpha+1)} \left(\frac{d}{dt}\right)^{\alpha+1} \left[t^{\frac{n}{2}+\alpha} P(tx,y)\right]_{t=1}. \end{aligned}$$

For $x \in B, x \neq 0$, let $\tilde{x} = x/|x|^2$ be the inversion of x. Notice that our reproducing kernels are slightly different from those in [3] and [8] because we choose $(1 - |x|^2)^\alpha$ as weights. We have the following lemma.

Lemma 6. *Let* $\alpha > -1$.
(i) $|R_\alpha(x,y)| \leqslant C|\tilde{x} - y|^{-n-\alpha}$ *for* $x, y \in B$ *with* $|x| > \frac{1}{2}$;
(ii) *If* $\alpha > n\left(\frac{1}{p} - 1\right) - \frac{1}{p}$, *then*

$$\int_S |R_\alpha(\zeta,y)|^p d\sigma(\zeta) \leqslant C(1-|y|)^{n-1-(n+\alpha)p}, \quad y \in B.$$

Proof. The same proofs as for Lemma 2.3 of [8] yields (i) (although only the case when $\alpha > 0$ was considered in [8]). Now (ii) follows from (i) by the proof for Lemma 3.2 of [3]. □

In order to prove our next result, we need the following simple estimate (see page 291 of [9]).

Lemma 7. *If* $\beta > -1$ *and* $m > 1 + \beta$, *then for* $0 \leqslant t < 1$,

$$\int_0^1 (1 - tr)^{-m} (1 - r)^\beta dr \leqslant C(1-t)^{1+\beta-m}.$$

The following is the last property for R_α in this section.

Proposition 8. *If* $p > \dfrac{n+\beta}{n+\alpha}, \beta > -1$, *and* $\alpha > -1$, *then*

$$\int_B |R_\alpha(x,y)|^p (1-|y|)^\beta dV(y) \approx \frac{1}{(1-|x|)^{(n+\alpha)p-(n+\beta)}}, \quad x \in B.$$

Proof. For $x \in B$, by Proposition 5, we have

$$\begin{aligned}\int_B |R_\alpha(x,y)|^p (1-|y|)^\beta dV(y) &\geqslant \int_{K_r(x)} |R_\alpha(x,y)|^p (1-|y|)^\beta dV(y) \\ &\geqslant \frac{C}{(1-|x|)^{(n+\alpha)p-(n+\beta)}}.\end{aligned}$$

To show the other direction, using Lemma 6 (ii) and the fact that $R_\alpha(rx,y) = R_\alpha(x,ry)$ for $x, y \in B, 0 < r < 1$ (which follows from Proposition 3), we have

$$\begin{aligned}\int_B |R_\alpha(x,y)|^p (1-|y|)^\beta dV(y) &= nV(B) \int_0^1 (1-r)^\beta r^{n-1} \left(\int_S |R_\alpha(x,r\zeta)|^p d\sigma(\zeta) \right) dr \\ &= nV(B) \int_0^1 (1-r)^\beta r^{n-1} \left(\int_S |R_\alpha(rx,\zeta)|^p d\sigma(\zeta) \right) dr \\ &\leqslant C \int_0^1 (1-r)^\beta (1-r|x|)^{n-1-(n+\alpha)p} dr \\ &\leqslant C(1-|x|)^{n+\beta-(n+\alpha)p},\end{aligned}$$

where we used Lemma 7 in the last step. □

4. Application to an Inequality on the Harmonic Bergman Space

The following result was proved in [4] and [10].

Theorem 9. *Suppose that* G *is a measurable subset of* B *and that* $p > 0, \beta > -1$. *Then the following conditions are equivalent:*

(i) *There is a constant* $C > 0$ *such that*

$$\int_B |f(y)|^p (1-|y|)^\beta dV(y) \leqslant C \int_G |f(y)|^p (1-|y|)^\beta dV(y)$$

for each harmonic function f *on* B *for which the left-hand side of the inequality is finite;*

(ii) *There is a constant* $\delta > 0$ *such that* $V(G \cap K) \geqslant \delta V(B \cap K)$ *for every ball* K *whose center lies on* S.

LUECKING [4] proved (ii) ⇒ (i), and (i) ⇒ (ii) only when $p = 2, \beta = 0$. Later SLEDD proved (i) ⇒ (ii) for all $p > 0, \beta > -1$ in [10] (I thank Professor William T. Sledd for this reference). To prove (i) ⇒ (ii) in the case when $p = 2, \beta = 0$, LUECKING [4] used $R_0(x,y)$ and suggested the use of $R_\beta(x,y)$ for the case when $p = 2, \beta > -1$. SLEDD [10] developed a different approach by constructing harmonic functions using the Poisson kernel.

