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Commutativity of Operators on the Lattice of Existence Varieties

By

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Abstract. In a previous paper the authors introduced seven complete congruences on the lattice $\mathscr{L}_{ev}(\mathscr{RS})$ of e-varieties of regular semigroups of the form $\rho_{\mathscr{P}}: \mathscr{U}\rho_{\mathscr{P}}\mathscr{V} \Leftrightarrow \mathscr{P}\circ\mathscr{U} = \mathscr{P}\circ\mathscr{V}$, where $\mathscr{P}$ is drawn from a small set of e-varieties: left zero, right zero, rectangular bands, groups, left groups, right groups and completely simple semigroups. Four new complete congruences are introduced here of the form $\alpha_{\mathscr{P}}: \mathscr{U}\alpha_{\mathscr{P}}\mathscr{V} \Leftrightarrow \mathscr{P}\cap\mathscr{U} = \mathscr{P}\cap\mathscr{V}$, where $\mathscr{P}$ is one of the following classes of regular semigroups: left monoids, right monoids, monoids, idempotent generated semigroups. For each complete congruence ρ on $\mathscr{L}_{ev}(\mathscr{RS})$ and each $\mathscr{U}\in\mathscr{L}_{ev}(\mathscr{RS})$, the ρ-class of $\mathscr{U}$ is an interval $[\mathscr{U}_\rho, \mathscr{U}^\rho]$ so that there is associated with each such congruence an idempotent operator $\mathscr{U}\to\mathscr{U}^\rho$ on $\mathscr{L}_{ev}(\mathscr{RS})$. This paper establishes numerous results concerning the commutativity of operators of this form.

1. Introduction

The importance of varieties (that is, classes of algebras that are closed under products, homomorphic images and subalgebras) as a tool in the study of various classes of algebras such as groups, lattices and semigroups is well established. For the study of certain classes of algebras, for example the class of finite semigroups and the class $\mathscr{RS}$ of all regular semigroups, the concept of a variety is not quite appropriate since these classes themselves fail to be closed under the operation of arbitrary direct products and subsemigroups, respectively. For some years, the concept of a pseudovariety has served well in the study of finite semigroups. However, it was only recently that HALL [11] and, independently, KADOUREK and SZENDREI [18] introduced the concept of an e-variety, that is a class of regular semigroups that is closed under direct products, homomorphic images and regular subsemigroups. The suitability of this concept for the study of classes of regular semigroups is attested to by the rapidly increasing body of results relating to e-varieties. As with varieties of any class of algebras, the e-varieties of regular semigroups form a lattice with respect to inclusion and it is this lattice that has been the main focus of attention and that is our main interest here.

This paper continues earlier work of the authors in [34] and [35]. In particular, in [34] the authors introduced seven complete congruences $\{T, T_l, T_r, K_l, K_r, K, M\}$

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on the lattice of e-varieties. In this paper we introduce two additional methods for finding such congruences.

Section 2 is devoted to preliminaries including the introduction of seven operators, one to be associated with each of the above mentioned congruences, each of which has a description in terms of Mal'cev products. In Section 3, we introduce a method for generating complete congruences on $\mathscr{L}_{ev}(\mathscr{RS})$ based on the techniques introduced by PETRICH and REILLY in [28].

The key to this approach is found in classes of regular semigroups (called *PH-closed classes*) that are closed under direct products and homomorphic images. For such a class $\mathscr{A}$, we refer to its members as *$\mathscr{A}$-semigroups* and to subclasses of $\mathscr{A}$ that are closed under the formation of direct products, homomorphic images and $\mathscr{A}$-subsemigroups as *$\mathscr{A}$-varieties*. If $\mathscr{A}$ has the property that

$$\mathscr{U}^{\mathscr{A}} = \{S \in \mathscr{RS} \mid \text{all } \mathscr{A}\text{-subsemigroups of } S \text{ belong to } \mathscr{U}\}$$

is an e-variety, for all $\mathscr{A}$-varieties $\mathscr{U}$, then we say that $\mathscr{A}$ is e-*varietal*. For an e-varietal PH-closed class $\mathscr{A}$ the main theorem of Section 3 establishes that the mapping $\mathscr{U} \to \mathscr{U} \cap \mathscr{A}$ is a complete lattice homomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$ onto the lattice $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ of $\mathscr{A}$-varieties.

Three examples of e-varietal PH-closed classes are considered in Section 4, namely, the classes of regular monoids, left monoids and right monoids. By the main theorem of Section 3, each of these classes determines a complete congruence on $\mathscr{L}_{ev}(\mathscr{RS})$. Of particular importance is the congruence L associated with the class $\mathscr{M}$ of all regular monoids. Various characterizations of these congruences are provided. A bonus in relation to the class $\mathscr{M}$ is the fact that the mapping

$$\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev}$$

is a complete endomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$ (where $\langle \mathscr{V} \rangle_{ev}$ denotes the smallest e-variety containing the class $\mathscr{V}$ of regular semigroups).

A similar but slightly different method, based on the techniques introduced by PETRICH and REILLY [26], is used in Section 5 to derive a complete congruence C on $\mathscr{L}_{ev}(\mathscr{RS})$ from the class $\mathscr{C}$ of all idempotent generated regular semigroups. In particular, it is shown that the mapping

$$\mathscr{U} \to \mathscr{U} \cap \mathscr{C}$$

is a complete lattice homomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$ onto the lattice of what we shall call (in keeping with [26]) *near* e-*varieties*.

For any complete congruence P on a complete lattice L and any $a \in L$, the class aP is an interval which we denote by $[a_P, a^P]$. This introduces two operators that are associated with P, namely

$$a \to a_P \quad \text{and} \quad a \to a^P \quad (a \in L).$$

Such operators illuminate the relationship between different classes and have proven to be extremely important in the study of completely regular semigroups (see [29]). These would appear to have good potential for the application of similar techniques to the lattice of e-varieties. Sections 6 and 7 are devoted to the study of the

commutativity of operators of the form

$$\mathscr{U} \to \mathscr{U}^P \quad (\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS}))$$

where $P \in \{T, T_l, T_r, K_l, K_r, K, M, L, C\}$ together with the operator C_∞ introduced by ZHANG [41]. The main results of these sections show that the operators of this form associated with $P \in \{L, C\}$ as well as C_∞ commute with the operators associated with any $Q \in \{T, T_l, T_r, K_l, K_r, K\}$.

In the final section it is shown how the action of the operator M (respectively, K) can in certain circumstances be captured as join of the results obtained from iterating the actions of T_l and T_r (respectively, K_l and K_r).

2. Preliminaries

In general we shall use the notation and terminology of CLIFFORD and PRESTON [5] and HOWIE [14]. For basic information on lattices and varieties of algebras, the reader is referred to GRÄTZER [7].

Let A and B be nonempty sets. If θ is a mapping from A to B, then $\bar{\theta}$ denotes the equivalence relation on A induced by θ. If θ is a relation on A, and C is a nonempty subset of A, then $\theta|_C$ denotes the restriction of θ to C. The equality relation on any set is denoted by ι. Throughout this paper $\mathscr{RS}$ stands for the class of all regular semigroups. If θ is an equivalence relation on S, $S \in \mathscr{RS}$, then θ^0 denotes the greatest congruence contained in θ. If θ_1 and θ_2 are equivalence relations, then $(\theta_1 \cap \theta_2)^0 = \theta_1^0 \cap \theta_2^0$ (see [21]).

Let $S \in \mathscr{RS}$. Then $E(S)$ denotes the set of all idempotents of S and $C(S)$ denotes the *core* of S, that is, the subsemigroup of S generated by $E(S)$. If $a \in S$, then $V(a)$ is the set of all inverses of a in S, that is, $V(a) = \{a' \in S \mid aa'a = a, a'aa' = a'\}$. Also, if $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}\}$, K_a stands for the $\mathscr{K}$-class of S containing a. For any $S \in \mathscr{RS}$, we introduce a relation $\leqslant$ on $E(S)$ as follows:

$$e \leqslant f \Leftrightarrow ef = fe = e$$

$(e, f \in E(S))$. Clearly $\leqslant$ is a partial order on $E(S)$.

For any nonempty class $\mathscr{U}$ of regular semigroups we define $\mathbf{H}\mathscr{U}$, $\mathbf{S}_r\mathscr{U}$ and $\mathbf{P}\mathscr{U}$ as follows: $\mathbf{H}\mathscr{U}$ is the class of all (regular) semigroups that are homomorphic images of semigroups in $\mathscr{U}$; $\mathbf{S}_r\mathscr{U}$ is the class of all *regular* subsemigroups of semigroups in $\mathscr{U}$; $\mathbf{P}\mathscr{U}$ is the class of all direct products of semigroups in $\mathscr{U}$.

As in [11], a class $\mathscr{U}$ of regular semigroups is called an *existence variety*, or e-variety, if $\mathbf{H}\mathscr{U} \subseteq \mathscr{U}$, $\mathbf{S}_r\mathscr{U} \subseteq \mathscr{U}$ and $\mathbf{P}\mathscr{U} \subseteq \mathscr{U}$. The class of all e-varieties of regular semigroups forms a complete lattice under inclusion, and is denoted by $\mathscr{L}_{ev}(\mathscr{RS})$.

Let $\mathscr{A}$ be a subclass of $\mathscr{RS}$. We denote by $\langle \mathscr{A} \rangle_{ev}$ the e-variety of regular semigroups *generated* by $\mathscr{A}$. Furthermore, the lattice of e-subvarieties of an e-variety $\mathscr{U}$ shall be denoted by $\mathscr{L}_{ev}(\mathscr{U})$.