We here provide another proof of (i) ⇒ (ii) for Theorem 9. Our method is similar to that in [4]. For $\beta \geqslant 0$, our proof is even shorter than that in [4], where the explicit formula for $R_0(x,y)$ was used. For $-1 < \beta < 0$, our proof uses a careful

argument. We believe the reproducing kernels are natural candidates for this type of inequality.

Proof of (i) $\Rightarrow$ (ii). By the argument in the proof of Lemma 3 of [4], we only need to show that given $\varepsilon > 0$, there is a constant C_ε (depending on ε) such that for every ball K with its center on S, there exists a harmonic function f (depending on ε and K) on B such that

(1) $\int_B |f(y)|^p (1-|y|)^\beta dV(y) \geqslant C$, where C does not depend on K, ε, and f;

(2) $\int_{B\setminus K} |f(y)|^p (1-|y|)^\beta dV(y) < \varepsilon$;

(3) $\int_{G\cap K} |f(y)|^p (1-|y|)^\beta dV(y) \leqslant C_\varepsilon (V(G\cap K)/V(K\cap B))^a$ for some $a > 0$, where a depends only on β.

Without loss generality let K have radius $h < 1$ and center $u = (1, 0, \ldots, 0)$. Choose α large enough so that $p > \dfrac{n+\beta}{n+\alpha}$. Let

$$f(y) = R_\alpha(x_k, y)(1-|x_k|)^{n+\alpha-\frac{n+\beta}{p}},$$

where $x_k = ru, r > 0$, and $1 - r = sh$ for small $s > 0$ to be chosen.

Condition (1) follows from Proposition 8.

The case $\beta \geqslant 0$ is easier to deal with in order to show (2) and (3). Let $\beta \geqslant 0$. If $y \in K$, then $1 - |y| < h$. By Proposition 4, for $y \in K$, we have

$$|f(y)|^p (1-|y|)^\beta \leqslant C \frac{(1-|y|)^\beta}{(1-|x_k|)^{n+\beta}} \leqslant C \frac{h^\beta}{(sh)^{n+\beta}} = C_s \frac{1}{V(K)}.$$

This implies (3) for $a = 1$.

By Lemma 6, we have $|R_\alpha(x_k, y)| \leqslant C/|\tilde{x}_k - y|^{n+\alpha}$ if $s < \frac{1}{2}$. Notice that $(1-|y|) < |\tilde{x}_k - y|, y \in B$. We have

$$\begin{aligned}\int_{B\setminus K} |f(y)|^p (1-|y|)^\beta dV(y) &\leqslant C(1-|x_k|)^{p(n+\alpha)-(n+\beta)} \int_{B\setminus K} \frac{1}{|\tilde{x}_k - y|^{(n+\alpha)p-\beta}} dV(y) \\ &\leqslant C(sh)^{p(n+\alpha)-(n+\beta)} \int_h^\infty \frac{r^{n-1}}{r^{p(n+\alpha)-\beta}} dr \\ &\leqslant C(s)^{p(n+\alpha)-(n+\beta)},\end{aligned}$$

where we used the fact that $B\setminus K \subset \{y \in \mathbf{R}^n : |y - \tilde{x}_k| > h\}$ in the second step. If s is chosen small, then we have condition (2).

The case when $-1 < \beta < 0$ requires more work. First we choose $q > 1$ such that $q\beta > -1$. Let q' denote the conjugate of q. Hölder's inequality gives

$$\begin{aligned}&\int_{B\setminus K} |f(y)|^p (1-|y|)^\beta dV(y) \\ &\leqslant (1-|x_k|)^{p(n+\alpha)-(n+\beta)} \left(\int_{B\setminus K} |R_\alpha(x_k, y)|^{\frac{pq}{2}} (1-|y|)^{\beta q} dV(y) \right)^{\frac{1}{q}} \\ &\quad \cdot \left(\int_{B\setminus K} |R_\alpha(x_k, y)|^{\frac{pq'}{2}} dV(y) \right)^{\frac{1}{q'}}.\end{aligned}$$