The following e-subvarieties of $\mathscr{RS}$ will figure prominently in our discussion:

$\mathscr{S}$ – the e-variety of all semilattices
$\mathscr{LZ}$ – the e-variety of all left zero semigroups
$\mathscr{RZ}$ – the e-variety of all right zero semigroups
$\mathscr{RB}$ – the e-variety of all rectangular bands

$\mathscr{B}$ – the e-variety of all bands
$\mathscr{G}$ – the e-variety of all groups
$\mathscr{LG}$ – the e-variety of all left groups
$\mathscr{RG}$ – the e-variety of all right groups
$\mathscr{SG}$ – the e-variety of all semilattices of groups
$\mathscr{ReG}$ – the e-variety of all rectangular groups
$\mathscr{CS}$ – the e-variety of all completely simple semigroups
$\mathscr{OG}$ – the e-variety of all orthogroups
$\mathscr{CR}$ – the e-variety of all completely regular semigroups
$\mathscr{I}$ – the e-variety of all inverse semigroups
$\mathscr{O}$ – the e-variety of all orthodox semigroups.

Let S be a regular semigroup. For any $a, b \in S$, we put

$$a\pi b \Leftrightarrow V(a) = V(b)$$

and

$$a\mathscr{M}b \Leftrightarrow \bigcup_{a' \in V(a)} H_{a'} = \bigcup_{b' \in V(b)} H_{b'}.$$

Clearly π and $\mathscr{M}$ are equivalence relations on S and $\pi \subseteq \mathscr{M} \subseteq \mathscr{D}$ (see [22], where notation $\mathscr{U}$ is used for $\mathscr{M}$).

Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})$. The *Mal'cev product* $\mathscr{U} \circ \mathscr{V}$ of $\mathscr{U}$ and $\mathscr{V}$ consists of those regular semigroups S for which there exists a congruence ρ on S such that $S/\rho \in \mathscr{V}$ and $e\rho \in \mathscr{U}$ for all $e \in E(S)$. In general $\mathscr{U} \circ \mathscr{V}$ is not an e-variety. But for certain choices of e-varieties $\mathscr{U}$ and $\mathscr{V}$, $\mathscr{U} \circ \mathscr{V}$ is indeed an e-variety.

Lemma 2.1 [34, Section 5]. *If* $\mathscr{U} \in \{\mathscr{LZ}, \mathscr{RZ}, \mathscr{RB}, \mathscr{G}, \mathscr{LG}, \mathscr{RG}, \mathscr{CS}\}$ *and* $\mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})$, *then* $\mathscr{U} \circ \mathscr{V}$ *is again an* e*-variety.*

The above lemma enables us to introduce seven operators on $\mathscr{L}_{ev}(\mathscr{RS})$ defined as follows: for any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$,

$$\mathscr{U}^{T_l} = \mathscr{LG} \circ \mathscr{U}, \quad \mathscr{U}^{T_r} = \mathscr{RG} \circ \mathscr{U}, \quad \mathscr{U}^{T} = \mathscr{G} \circ \mathscr{U},$$
$$\mathscr{U}^{K_l} = \mathscr{LZ} \circ \mathscr{U}, \quad \mathscr{U}^{K_r} = \mathscr{RZ} \circ \mathscr{U}, \quad \mathscr{U}^{K} = \mathscr{RB} \circ \mathscr{U},$$
$$\mathscr{U}^{M} = \mathscr{CS} \circ \mathscr{U}.$$

It is easily verified that T_l, T_r, T, K_l, K_r, K and M are closure operators on $\mathscr{L}_{ev}(\mathscr{RS})$. We shall also have the following useful characterizations.

Lemma 2.2 [34, Section 5]. *Let* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$. *Then*

(i) $\mathscr{U}^{T_l} = \{S \in \mathscr{RS} \mid S/\mathscr{L}^0 \in \mathscr{U}\}$.
(ii) $\mathscr{U}^{T_r} = \{S \in \mathscr{RS} \mid S/\mathscr{R}^0 \in \mathscr{U}\}$.
(iii) $\mathscr{U}^{T} = \{S \in \mathscr{RS} \mid S/\mathscr{H}^0 \in \mathscr{U}\}$.
(iv) $\mathscr{U}^{K_l} = \{S \in \mathscr{RS} \mid S/(\pi \cap \mathscr{L})^0 \in \mathscr{U}\}$.
(v) $\mathscr{U}^{K_r} = \{S \in \mathscr{RS} \mid S/(\pi \cap \mathscr{R})^0 \in \mathscr{U}\}$.
(vi) $\mathscr{U}^{K} = \{S \in \mathscr{RS} \mid S/\pi^0 \in \mathscr{U}\}$.
(vii) $\mathscr{U}^{M} = \{S \in \mathscr{RS} \mid S/\mathscr{M}^0 \in \mathscr{U}\}$.

As shown in [34], associated with each of the operators $P \in \{T, T_l, T_r, K_l, K_r, K, M\}$ there is a complete congruence on $\mathscr{L}_{ev}(\mathscr{RS})$, which we

also denote by P, defined by

$$\mathscr{U} P \mathscr{V} \Leftrightarrow \mathscr{U}^P = \mathscr{V}^P \quad (\mathscr{U}, \mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})).$$

For these congruences, from [34, Section 6] we have

$$T = T_l \cap T_r, \quad K_l = T_l \cap K, \quad K_r = T_r \cap K.$$

It then follows from purely lattice theoretic considerations that

$$\mathscr{U}^T = \mathscr{U}^{T_l} \cap \mathscr{U}^{T_r}, \quad \mathscr{U}^{K_l} = \mathscr{U}^{T_l} \cap \mathscr{U}^K, \quad \mathscr{U}^{K_r} = \mathscr{U}^{T_r} \cap \mathscr{U}^K$$

for all $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$.

We conclude this section with the following two lattice theoretic results that will be used later. The first observation, which is a strengthened form of [22, Lemma 4.13], is due to R. Broeksteeg (unpublished).

Lemma 2.3. *Let L_1 and L_2 be complete lattices and $\theta: L_1 \to L_2$ be a surjective mapping such that for all $m \in L_2$ there exist $m_\theta, m^\theta \in L_1$ with*

$$m\theta^{-1} = [m_\theta, m^\theta].$$

Then θ is a complete lattice homomorphism if and only if the following hold:

(a) *for all $a, b \in L_1$,*

$$a \leqslant b \Rightarrow a\theta \leqslant b\theta;$$

(b) *for all $m, n \in L_2$,*

$$m \leqslant n \Rightarrow m_\theta \leqslant n_\theta \quad \text{and} \quad m^\theta \leqslant n^\theta.$$

Lemma 2.4 [21]. *Let L be a complete lattice and γ be a complete congruence on L. For $a \in L$, let a_γ and a^γ be defined by $a\gamma = [a_\gamma, a^\gamma]$. Then the mapping*

$$a \to a_\gamma$$

is a complete $\vee$-homomorphism of L into itself and the mapping

$$a \to a^\gamma$$

is a complete $\wedge$-homomorphism of L into itself.

3. E-varietal PH-closed Classes

Let $\mathscr{A}$ be a class of regular semigroups. Call its members *$\mathscr{A}$-semigroups*. We will say that $\mathscr{A}$ is a *PH-closed class* if it is closed under direct products and homomorphic images. Clearly each e-variety is a PH-closed class. Let $\mathscr{A}$ be a PH-closed class. A subclass $\mathscr{U}$ of $\mathscr{A}$ is an *$\mathscr{A}$-variety* if it is closed under the formation of direct products, homomorphic images and $\mathscr{A}$-subsemigroups. Denote by $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ the class of all $\mathscr{A}$-varieties. Note that if $\mathscr{A}$ is an e-variety, then all $\mathscr{A}$-varieties are e-subvarieties of $\mathscr{A}$.

For any class $\mathscr{U}$ of regular semigroups, we define

$$\mathscr{U}^{\mathscr{A}} = \{S \in \mathscr{RS} \mid \text{all } \mathscr{A}\text{-subsemigroups of } S \text{ belong to } \mathscr{U}\}.$$

It is clear that $\mathscr{U}^{\mathscr{A}} = (\mathscr{U} \cap \mathscr{A})^{\mathscr{A}}$. We will say that $\mathscr{A}$ is e-*varietal* if for every $\mathscr{U} \in \mathscr{L}_{\mathscr{A}}(\mathscr{RS})$, $\mathscr{U}^{\mathscr{A}}$ is an e-variety.

Theorem 3.1. *Let $\mathscr{A}$ be an* e*-varietal PH-closed class of regular semigroups.*

(i) *$\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ is a complete lattice with maximum element $\mathscr{A}$.*

(ii) *The mapping*

$$\theta_{\mathscr{A}}: \mathscr{U} \to \mathscr{U} \cap \mathscr{A} \quad (\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS}))$$

is a complete homomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ onto $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$. Moreover, for any $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$, we have $\mathscr{U}\bar{\theta}_{\mathscr{A}} = [\langle \mathscr{U} \cap \mathscr{A} \rangle_{\mathrm{ev}}, \mathscr{U}^{\mathscr{A}}]$.

Proof. (i) Let $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$. Since both $\mathscr{U}$ and $\mathscr{A}$ are closed under direct products, homomorphic images and $\mathscr{A}$-subsemigroups, it follows that $\mathscr{U} \cap \mathscr{A} \in \mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ and that $\theta_{\mathscr{A}}$ maps $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ into $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$. Now let $\mathscr{V} \in \mathscr{L}_{\mathscr{A}}(\mathscr{RS})$. Since $\mathscr{A}$ is e-varietal, $\mathscr{V}^{\mathscr{A}} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$. By the definition of $\mathscr{V}^{\mathscr{A}}$, we clearly have that $\mathscr{V}^{\mathscr{A}}\theta_{\mathscr{A}} = \mathscr{V}^{\mathscr{A}} \cap \mathscr{A} = \mathscr{V}$. Thus $\theta_{\mathscr{A}}$ is a surjection of $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ onto $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ and therefore $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ is a set. It is evident that $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ is closed under arbitrary intersections and has $\mathscr{A}$ as the largest element. Hence $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$ is a complete lattice.

(ii) Let $\mathscr{V} \in \mathscr{L}_{\mathscr{A}}(\mathscr{RS})$. We have

$$\mathscr{V} \subseteq \langle \mathscr{V} \rangle_{\mathrm{ev}} \subseteq \mathscr{V}^{\mathscr{A}}$$

so that $\langle \mathscr{V} \rangle_{\mathrm{ev}} \cap \mathscr{A} = \mathscr{V} = \mathscr{V}^{\mathscr{A}} \cap \mathscr{A}$. Thus $[\langle \mathscr{V} \rangle_{\mathrm{ev}}, \mathscr{V}^{\mathscr{A}}] \subseteq \mathscr{V}\theta_{\mathscr{A}}^{-1}$. On the other hand, if $\mathscr{W} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ and $\mathscr{W} \in \mathscr{V}\theta_{\mathscr{A}}^{-1}$, then $\mathscr{W} \cap \mathscr{A} = \mathscr{V}$ and $\langle \mathscr{V} \rangle_{\mathrm{ev}} \subseteq \mathscr{W} \subseteq$ $\subseteq \mathscr{W}^{\mathscr{A}} = (\mathscr{W} \cap \mathscr{A})^{\mathscr{A}} = \mathscr{V}^{\mathscr{A}}$. Hence $\mathscr{V}\theta_{\mathscr{A}}^{-1} = [\langle \mathscr{V} \rangle_{\mathrm{ev}}, \mathscr{V}^{\mathscr{A}}]$. It is now clear that $\theta_{\mathscr{A}}$ satisfies the conditions in Lemma 2.3, and hence $\theta_{\mathscr{A}}$ is a complete lattice homomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ onto $\mathscr{L}_{\mathscr{A}}(\mathscr{RS})$.

For any $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$, we then have

$$\begin{aligned} \mathscr{U}\bar{\theta}_{\mathscr{A}} &= (\mathscr{U}\theta_{\mathscr{A}})\theta_{\mathscr{A}}^{-1} \\ &= [\langle \mathscr{U} \cap \mathscr{A} \rangle_{\mathrm{ev}}, (\mathscr{U} \cap \mathscr{A})^{\mathscr{A}}] \\ &= [\langle \mathscr{U} \cap \mathscr{A} \rangle_{\mathrm{ev}}, \mathscr{U}^{\mathscr{A}}] \end{aligned}$$

since clearly, from their definitions, $\mathscr{U}^{\mathscr{A}} = (\mathscr{U} \cap \mathscr{A})^{\mathscr{A}}$. The proof of the theorem is completed. □

Corollary 3.2. *$\bar{\theta}_{\mathscr{A}}$ is a complete congruence on $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$.*

Proof. This follows immediately from Theorem 3.1. □

Corollary 3.3. *Let $\mathscr{A}$ be an* e*-varietal PH-closed class of regular semigroups.*

(a) *The mapping*

$$\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{A} \rangle_{\mathrm{ev}} \quad (\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS}))$$

is a complete $\vee$-endomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$.

(b) *The mapping*

$$\mathscr{U} \to \mathscr{U}^{\mathscr{A}} \quad (\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS}))$$

is a complete $\wedge$-endomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$.

Proof. This follows directly from Lemma 2.4, Theorem 3.1 and Corollary 3.2. □

We shall encounter some PH-closed classes in the next section.

Remark 3.4. The class $\mathscr{CR}$ of all completely regular semigroups, considered as algebras with the binary operation of multiplication and the unary operation of inversion, is a variety. Considerable progress has been made in the study of the lattice $\mathscr{L}(\mathscr{CR})$ of varieties of completely regular semigroups over the last two decades. For an extensive bibliography, see PETRICH and REILLY [29].

By restricting our attention to $\mathscr{CR}$, we can introduce the concept of a PH-closed class, and perform a similar analysis; that is, if $\mathscr{A}$ is a varietal PH-closed class of completely regular semigroups, then the mapping

$$\theta_{\mathscr{A}}: \mathscr{U} \to \mathscr{U} \cap \mathscr{A} \quad (\mathscr{U} \in \mathscr{L}(\mathscr{CR}))$$

is a complete homomorphism of $\mathscr{L}(\mathscr{CR})$ onto $\mathscr{L}_{\mathscr{A}}(\mathscr{CR})$. The result can then be used to show that many mappings on $\mathscr{L}(\mathscr{CR})$ induced by taking intersection with subvarieties of $\mathscr{CR}$ are complete endomorphisms of $\mathscr{L}(\mathscr{CR})$. For example, it follows easily from [31, Theorem 4.4] and [16, Section 3] that $\mathscr{CS}$ and $\mathscr{G}$ are varietal PH-closed classes, so that the mappings

$$\mathscr{U} \to \mathscr{U} \cap \mathscr{CS}$$
$$\mathscr{U} \to \mathscr{U} \cap \mathscr{G} \qquad (\mathscr{U} \in \mathscr{L}(\mathscr{CR}))$$

are complete endomorphisms of $\mathscr{L}(\mathscr{CR})$ (see [15, Theorems 3.1 and 3.3]).

4. Operators L, L_l and L_r

In this section we provide examples of PH-closed classes.

Let S be a regular semigroup. It is easily verified that for any $e \in E(S)$, eSe is a regular subsemigroup of S. For any class $\mathscr{U}$ of regular semigroups, we let

$$\mathscr{U}L = \{S \in \mathscr{RS} \mid eSe \in \mathscr{U} \text{ for all } e \in E(S)\}.$$

Equivalently, $\mathscr{U}L$ consists of those regular semigroups S all of whose regular submonoids are in $\mathscr{U}$. In the context of e-varieties of regular semigroups this operator was introduced by HALL [11, Lemma 4.6.1], where the notation $\mathscr{U}^{\text{loc}}$ is used and it is shown that if $\mathscr{U} \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$, then $\mathscr{U}L \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$. Its restriction to varieties of completely regular semigroups was studied in [27, 28, 30, 42]. It is easy to see that the operator L is a complete $\wedge$-endomorphism of $\mathscr{L}_{\text{ev}}(\mathscr{RS})$. However, it remains an open question whether or not L is a $\vee$-endomorphism of $\mathscr{L}_{\text{ev}}(\mathscr{RS})$.

We denote by $\mathscr{M}$ the class of all regular monoids.

Theorem 4.1. (i) *$\mathscr{M}$ is an* e*-varietal PH-closed class.*

(ii) *The mapping*

$$\theta_{\mathscr{M}}: \mathscr{U} \to \mathscr{U} \cap \mathscr{M} \quad (\mathscr{U} \in \mathscr{L}_{\text{ev}}(\mathscr{RS}))$$

is a complete homomorphism of $\mathscr{L}_{\text{ev}}(\mathscr{RS})$ onto $\mathscr{L}_{\mathscr{M}}(\mathscr{RS})$. Moreover, for any $\mathscr{U} \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$, we have $\mathscr{U}\bar{\theta}_{\mathscr{M}} = [\langle \mathscr{U} \cap \mathscr{M} \rangle_{\text{ev}}, \mathscr{U}L]$.

Proof. (i) It is straightforward to see that $\mathscr{M}$ is closed under direct products and homomorphic images, that is, $\mathscr{M}$ is a PH-closed class. Now let $\mathscr{U} \in \mathscr{L}_{\mathscr{M}}(\mathscr{RS})$. Then

clearly we have that

$$\mathscr{U}^{\mathscr{M}} = \mathscr{U}L = \{S \in \mathscr{RS} \mid eSe \in \mathscr{U} \text{ for all } e \in E(S)\}.$$

The same type of argument as in the proof of [11, Lemma 4.6.1] also yields that $\mathscr{U}L$ is an e-variety. Hence $\mathscr{M}$ is e-varietal.

(ii) This part follows immediately from (i) and Theorem 3.1. □

Corollary 4.2. *Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})$. Then*

$$\mathscr{U} \cap \mathscr{M} = \mathscr{V} \cap \mathscr{M} \Leftrightarrow \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev} = \langle \mathscr{V} \cap \mathscr{M} \rangle_{ev} \Leftrightarrow \mathscr{U}L = \mathscr{V}L.$$

Proof. This follows directly from Theorem 4.1. □

In order to study the mapping $\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev}$, we need some preparation.

Lemma 4.3. *For any $\mathscr{U} \in \mathscr{L}_{\mathscr{M}}(\mathscr{RS})$, we have $\langle \mathscr{U} \rangle_{ev} = \mathbf{S}_r\mathscr{U}$.*

Proof. Let $\mathscr{U} \in \mathscr{L}_{\mathscr{M}}(\mathscr{RS})$. It suffices to show that $\mathbf{S}_r\mathscr{U}$ is an e-variety. Clearly $\mathbf{S}_r\mathscr{U}$ is closed under regular subsemigroups. Let $S_\alpha \in \mathbf{S}_r\mathscr{U}$, for $\alpha \in A$. Then for each $\alpha \in A$, there exists $T_\alpha \in \mathscr{U}$ such that S_α is a regular subsemigroup of T_α. Thus $\prod_{\alpha \in A} S_\alpha$ is a regular subsemigroup of $\prod_{\alpha \in A} T_\alpha$. Since $\mathscr{U}$ is closed under direct products, then $\prod_{\alpha \in A} T_\alpha \in \mathscr{U}$ so that $\prod_{\alpha \in A} S_\alpha \in \mathbf{S}_r\mathscr{U}$, and hence $\mathbf{S}_r\mathscr{U}$ is closed under direct products. Let $S \in \mathbf{S}_r\mathscr{U}$, $T \in \mathscr{RS}$ and any homomorphism $\varphi: S \to T$ of S onto T. Then there exists a regular monoid $M \in \mathscr{U}$ such that S is a regular subsemigroup of M. If S has an identity, then $S \in \mathscr{U}$ and $T \in \mathscr{M}$, whence $T \in \mathscr{U} \subseteq \mathbf{S}_r\mathscr{U}$, since $\mathscr{U}$ is an $\mathscr{M}$-variety. Thus suppose that S does not have an identity. Note that $S^1 = S \cup \{1\}$, where 1 is the identity of M, is a regular submonoid of M. Hence $S^1 \in \mathscr{U}$. We extend φ to an epimorphism $\varphi^1: S^1 \to T^1$ by defining $1\varphi^1 = 1$. Then $T^1 \in \mathscr{U}$ and $T \in \mathbf{S}_r\mathscr{U}$. Hence $\mathbf{S}_r\mathscr{U}$ is closed under homomorphic images, and therefore $\mathbf{S}_r\mathscr{U}$ is an e-variety. □

The next result extends [28, Theorem 5.3] to e-varieties of regular semigroups.