If $(n+\alpha)p > 2\left(\frac{n}{q}+\beta\right)$, then by Proposition 8

$$\left(\int_{B\setminus K} |R_\alpha(x_k,y)|^{\frac{pq}{2}}(1-|y|)^{\beta q}dV(y)\right)^{\frac{1}{q}} \leqslant C\frac{1}{(1-|x_k|)^{(n+\alpha)\frac{p}{q}-(\frac{n}{q}+\beta)}}.$$

If $(n+\alpha)p > 2\left(\frac{n}{q'}\right)$, then we have

$$\left(\int_{B\setminus K} |R_\alpha(x_k,y)|^{\frac{pq'}{2}}dV(y)\right)^{\frac{1}{q'}} \leqslant C\left(\int_h^\infty \frac{r^{n-1}}{r^{(n+\alpha)\frac{pq'}{2}}}dr\right)^{\frac{1}{q'}} = C\frac{1}{h^{(n+\alpha)\frac{p}{2}-\frac{n}{q'}}}.$$

Combining the inequalities above, we get

$$\int_{B\setminus K} |f(y)|^p(1-|y|)^\beta dV(y) \leqslant C(s)^{(n+\alpha)\frac{p}{2}-\frac{n}{q'}},$$

provided that α is large enough. This gives (2) if s is small enough.

We now show (3). We have

$$\begin{aligned}&\int_{G\cap K} |f(y)|^p(1-|y|)^\beta dV(y)\\ &\quad = (1-|x_k|)^{p(n+\alpha)-(n+\beta)}\int_{G\cap K} |R_\alpha(x_k,y)|^p(1-|y|)^\beta dV(y).\end{aligned}$$

By Hölder's inequality and Proposition 8, we get

$$\begin{aligned}&\int_{G\cap K} |R_\alpha(x_k,y)|^p(1-|y|)^\beta dV(y)\\ &\quad \leqslant \left(\int_B |R_\alpha(x_k,y)|^{qp}(1-|y|)^{q\beta}dV(y)\right)^{\frac{1}{q}}(V(G\cap K))^{\frac{1}{q'}}\\ &\quad \leqslant \frac{C}{(1-|x_k|)^{p(n+\alpha)-(\frac{n}{q}+\beta)}}(V(G\cap K))^{\frac{1}{q'}}.\end{aligned}$$

Hence we obtain that

$$\int_{G\cap K} |f(y)|^p(1-|y|)^\beta dV(y) \leqslant C\left(\frac{V(G\cap K)}{(1-|x_k|)^n}\right)^{\frac{1}{q'}} \leqslant C_s\left(\frac{V(G\cap K)}{V(B\cap K)}\right)^{\frac{1}{q'}}.$$

Thus the condition (3) is satisfied with $a = 1/q'$. □

5. Application to Toeplitz Operators on the Harmonic Bergman Space

Let μ be a finite complex Borel measure on B. We densely define the Toeplitz operator on $b^2_\alpha(B)$ with symbol μ by

$$T_\mu u(x) = \int_B R_\alpha(x,y)u(y)d\mu(y)$$

for $u \in b^2_\alpha(B)\cap L^\infty(B,(1-|x|^2)^\alpha dV(x))$. If $d\mu(y) = f(y)(1-|x|^2)^\alpha dV(y)$, then we write $T_\mu = T_f$. Let $\langle\cdot,\cdot\rangle_\alpha$ denote the inner product for $L^2(B,(1-|x|^2)^\alpha dV(x))$.

For bounded $u, v \in b_\alpha^2(B)$, it follows from Fubini's Theorem that

$$\langle T_\mu u, v\rangle_\alpha = \int_B u\bar{v}d\mu.$$

Suppose $\mu \geqslant 0$ and let I denote the embedding operator from $b_\alpha^2(B)$ into $L^2(B, d\mu)$. It is clear that T_μ is bounded (compact) on $b_\alpha^2(B)$ if and only if I is bounded (compact).

The characterization of boundedness and compactness for the embedding operator was given in [7], where more general domains in $\mathbf{R}^n$ and more general spaces were considered, except for $-1 < \alpha < 0$. We can extend the characterization to all $\alpha > -1$ in our case. From here on we always assume r is the number given in Proposition 5.