Proposition 4.4. *The mapping*

$$\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev} \quad (\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS}))$$

is a complete endomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$.

Proof. By Theorem 4.1 and Corollary 3.3, the mapping $\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev}$ is a complete $\vee$-endomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$. Let $\mathscr{U}_\alpha \in \mathscr{L}_{ev}(\mathscr{RS})$, for $\alpha \in A$. We then have

$$\begin{aligned}
\left\langle \left(\bigcap_{\alpha \in A} \mathscr{U}_\alpha \right) \cap \mathscr{M} \right\rangle_{ev} &= \mathbf{S}_r \left[\left(\bigcap_{\alpha \in A} \mathscr{U}_\alpha \right) \cap \mathscr{M} \right] \quad \text{by Lemma 4.3} \\
&= \mathbf{S}_r \left[\bigcap_{\alpha \in A} (\mathscr{U}_\alpha \cap \mathscr{M}) \right] \\
&= \bigcap_{\alpha \in A} \mathbf{S}_r(\mathscr{U}_\alpha \cap \mathscr{M}) \\
&= \bigcap_{\alpha \in A} \langle \mathscr{U}_\alpha \cap \mathscr{M} \rangle_{ev} \quad \text{by Lemma 4.3,}
\end{aligned}$$

so that the mapping is also a complete $\wedge$-endomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$. □

Corollary 4.5. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$. Then the following two conditions are equivalent.*

(a) $\langle \mathscr{U} \cap \mathscr{M} \rangle_{ev} = \mathscr{U}$.
(b) *If* $S \in \mathscr{U}$, *then* $S^1 \in \mathscr{U}$.

Proof. (a)⇒(b). Let $S \in \mathscr{U}$. Then $S \in \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev} = \mathbf{S}_r(\mathscr{U} \cap \mathscr{M})$ and hence there exists a regular monoid $M \in \mathscr{U} \cap \mathscr{M}$ such that S is a regular subsemigroup of M. Assume that S does not have an identity and let 1 be the identity of M, then $S^1 = S \cup \{1\}$ is a regular submonoid of M, so that $S^1 \in \mathscr{U} \cap \mathscr{M} \subseteq \mathscr{U}$, since $\mathscr{U} \cap \mathscr{M}$ is an $\mathscr{M}$-variety. Hence (b) holds.

(b)⇒(a). Clearly $\langle \mathscr{U} \cap \mathscr{M} \rangle_{ev} \subseteq \mathscr{U}$. Let $S \in \mathscr{U}$. Then $S^1 \in \mathscr{U}$ and $S^1 \in \mathscr{M}$ so that $S \in \mathbf{S}_r(\mathscr{U} \cap \mathscr{M}) = \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev}$. Hence $\mathscr{U} \subseteq \langle \mathscr{U} \cap \mathscr{M} \rangle_{ev}$ and (a) holds. □

For any semigroup S, we denote by $\mathrm{Reg}(S)$ the set of all regular elements of S, that is, $\mathrm{Reg}(S) = \{a \in S \mid axa = a \text{ for some } x \in S\}$. Note that $\mathrm{Reg}(S)$ need not be a subsemigroup of S.

Let $S \in \mathscr{RS}$ and $e \in E(S)$. In general, eS (respectively, Se) need not be a regular subsemigroup of S. However, from [10, Theorem 1(iv)], $\mathrm{Reg}(eS)$ (respectively, $\mathrm{Reg}(Se)$) is a regular subsemigroup of S with e as a left (respectively, right) identity.

A semigroup S is a *left* (respectively, *right*) *monoid* if S has a left (respectively, right) identity. We denote by $\mathscr{LM}$ (respectively, $\mathscr{RM}$) the class of all regular left (respecctively, right) monoids. Thus we clearly have

$$\mathscr{LM} = \{\mathrm{Reg}(eS) \mid S \in \mathscr{RS} \text{ and } e \in E(S)\},$$
$$\mathscr{RM} = \{\mathrm{Reg}(Se) \mid S \in \mathscr{RS} \text{ and } e \in E(S)\}.$$

The next observation and its dual will be required later.

Lemma 4.6. *Let $S, T \in \mathscr{RS}$ and let $\varphi: S \to T$ be a homomorphism of S onto T. Then for each $f \in E(T)$, there exists $e \in E(S)$ such that $e\varphi = f$ and the restriction mapping $\psi = \varphi|_{\mathrm{Reg}(eS)}$ maps* $\mathrm{Reg}(eS)$ *onto* $\mathrm{Reg}(fT)$.

Proof. Let $f \in E(T)$. Then by Lallement's Lemma (see [14, Lemma II.4.6]), there exists $e \in E(S)$ such that $e\varphi = f$. Clearly $\psi = \varphi|_{\mathrm{Reg}(eS)}$ maps $\mathrm{Reg}(eS)$ into $\mathrm{Reg}(fT)$. It remains to show that ψ is surjective. So let $x \in \mathrm{Reg}(fT)$. Since $\mathrm{Reg}(fT)$ is a regular subsemigroup of T, there exists $y \in V(x) \cap fT$. Let $a, b \in S$ be such that $a\varphi = x$ and $b\varphi = y$. Then $ea, eb \in eS$ and $(ea)\varphi = x$ and $(eb)\varphi = y$. From [14, Exercise II.14], there exist $s, r \in eS$ such that $s \in V(r)$, $s\varphi = x$ and $r\varphi = y$. Thus $s \in \mathrm{Reg}(eS)$ and $s\psi = x$. Hence ψ is a homomorphism of $\mathrm{Reg}(eS)$ onto $\mathrm{Reg}(fT)$. □

For any class $\mathscr{U}$ of regular semigroups, let

$$\mathscr{U}L_l = \{S \in \mathscr{RS} \mid \mathrm{Reg}(eS) \in \mathscr{U} \text{ for all } e \in E(S)\},$$
$$\mathscr{U}L_r = \{S \in \mathscr{RS} \mid \mathrm{Reg}(Se) \in \mathscr{U} \text{ for all } e \in E(S)\}.$$

It is obvious that $\mathscr{U}L_l \subseteq \mathscr{U}L$ and $\mathscr{U}L_r \subseteq \mathscr{U}L$, and that $L_l^2 = L_l$ and $L_r^2 = L_r$. In the context of varieties of completely regular semigroups these two operators were introduced and studied in [42].

By duality, in the following we may limit our deliberations to "left" concepts.

Theorem 4.7. (i) *$\mathscr{L}\mathscr{M}$ is an* e*-varietal PH-closed class.*
(ii) *The mapping*

$$\theta_{\mathscr{L}\mathscr{M}}: \mathscr{U} \to \mathscr{U} \cap \mathscr{L}\mathscr{M} \quad (\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S}))$$

is a complete homomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$ onto $\mathscr{L}_{\mathscr{L}\mathscr{M}}(\mathscr{R}\mathscr{S})$. Moreover, for any $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$, we have $\mathscr{U}\bar{\theta}_{\mathscr{L}\mathscr{M}} = [\langle \mathscr{U} \cap \mathscr{L}\mathscr{M} \rangle_{\mathrm{ev}}, \mathscr{U}L_l]$.

Proof. (i) Clearly $\mathscr{L}\mathscr{M}$ is closed under direct product and homomorphic images. Thus $\mathscr{L}\mathscr{M}$ is a PH-closed class. To show that $\mathscr{L}\mathscr{M}$ is e-varietal, let $\mathscr{U} \in \mathscr{L}_{\mathscr{L}\mathscr{M}}(\mathscr{R}\mathscr{S})$. It is clear that

$$\mathscr{U}^{\mathscr{L}\mathscr{M}} = \mathscr{U}L_l = \{S \in \mathscr{R}\mathscr{S} \mid \mathrm{Reg}(eS) \in \mathscr{U} \text{ for all } e \in E(S)\}.$$

Let $S \in \mathscr{U}L_l$. If T is a regular subsemigroup of S and $e \in E(T)$, then $\mathrm{Reg}(eT)$ is a regular subsemigroup of $\mathrm{Reg}(eS)$ and so belongs to $\mathscr{U}$. Thus $T \in \mathscr{U}L_l$ and $\mathscr{U}L_l$ is closed under regular subsemigroups. If T is a homomorphic image of S, then by Lemma 4.6, $T \in \mathscr{U}L_l$. Hence $\mathscr{U}L_l$ is closed under homomorphic images. Let $S_\alpha \in \mathscr{U}L_l$ for $\alpha \in A$, and let $T = \prod_{\alpha \in A} S_\alpha$. For any $e = (e_\alpha)_{\alpha \in A} \in E(T)$, we have $e_\alpha \in E(S_\alpha)$ and $\mathrm{Reg}(eT) = \prod_{\alpha \in A} \mathrm{Reg}(e_\alpha S_\alpha) \in \mathscr{U}$, so that $T \in \mathscr{U}L_l$. Hence $\mathscr{U}L_l$ is closed under direct products and therefore $\mathscr{U}L_l$ is an e-variety.

(ii) This follows immediately from (i) and Theorem 3.1. □

Corollary 4.8. *Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$. Then*

$$\mathscr{U} \cap \mathscr{L}\mathscr{M} = \mathscr{V} \cap \mathscr{L}\mathscr{M} \Leftrightarrow \langle \mathscr{U} \cap \mathscr{L}\mathscr{M} \rangle_{\mathrm{ev}} = \langle \mathscr{V} \cap \mathscr{L}\mathscr{M} \rangle_{\mathrm{ev}} \Leftrightarrow \mathscr{U}L_l = \mathscr{V}L_l.$$

Proof. This follows directly from Theorem 4.7. □

In conclusion, we give some properties of the mappings related to the upper and lower bounds of the intervals $\mathscr{U}\bar{\theta}_{\mathscr{L}\mathscr{M}}$.

Proposition 4.9. *The operator L_l is a complete $\wedge$-endomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$ but is not a $\vee$-endomorphism.*

Proof. It is clear from the definition that L_l is a complete $\wedge$-endomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$.

Note that $\mathscr{C}\mathscr{S}L = \mathscr{C}\mathscr{S}$ and $\mathscr{U}L_l \subseteq \mathscr{U}L$ for any $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{G})$. From this it follows easily that $\mathscr{L}\mathscr{G}L_l = \mathscr{L}\mathscr{G}$, $\mathscr{R}\mathscr{G}L_l = \mathscr{C}\mathscr{S}$ and $\mathscr{R}\mathscr{Z}L_l = \mathscr{R}\mathscr{B}$. We thus have

$$(\mathscr{L}\mathscr{G} \vee \mathscr{R}\mathscr{Z})L_l = \mathscr{R}e\mathscr{G}L_l = \mathscr{R}\mathscr{G}L_l = \mathscr{C}\mathscr{S}$$

and

$$\mathscr{L}\mathscr{G}L_l \vee \mathscr{R}\mathscr{Z}L_l = \mathscr{L}\mathscr{G} \vee \mathscr{R}\mathscr{B} = \mathscr{R}e\mathscr{G},$$

so that $(\mathscr{L}\mathscr{G} \vee \mathscr{R}\mathscr{Z})L_l \neq \mathscr{L}\mathscr{G}L_l \vee \mathscr{R}\mathscr{Z}L_l$. Hence L_l is not a $\vee$-homomorphism. □

Remark 4.10. It follows from Theorem 4.7 and Corollary 3.3 that the mapping $\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{L}\mathscr{M} \rangle_{\mathrm{ev}}$ is a complete $\vee$-endomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$. Whether this mapping is a (complete) $\wedge$-endomorphism of $\mathscr{L}_{\mathrm{ev}}(\mathscr{R}\mathscr{S})$ remains an open question. □

5. Near Existence Varieties of Idempotent Generated Regular Semigroups

The concept of a near variety for idempotent generated completely simple semigroups was introduced by PETRICH and REILLY in [26]. In this section we generalize this concept to idempotent generated regular semigroups.

We denote by $\mathscr{C}$ the class of all idempotent generated regular semigroups. Then we clearly have

$$\mathscr{C} = \{C(S) | S \in \mathscr{RS}\} = \{S \in \mathscr{RS} | S = C(S)\}.$$

It is easily verified that $\mathscr{C}$ is closed under finite direct products, idempotent generated regular subsemigroups and homomorphic images. However, as pointed out by HALL in the proof of [11, Lemma 4.3.1], $\mathscr{C}$ is not closed under *infinite* direct products.

A subclass $\mathscr{U}$ of $\mathscr{C}$ is a *near existence variety*, or *near* e-*variety* if it satisfies the following conditions:

(i) $\mathscr{U}$ is closed under idempotent generated subsemigroups;
(ii) $\mathscr{U}$ is closed under homomorphic images;
(iii) if $\{S_\alpha | \alpha \in A\} \subseteq \mathscr{U}$, then $C(\prod_{\alpha \in A} S_\alpha) \in \mathscr{U}$.

Lemma 5.1. *$\mathscr{C}$ is a near* e-*variety.*

Proof. This is obvious. □

Let $\mathscr{L}_{\text{nev}}(\mathscr{C})$ denote the class of all near e-varieties of idempotent generated regular semigroups. There is a natural order structure on $\mathscr{L}_{\text{nev}}(\mathscr{C})$, determined by containment, with respect to which $\mathscr{L}_{\text{nev}}(\mathscr{C})$ is complete lattice with the greatest element $\mathscr{C}$.

For any class $\mathscr{U}$ of regular semigroups, we let

$$\mathscr{U}C = \{S \in \mathscr{RS} | C(S) \in \mathscr{U}\}.$$

Obviously, $\mathscr{U}C = (\mathscr{U} \cap \mathscr{C})C$. On e-varieties of regular semigroups this operator was introduced by HALL in [11, Lemma 4.3.1], where the notation $\mathscr{U}^{ig}$ is used and it is shown that if $\mathscr{U} \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$, then $\mathscr{U}C \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$. Its restriction to varieties of completely regular semigroups was studied in [27, 28, 30].

Lemma 5.2. (i) *Let $\mathscr{U} \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$. Then $\mathscr{U} \cap \mathscr{C} \in \mathscr{L}_{\text{nev}}(\mathscr{C})$.*
(ii) *Let $\mathscr{U} \in \mathscr{L}_{\text{nev}}(\mathscr{C})$. Then $\mathscr{U}C \in \mathscr{L}_{\text{ev}}(\mathscr{RS})$ and $\mathscr{U}C \cap \mathscr{C} = \mathscr{U}$.*

Proof. (i) This is clear.

(ii) The argument of [11, Lemma 4.3.1] applies to yield that $\mathscr{U}C$ is an e-variety. Clearly $\mathscr{U} \subseteq \mathscr{U}C \cap \mathscr{C}$. Conversely, if $S \in \mathscr{U}C \cap \mathscr{C}$, then $S = C(S) \in \mathscr{U}$ so that $\mathscr{U}C \cap \mathscr{C} \subseteq \mathscr{U}$. Hence $\mathscr{U}C \cap \mathscr{C} = \mathscr{U}$, as required. □

We are now ready for the first main result of this section, which generalizes Theorem 4.4 of [28] to e-varieties of regular semigroups.

Theorem 5.3. *The mapping*

$$\theta_{\mathscr{C}}: \mathscr{U} \to \mathscr{U} \cap \mathscr{C} \quad (\mathscr{U} \in \mathscr{L}_{\text{ev}}(\mathscr{RS}))$$

is a complete homomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$ *onto* $\mathscr{L}_{nev}(\mathscr{C})$. *Moreover, for any* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, *we have* $\mathscr{U}\bar{\theta}_{\mathscr{C}} = [\langle \mathscr{U} \cap \mathscr{C} \rangle_{ev}, \mathscr{U}C]$.

Proof. By Lemma 5.2, $\theta_{\mathscr{C}}$ is a surjection of $\mathscr{L}_{ev}(\mathscr{RS})$ onto $\mathscr{L}_{nev}(\mathscr{C})$. Let $\mathscr{V} \in \mathscr{L}_{nev}(\mathscr{C})$. The same type of argument as in the proof of Theorem 3.1(ii) also yields that

$$\mathscr{V}\theta_{\mathscr{C}}^{-1} = [\langle \mathscr{V} \rangle_{ev}, \mathscr{V}C]. \tag{1}$$

Thus $\theta_{\mathscr{C}}$ satisfies the conditions in Lemma 2.3, and hence $\theta_{\mathscr{C}}$ is a complete lattice homomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$ onto $\mathscr{L}_{nev}(\mathscr{C})$. The last assertion follows easily from (1). □

Corollary 5.4. *Let* $\mathscr{U}, \mathscr{V} \in \mathscr{L}_{ev}(\mathscr{RS})$. *Then*

$$\mathscr{U} \cap \mathscr{C} = \mathscr{V} \cap \mathscr{C} \Leftrightarrow \langle \mathscr{U} \cap \mathscr{C} \rangle_{ev} = \langle \mathscr{V} \cap \mathscr{C} \rangle_{ev} \Leftrightarrow \mathscr{U}C = \mathscr{V}C.$$

Proof. This follows directly from Theorem 5.3. □

Our next goal is to study the mappings related to the lower and upper bounds of the intervals $\mathscr{U}\bar{\theta}_{\mathscr{C}}$. The arguments of [28, Lemmas 3.3 and 3.4] apply to yield the following two results using [39, Lemma 4.8].

Lemma 5.5. *For any* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$ *the following conditions are equivalent*:

(i) $\mathscr{U} \subseteq \mathscr{O}$;
(ii) $\langle \mathscr{U} \cap \mathscr{C} \rangle_{ev} \subseteq \mathscr{B}$;
(iii) $\mathscr{U} \cap \mathscr{C} = \mathscr{U} \cap \mathscr{B}$.

Lemma 5.6. *The mapping*

$$\mathscr{U} \to \mathscr{U} \cap \mathscr{B} \quad (\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{O}))$$

is a complete endomorphism of $\mathscr{L}_{ev}(\mathscr{O})$.

The next result generalizes Theorem 3.5 of [28] to e-varieties of regular semigroups.

Proposition 5.7. *The mapping*

$$\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{C} \rangle_{ev} \quad (\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS}))$$

is a complete $\vee$*-endomorphism of* $\mathscr{L}_{ev}(\mathscr{RS})$ *but is not a* $\wedge$*-homomorphism. Its restriction to* $\mathscr{L}_{ev}(\mathscr{O})$ *is a complete endomorphism of* $\mathscr{L}_{ev}(\mathscr{O})$.

Proof. The first assertion follows immediately from Theorem 5.3 and Lemma 2.4.

Since each variety of completely simple semigroups is an e-variety of regular semigroups and each e-subvariety of $\mathscr{CS}$ consists of completely simple semigroups, it follows that the counterexample in the proof of [28, Theorem 3.5] also holds for e-varieties. Hence the mapping $\mathscr{U} \to \langle \mathscr{U} \cap \mathscr{C} \rangle_{ev}$ is not a $\wedge$-homomorphism.