Proposition 10. *Let $\alpha > -1$ and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *I is bounded (compact);*

(ii) *$\mu(K_r(x))/V(K_r(x))^{1+\frac{\alpha}{n}}$ is bounded, for $x \in B(\to 0$ as $|x| \to 1)$.*

Proof. Oleinik and Pavlov [7] proved that (ii) $\Rightarrow$ (i). To prove the implication in the other direction, suppose I is bounded. Then

$$\int_B |u|^2 d\mu \leqslant C \int_B |u(y)|^2 (1 - |y|^2)^\alpha dV(y)$$

for all $u \in b_\alpha^2(B)$. For $x \in B$, let $u(y) = R_\alpha(x, y) \in b_\alpha^2(B)$. Then

$$\begin{aligned}
\frac{\mu(K_r(x))}{(1 - |x|)^{2(n+\alpha)}} &\leqslant C \int_{K_r(x)} |R_\alpha(x, y)|^2 d\mu(y) \\
&\leqslant C \int_B |R_\alpha(x, y)|^2 d\mu(y) \\
&\leqslant C \int_B |R_\alpha(x, y)|^2 (1 - |y|^2)^\alpha dV(y) \\
&\leqslant \frac{C}{(1 - |x|)^{n+\alpha}},
\end{aligned}$$

where we used Proposition 4 in the last step. A modification of this argument shows that compactness of I implies the little o condition; we omit the details. This proves (ii). □

We need the following decomposition of B (see Lemma 4 of [6]).

Lemma 11. *There exists a sequence $\{x_i\}$ in B such that*

(i) $\bigcup K_{\frac{r}{3}}(x_i) = B$;

(ii) *There exists a positive integer N such that each $K_r(x_i)$ intersects at most N spheres of $\{K_r(x_j)\}$.*

Now we can state Proposition 10 in terms of Toeplitz operators. Although [7] only gives the continuous version, a discrete version can be easily obtained (see, for example, Lemma 5 of [6]).

Proposition 12. *Let $\alpha > -1$ and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *T_μ is bounded (compact) on $b^2_\alpha(B)$;*

(ii) *$\mu(K_r(x))/V(K_r(x))^{1+\frac{\alpha}{n}}$ is bounded for $x \in B (\to 0$ as $|x| \to 1)$;*

(iii) *$\mu(K_r(x_i))/V(K_r(x_i))^{1+\frac{\alpha}{n}}$ is bounded for $i = 1, 2, \ldots (\to 0$ as $i \to \infty)$.*

Now we can establish a trace ideal criteria for positive Toeplitz operators on $b^2_\alpha(B)$. The case $\alpha = 0$ was proved in [6] using ideas from [5] and [12] (see Theorem 11 of [6]). That result can be extended to all $\alpha > -1$.

First let us recall the definition for the Schatten ideal. If T is a compact operator on a separable Hilbert space H, then there exist numbers $s_0(T) \geqslant s_1(T) \geqslant \cdots \geqslant 0$, called the singular numbers of T, and orthonormal sets of vectors $\{e_i\}$ and $\{f_i\}$ such that

$$Tx = \sum_{i=0}^{\infty} S_i(T)\langle x, e_i\rangle f_i, \quad x \in H.$$

For $1 \leqslant p < \infty$, the Schatten ideal $S_p(H)$ is defined to be the set of all compact operators for which $\|T\|_{S_p} = (\sum_{i=0}^{\infty} S_i(T)^p)^{\frac{1}{p}} < \infty$. As is well known, $S_p(H)$ is a Banach space with the norm $\|\cdot\|_{S_p}$ and is a two-sided ideal in the space of bounded linear operators on H.

Theorem 13. *Let $1 \leqslant p < \infty, \alpha > -1$, and μ be a finite positive Borel measure on B. Then the following conditions are equivalent:*

(i) *$T_\mu \in S_p(b^2_\alpha(B))$;*

(ii) *$\mu(K_r(x))/V(K_r(x))^{1+\frac{\alpha}{n}} \in L^p(B, (1-|x|^2)^{-n} dV(x))$;*

(iii) *$\sum_{i=1}^{\infty} (\mu(K_r(x_i))/V(K_r(x_i))^{1+\frac{\alpha}{n}})^p < \infty$.*

The proof of the theorem above is entirely analogous to that for Theorem 11 of [6], so we will not give a proof for it. We remark that the two properties for the reproducing kernels needed for the proof are supplied by Proposition 4 and 5, and the S_p-norm of T_μ is related to the reproducing kernels by the following identity:

$$\|T_\mu\|^p_{S_p} = \int_B \langle T^p_\mu R_\alpha(x, \cdot), R_\alpha(x, \cdot)\rangle_\alpha (1-|x|^2)^\alpha dV(x).$$

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