The final assertion of the theorem follows from Lemmas 5.5 and 5.6. □

Before proceeding, we introduce another operator on $\mathscr{L}_{ev}(\mathscr{RS})$, which will play an important role in the subsequent text. Let $S \in \mathscr{RS}$. A subsemigroup R of S is *full* if

$E(S) \subseteq R$; R is *self-conjugate* if $a'Ra \subseteq R$ for each $a \in S$ and $a' \in V(a)$. We denote by R_c the subsemigroup of S generated by the conjugates of R in S^1; that is

$$R_c = \langle a'ra | r \in R, a \in S^1, a' \in V(a) \rangle.$$

Let $C_\omega(S)$ denote the *least full and self-conjugate subsemigroup* of S. So

$$C_\omega(S) = C(S)_{ccc} \cdots.$$

By [37, Lemma 2.3], $C_\omega(S)$ is a full regular subsemigroup of S.

For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, let

$$\mathscr{U}C_\omega = \{S \in \mathscr{RS} \mid C_\omega(S) \in \mathscr{U}\}.$$

Then $\mathscr{U}C_\omega$ is again an e-variety. This operator was introduced in [17, 35]. Its restriction to $\mathscr{L}(\mathscr{CR})$ was studied in [41, 43], where the notation C^* is used.

Result 5.8 [17, 35]. *For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, $\mathscr{U}C_\omega = \langle \mathscr{U} \circ \mathscr{G} \rangle_{ev}$.*

We put together some general relationships between the operators C and C_ω in the next result.

Lemma 5.9. (i) *$\mathscr{U}C_\omega \subseteq \mathscr{U}C$ for any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$. However, $C \neq C_\omega$ on $\mathscr{L}_{ev}(\mathscr{RS})$.*
(ii) *$C = C_\omega$ on $\mathscr{L}_{ev}(\mathscr{O})$.*
(iii) *$CC_\omega = C_\omega C = C$ on $\mathscr{L}_{ev}(\mathscr{RS})$.*

Proof. See [43, Lemma 3.4 and Corollary 3.8]. □

We are now ready to prove the final result of this section, which generalizes Theorem 4.1 of [28].

Proposition 5.10. *The operator C is a complete $\wedge$-endomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$ but is not a $\vee$-homomorphism. Its restriction to $\mathscr{L}_{ev}(\mathscr{O})$ is a complete lattice endomorphism of $\mathscr{L}_{ev}(\mathscr{O})$.*

Proof. It is clear from the definition that C is a complete $\wedge$-endomorphism of $\mathscr{L}_{ev}(\mathscr{RS})$. That C is not a $\vee$-homomorphism was shown by PETRICH and REILLY in [28, Theorem 4.1].

For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{O})$, we then have

$$\begin{aligned} \mathscr{U}C &= \mathscr{U}C_\omega && \text{by Lemma 5.9 (ii)} \\ &= \langle \mathscr{U} \circ \mathscr{G} \rangle_{ev} && \text{by Result 5.8} \\ &= \mathscr{U} \otimes \mathscr{G} && \text{by [17, Theorem 5.2],} \end{aligned}$$

where $\mathscr{U} \otimes \mathscr{G}$ is the e-variety generated by all wreath products of elements from $\mathscr{U}$ with elements from $\mathscr{G}$. On the other hand, it follows from [36, Proposition 3.13(c)] that $\mathscr{U} \to \mathscr{U} \otimes \mathscr{G}$ is a complete $\vee$-endomorphism of $\mathscr{L}_{ev}(\mathscr{O})$. Hence C is a complete endomorphism of $\mathscr{L}_{ev}(\mathscr{O})$. □

6. L Commutes with T, T_l, T_r, K_l, K_r and K

In order to prove the statement in the title, we require some preparation.

Lemma 6.1. *Let $S \in \mathscr{RS}$ and $e \in E(S)$. Then*

$$\mathscr{R}^0|_{eSe} = (\mathscr{R}_{eSe})^0, \quad \mathscr{L}^0|_{eSe} = (\mathscr{L}_{eSe})^0, \quad \mathscr{H}^0|_{eSe} = (\mathscr{H}_{eSe})^0.$$

Proof. Consider $\mathscr{R}^0$. Clearly $\mathscr{R}^0|_{eSe} \subseteq (\mathscr{R}_{eSe})^0$. So let $a, b \in eSe$, $a(\mathscr{R}_{eSe})^0 b$ and $x \in S$. We wish to show that $ax \mathscr{R} bx$. Since $ax = aex$ and $bx = bex$, we can assume without loss of generality that $x = ex$. Take any $x' \in V(x)$. Then $x'e \in V(x)$ so we can assume that $x' = x'e$. Thus $xx' \in eSe$, and $a(\mathscr{R}_{eSe})^0 b$ implies that $axx' \mathscr{R}_{eSe} bxx'$. From this it follows that $axx' = bxx'y$ for some $y \in eSe$, so that $ax = axx'x = bxx'yx \in bxS$ and hence $axS \subseteq bxS$. Similary, $bxS \subseteq axS$. Hence $ax \mathscr{R} bx$, and therefore $a \mathscr{R}^0 b$ since $\mathscr{R}$ is a left congruence on S.

The case of $\mathscr{L}^0$ is symmetric and the case of $\mathscr{H}^0$ follows from the fact that $\mathscr{H}^0 = \mathscr{R}^0 \cap \mathscr{L}^0$. □

Lemma 6.2. *Let $S \in \mathscr{RS}$ and $e \in E(S)$. Then $\pi^0|_{eSe} = (\pi_{eSe})^0$.*

Proof. Clearly $\pi^0|_{eSe} \subseteq (\pi_{eSe})^0$. So let $a, b \in eSe$, $a(\pi_{eSe})^0 b$ and $x, y \in S$. We are going to show that $xay \pi xby$. Since $a, b \in eSe$, we can assume without loss of generality that $x = xe$ and $y = ey$. Take $x' \in V(x)$ and $y' \in V(y)$. Since $ex' \in V(x)$ and $y'e \in V(y)$, we can assume that $x' = ex'$ and $y' = y'e$. Thus we have $x'x, yy' \in E(eSe)$, so that $x'xayy' \pi_{eSe} x'xbyy'$, whence

$$V(x'xayy') \cap eSe = V(x'xbyy') \cap eSe. \tag{2}$$

It remains to show that $V(xay) = V(xby)$.

Let $z \in V(xay)$. Then we can easily verify that $yzx \in V(x'xayy')$. Note that yzx is also in eSe, so by (2), $yzx \in V(x'xbyy') \cap eSe$. From this it follows that $y'yzxx' \in V(xby)$. Thus we have

$$xby = xby(y'yzxx')xby = (xby)z(xby), \tag{3}$$

and

$$y'yzxx' = y'yzxx'xbyy'yzxx' = y'yzxbyzxx',$$

so that

$$yzx = y(y'yzxx')x = yy'yzxbyzxx'x = yzxbyzx. \tag{4}$$

On the other hand, we have

$$\begin{aligned}
z &= zxayz && \text{since } z \in V(xay) \\
&= zx(x'xayy')yz \\
&= zx(x'xayy')(yzx)(x'xayy')yz && \text{since } yzx \in V(x'xayy') \\
&= zxayy'(yzxbyzx)x'xayz && \text{from (4)} \\
&= zxayzxbyzxayz \\
&= zxbyz. && (5)
\end{aligned}$$

From (3) and (5), we conclude that $z \in V(xby)$ and whence $V(xay) \subseteq V(xby)$. Similarly, $V(xby) \subseteq V(xay)$. Hence $V(xay) = V(xby)$, that is $xay \; \pi \; xby$. Therefore $(\pi_{eSe})^0 \subseteq \pi^0$ and $\pi^0|_{eSe} = (\pi_{eSe})^0$. □

We can now show the desired result.

Proposition 6.3. *L commutes with T, T_l, T_r, K_l, K_r and K.*

Proof. Consider K. For any $S \in \mathscr{RS}$ and $e \in E(S)$, it follows from Lemma 6.2 that

$$(e\pi^0)(S/\pi^0)(e\pi^0) \cong eSe/(\pi_{eSe})^0.$$

Thus for any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, we obtain

$$\begin{aligned} S \in (\mathscr{U}^K)L &\Leftrightarrow eSe \in \mathscr{U}K && \text{for all } e \in E(S) \\ &\Leftrightarrow eSe/(\pi_{eSe})^0 \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Leftrightarrow (e\pi^0)(S/\pi^0)(e\pi^0) \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Leftrightarrow S/\pi^0 \in \mathscr{U}L \\ &\Leftrightarrow S \in (\mathscr{U}L)^K. \end{aligned}$$

Hence L commutes with K. Similarly for T, T_l and T_r by using Lemma 6.1. That L commutes with K_l and K_r follows easily from the above cases and the fact that $\mathscr{U}^{K_l} = \mathscr{U}^{T_l} \cap \mathscr{U}^K$ and $\mathscr{U}^{K_r} = \mathscr{U}^{T_r} \cap \mathscr{U}^K$, for all $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$. □

Corollary 6.4 [20, Theorem 1.5]. $\mathscr{O}L = (\mathscr{I}L)^K$.

Proof. By [23, Theorem 4], $\mathscr{O} = \mathscr{RB} \circ \mathscr{I} = \mathscr{I}^K$. From this and Proposition 6.3 the required result follows. □

Recall that a regular semigroup S is *strict* if it is a subdirect product of completely 0-simple and completely simple semigroups. We denote by $\mathscr{SR}$ the class of all strict regular semigroups.

A *combinatorial* semigroup is one in which all subgroups are trivial groups. We let $\mathscr{CSR}$ denote the class of all combinatorial strict regular semigroups. From [11, Corollary 4.6.4], $\mathscr{SR} = (\mathscr{SG})L$ and $\mathscr{CSR} = \mathscr{S}L$ so that both $\mathscr{SR}$ and $\mathscr{CSR}$ are e-subvarieties of regular semigroups. As in [12], we write $C_2 = \mathscr{M}^0\left(\langle 1 \rangle; 2, 2; \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)$. According to [12, Theorem 3.5], $\langle C_2 \rangle_{ev}$ is the least e-variety of regular semigroups which is not E-solid. Furthermore we have

Result 6.5 [12, Theorem 3.3]. $\langle C_2 \rangle_{ev} = \mathscr{CSR} = \mathscr{S}L$.

Corollary 6.6. (i) $(\mathscr{OG})L = (\mathscr{SR})^K = \mathscr{RB} \circ (\mathscr{SG})L$.
(ii) $\mathscr{B}L = (\mathscr{S}L)^K = \langle C_2 \rangle_{ev}^K$.

Proof. (i) This follows from Proposition 6.3 and the fact that $\mathscr{OG} = \mathscr{RB} \circ (\mathscr{SG}) = (\mathscr{SG})^K$.

(ii) This follows from Proposition 6.3 and the fact that $\mathscr{B} = \mathscr{RB} \circ \mathscr{S} = \mathscr{S}^K$. □

In the following we discuss the commutativity of the operator L and the operator M. As in [5] or [25], a semigroup is called *completely semisimple* if all of its principal factors are completely 0-simple or completely simple. From [9, Result 6], a regular semigroup S is completely semisimple if and only if it has no pair of distinct idempotents e, f such that $e \leqslant f$ and $e \mathscr{D} f$. We denote by $\mathscr{CSS}$ the class of all completely semisimple semigroups.

Lemma 6.7. *Let $S \in \mathscr{RS}$, $e \in E(S)$ and ρ^e be the congruence on S generated by $(\mathscr{M}_{eSe})^0$. Then $\rho^e \subseteq \mathscr{D}$.*

Proof. From the definition, we see that $\mathscr{M}_{eSe} \subseteq \mathscr{D}_{eSe} \subseteq \mathscr{D}$ so that $(\mathscr{M}_{eSe})^0 \subseteq \mathscr{D}$. Let $a, b \in eSe$ be such that $a(\mathscr{M}_{eSe})^0 b$, and let $x, y \in S^1$. We then have

$$\begin{aligned} xay = xeaey &\mathscr{L} e(xe)'xeaey && \text{for any } (xe)' \in V(xe) \\ &\mathscr{R} e(xe)'xeaey(ey)'e && \text{for any } (ey)' \in V(ey) \\ &(\mathscr{M}_{eSe})^0 e(xe)'xebey(ey)'e \\ &\mathscr{R} e(xe)'xebey \\ &\mathscr{L} xebey = xby. \end{aligned}$$

From this it follows that $xay \mathscr{D} xby$.

Let $u, v \in S$ and $u\rho^e v$. Then there exist $a_i, b_i \in eSe, x_i, y_i \in S^1$ with $(a_i, b_i) \in (\mathscr{M}_{eSe})^0$ such that

$$\begin{aligned} u &= x_1 a_1 y_1 \\ x_1 b_1 y_1 &= x_2 a_2 y_2 \\ x_2 b_2 y_2 &= x_3 a_3 y_3 \\ &\vdots \\ x_n b_n y_n &= v. \end{aligned} \tag{6}$$

Applying the above observation to (6), we can easily see that $u \mathscr{D} v$. Hence $\rho^e \subseteq \mathscr{D}$, as required. □

Lemma 6.8. *Let $S \in \mathscr{R}\mathscr{S}$ and $e \in E(S)$. If S is completely semisimple, then $\mathscr{M}^0|_{eSe} = (\mathscr{M}_{eSe})^0$.*

Proof. Clearly $\mathscr{M}^0|_{eSe} \subseteq (\mathscr{M}_{eSe})^0$. Since $(\mathscr{M}_{eSe})^0 \subseteq \rho^e$, to show that $(\mathscr{M}_{eSe})^0 \subseteq \mathscr{M}^0$, it suffices to show that $\rho^e \subseteq \mathscr{M}^0$, that is, that ρ^e is a congruence on S over $\mathscr{C}\mathscr{S}$. Let $f, g \in E(S)$ be such that $f(\rho^e \cap \leqslant)g$. Then by Lemma 6.7, $f(\mathscr{D} \cap \leqslant)g$ and so $f = g$, since S is completely semisimple. It follows from [22, Theorem 2.8] that ρ^e is over $\mathscr{C}\mathscr{S}$. □

Proposition 6.9. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$. Then*

$$\mathscr{C}\mathscr{S} \circ (\mathscr{U}L) \subseteq (\mathscr{C}\mathscr{S} \circ \mathscr{U})L \quad \text{and} \quad (\mathscr{C}\mathscr{S} \circ \mathscr{U})L \cap \mathscr{C}\mathscr{S}\mathscr{S} \subseteq \mathscr{C}\mathscr{S} \circ (\mathscr{U}L).$$

Proof. For any $S \in \mathscr{R}\mathscr{S}$, we have

$$\begin{aligned} S \in \mathscr{C}\mathscr{S} \circ (\mathscr{U}L) &\Rightarrow S/\mathscr{M}^0 \in \mathscr{U}L \\ &\Rightarrow (e\mathscr{M}^0)(S/\mathscr{M}^0)(e\mathscr{M}^0) \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow eSe/(\mathscr{M}^0|_{eSe}) \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow eSe/(\mathscr{M}_{eSe})^0 \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow eSe \in \mathscr{C}\mathscr{S} \circ \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow S \in (\mathscr{C}\mathscr{S} \circ \mathscr{U})L, \end{aligned}$$

since $(e\mathscr{M}^0)(S/\mathscr{M}^0)(e\mathscr{M}^0) \cong eSe/(\mathscr{M}^0|_{eSe})$ and $\mathscr{M}^0|_{eSe} \subseteq (\mathscr{M}_{eSe})^0$. Hence $\mathscr{C}\mathscr{S} \circ (\mathscr{U}L) \subseteq (\mathscr{C}\mathscr{S} \circ \mathscr{U})L$.

Now let $S \in (\mathscr{C}\mathscr{S} \circ \mathscr{U})L \cap \mathscr{C}\mathscr{S}\mathscr{S}$. By Lemma 6.8, $\mathscr{M}^0|_{eSe} = (\mathscr{M}_{eSe})^0$ for all $e \in E(S)$,

so that $(e\mathscr{M}^0)(S/\mathscr{M}^0)(e\mathscr{M}^0) \cong eSe/(\mathscr{M}_{eSe})^0$. Hence we have

$$\begin{aligned} S \in (\mathscr{CS} \circ \mathscr{U})L \cap \mathscr{CSS} &\Rightarrow eSe \in \mathscr{CS} \circ \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow eSe/(\mathscr{M}_{eSe})^0 \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow (e\mathscr{M}^0)(S/\mathscr{M}^0)(e\mathscr{M}^0) \in \mathscr{U} && \text{for all } e \in E(S) \\ &\Rightarrow S/\mathscr{M}^0 \in \mathscr{U}L \\ &\Rightarrow S \in \mathscr{CS} \circ (\mathscr{U}L). \end{aligned}$$

Thus $(\mathscr{CS} \circ \mathscr{U})L \cap \mathscr{CSS} \subseteq \mathscr{CS} \circ (\mathscr{U}L)$, as required. □

Corollary 6.10. *Let $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ be such that $(\mathscr{CS} \circ \mathscr{U})L$ is generated by its completely semisimple members. Then*

$$(\mathscr{CS} \circ \mathscr{U})L = \mathscr{CS} \circ (\mathscr{U}L).$$

Proof. This follows immediately from Proposition 6.9. □

Corollary 6.11. $(\mathscr{CR})L = \mathscr{CS} \circ (\mathscr{S}L) = \mathscr{CS} \circ \langle C_2 \rangle_{\mathrm{ev}} = \langle C_2 \rangle_{\mathrm{ev}}^{M}$.

Proof. Observe that if a regular semigroup S is locally completely regular then it cannot contain the bicyclic semigroup and must therefore be completely semisimple. From this and Corollary 6.10 it follows that $(\mathscr{CR})L = (\mathscr{CS} \circ \mathscr{S})L = \mathscr{CS} \circ (\mathscr{S}L) = \mathscr{CS} \circ \langle C_2 \rangle_{\mathrm{ev}}$. □

A regular semigroup S is called *E-solid* if for all idempotents $e, f, g \in S$ such that $e \mathscr{L} f \mathscr{R} g$ there exists an idempotent $h \in S$ such that $e \mathscr{R} h \mathscr{L} g$. We denote by $\mathscr{ES}$ the class of all E-solid regular semigroups.

From [9, Theorem 4] and [37, Proposition 5.1 and Corollary 5.2], it follows easily that $\mathscr{ES} = (\mathscr{CR})C = (\mathscr{CR})C_\infty = \mathscr{CS} \circ \mathscr{I}$. Thus $\mathscr{ES}$ forms an e-variety, and includes all inverse semigroups (in fact, orthodox semigroups) and completely regular semigroups.

Lemma 6.12. $(\mathscr{ES})L = (\mathscr{CS} \circ \mathscr{I})L = \mathscr{CS} \circ (\mathscr{I}L)$.

Proof. This follows directly from HALL [10, Theorems 7 and 9]. □

We conclude this section with the following question.

Question 6.13. Does L commute with M on $\mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$?(†)

7. C and C_∞ Commute with T, T_l, T_r, K_l, K_r and K

In order to prove the statement in the title, we require some preparation.

Lemma 7.1. *Let $S \in \mathscr{RS}$ and $C_\infty = C_\infty(S)$. Then*

$$\mathscr{R}^0|_{C_\infty} = (\mathscr{R}_{C_\infty})^0, \quad \mathscr{L}^0|_{C_\infty} = (\mathscr{L}_{C_\infty})^0, \quad \mathscr{H}^0|_{C_\infty} = (\mathscr{H}_{C_\infty})^0.$$

Proof. The proof is entirely analogous to that of [27, Lemma 5.4]. □

(†)Added in proof: For an affirmative answer to this question see: K. Auinger and T. E. Hall, "Representations of semigroups by transformations and the congruence lattice of an eventually regular semigroup," Internat. J. Algebra Comput. 6 (1996), 655–685.

Lemma 7.2. *Let $S \in \mathscr{RS}$ and $a, b, x, y \in S$. If there exist $x' \in V(x)$ and $y' \in V(y)$ such that $V(x'xayy') = V(x'xbyy')$, then $V(xay) = V(xby)$.*

Proof. Let $x' \in V(x)$ and $y' \in V(y)$ be such that $V(x'xayy') = V(x'xbyy')$. By symmetry, it suffices to show that $V(xay) \subseteq V(xby)$. So let $z \in V(xay)$. It is easily verified that $yzx \in V(x'xayy') = V(x'xbyy')$ and that $y'yzxx' \in y'V(x'xbyy')x' \subseteq$ $\subseteq V(xby)$. Hence

$$xby = xby(y'yzxx')xby = xbyzxby \tag{7}$$

and

$$y'yzxx' = y'yzxx'xbyy'yzxx' = y'yzxbyzxx'.$$

From this it follows that

$$yzx = y(y'yzxx')x = y(y'yzxbyzxx')x = yz(xby)zx. \tag{8}$$

Since $z \in V(xay)$, we have

$$\begin{aligned} z &= z(xay)z \\ &= zx(x'xayy')yz \\ &= zx(x'xayy')(yzx)(x'xayy')yz \quad \text{since } yzx \in V(x'xayy') \\ &= zxayy'(yzxbyzx)x'xayz \quad \text{by (8)} \\ &= zxayz(xby)zxayz \\ &= z(xby)z, \end{aligned}$$

so that $z \in V(xby)$ by (7). Hence $V(xay) \subseteq V(xby)$, as required. □

Lemma 7.3. *Let $S \in \mathscr{RS}$, and let $C = C(S)$ and $C_\infty = C_\infty(S)$. Then*

$$\pi^0|_C = (\pi_C)^0, \quad \pi^0|_{C_\infty} = (\pi_{C_\infty})^0.$$

Proof. Clearly $\pi^0|_C \subseteq (\pi_C)^0$. So let $a, b \in C$, $a(\pi_C)^0 b$ and $x, y \in S$. Let $x' \in V(x)$ and $y' \in V(y)$. Then $x'x, yy' \in C$, and thus $x'xayy' \pi_C x'xbyy'$. Hence $V(x'xayy') \cap C =$ $= V(x'xbyy') \cap C$. Since C is a full regular subsemigroup of S, then by [37, Lemma 2.3 (i)], we have $V(x'xayy') = V(x'xayy') \cap C = V(x'xbyy') \cap C = V(x'xbyy')$. From this and Lemma 7.2, $V(xay) = V(xby)$, that is, $xay \pi xby$. Hence $a\pi^0 b$ and therefore $\pi^0|_C = (\pi_C)^0$.

The second equality follows similarly. □

We are now ready for the main result of this section.

Proposition 7.4. *C and C_∞ commute with T, T_l, T_r, K_l, K_r and K.*

Proof. That C commutes with T, T_l and T_r was shown in [34, Lemma 7.3]. We now show that C commutes with K. For any $S \in \mathscr{RS}$, it follows from Lemma 7.3 that

$$C(S/\pi^0) \cong C(S)/(\pi_{C(S)})^0.$$

Thus, for any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, we obtain

$$\begin{aligned} S \in (\mathscr{U}^K)C &\Leftrightarrow C(S) \in \mathscr{U}^K \\ &\Leftrightarrow C(S)/(\pi_{C(S)})^0 \in \mathscr{U} \\ &\Leftrightarrow C(S/\pi^0) \in \mathscr{U} \\ &\Leftrightarrow S/\pi^0 \in \mathscr{U}C \\ &\Leftrightarrow S \in (\mathscr{U}C)^K. \end{aligned}$$

Hence C commutes with K. That C commutes with K_l and K_r now follows easily from the above cases and the fact that $\mathscr{U}^{K_l} = \mathscr{U}^{T_l} \cap \mathscr{U}^K$ and $\mathscr{U}^{K_r} = \mathscr{U}^{T_r} \cap \mathscr{U}^K$, for all $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$.

Similarly, we can show that C_∞ commutes with T, T_l, T_r, K_l, K_r and K by using Lemmas 7.1 and 7.3.

Corollary 7.5 [23, Theorem 4]. $\mathscr{O} = \mathscr{RB} \circ \mathscr{I}$.

Proof. Since $\mathscr{I} = \mathscr{S}C$ and $\mathscr{B} = \mathscr{S}^K$, it follows from Proposition 7.4 that

$$\mathscr{O} = \mathscr{B}C = (\mathscr{S}^K)C = (\mathscr{S}C)^K = \mathscr{RB} \circ \mathscr{I}. \qquad \square$$

We now consider the commutativity of the operator C (respectively, C_∞) and the operator M.

Lemma 7.6. *Let $S \in \mathscr{RS}$ and ρ^C (respectively, ρ^{C_∞}) be the congruence on S generated by $(\mathscr{M}_{C(S)})^0$ (respectively, $(\mathscr{M}_{C_\infty(S)})^0$). Then ρ^C (respectively, ρ^{C_∞}) $\subseteq \mathscr{D}$.*

Proof. The same type of argument as in the proof of Lemma 6.7 yields this result. $\square$

Lemma 7.7. *Let $S \in \mathscr{RS}$, and let $C = C(S)$ and $C_\infty = C_\infty(S)$. If S is completely semisimple, then*

$$\mathscr{M}^0|_C = (\mathscr{M}_C)^0, \quad \mathscr{M}^0|_{C_\infty} = (\mathscr{M}_{C_\infty})^0.$$

Proof. The proof is entirely similar to that of Lemma 6.8. $\square$

Proposition 7.8. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$. Then*

(i) $\mathscr{CS} \circ (\mathscr{U}C) \subseteq (\mathscr{CS} \circ \mathscr{U})C$ *and* $(\mathscr{CS} \circ \mathscr{U})C \cap \mathscr{CSS} \subseteq \mathscr{CS} \circ (\mathscr{U}C)$.

(ii) $\mathscr{CS} \circ (\mathscr{U}C_\infty) \subseteq (\mathscr{CS} \circ \mathscr{U})C_\infty$ *and* $(\mathscr{CS} \circ \mathscr{U})C_\infty \cap \mathscr{CSS} \subseteq \mathscr{CS} \circ (\mathscr{U}C_\infty)$.

Proof. (i) For any $S \in \mathscr{RS}$, we have

$$\begin{aligned} S \in \mathscr{CS} \circ (\mathscr{U}C) &\Rightarrow S/\mathscr{M}^0 \in \mathscr{U}C \\ &\Rightarrow C(S/\mathscr{M}^0) \in \mathscr{U} \\ &\Rightarrow C(S)/(\mathscr{M}^0|_{C(S)}) \in \mathscr{U} \\ &\Rightarrow C(S)/(\mathscr{M}_{C(S)})^0 \in \mathscr{U} \\ &\Rightarrow C(S) \in \mathscr{CS} \circ \mathscr{U} \\ &\Rightarrow S \in (\mathscr{CS} \circ \mathscr{U})C, \end{aligned}$$

since $C(S/\mathscr{M}^0) \cong C(S)/(\mathscr{M}^0|_{C(S)})$ and $\mathscr{M}^0|_{C(S)} \subseteq (\mathscr{M}_{C(S)})^0$. Hence $\mathscr{CS} \circ (\mathscr{U}C) \subseteq (\mathscr{CS} \circ \mathscr{U})C$.

Now let $S \in (\mathscr{CS} \circ \mathscr{U})C \cap \mathscr{CSS}$. By Lemma 7.7, $\mathscr{M}^0|_{C(S)} = (\mathscr{M}_{C(S)})^0$ so that $C(S/\mathscr{M}^0) \cong C(S)/(\mathscr{M}_{C(S)})^0$. Thus we have

$$\begin{aligned} S \in (\mathscr{CS} \circ \mathscr{U})C \cap \mathscr{CSS} &\Rightarrow C(S) \in \mathscr{CS} \circ \mathscr{U} \\ &\Rightarrow C(S)/(\mathscr{M}_{C(S)})^0 \in \mathscr{U} \\ &\Rightarrow C(S/\mathscr{M}^0) \in \mathscr{U} \\ &\Rightarrow S/\mathscr{M}^0 \in \mathscr{U}C \\ &\Rightarrow S \in \mathscr{CS} \circ (\mathscr{U}C). \end{aligned}$$

Hence $(\mathscr{CS} \circ \mathscr{U})C \cap \mathscr{CSS} \subseteq \mathscr{CS} \circ (\mathscr{U}C)$, as required.

(ii) The proof is entirely analogous to that of (i). □

Corollary 7.9. *Let* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$.

(i) *If* $(\mathscr{CS} \circ \mathscr{U})C$ *is generated by its completely semisimple members, then*

$$(\mathscr{CS} \circ \mathscr{U})C = \mathscr{CS} \circ (\mathscr{U}C).$$

(ii) *If* $(\mathscr{CS} \circ \mathscr{U})C_\infty$ *is generated by its completely semisimple members, then*

$$(\mathscr{CS} \circ \mathscr{U})C_\infty = \mathscr{CS} \circ (\mathscr{U}C_\infty).$$

Proof. This follows immediately from Proposition 7.8. □

From Corollary 7.9, we have $\mathscr{ES} = (\mathscr{CS} \circ \mathscr{S})C = \mathscr{CS} \circ (\mathscr{S}C) = \ = \mathscr{CS} \circ \mathscr{I}$, as noted previously. In conclusion, we pose the following.

Question 7.10. Does C (respectively, C_∞) commute with M on $\mathscr{L}_{ev}(\mathscr{RS})$?(†)

8. Infinite Iterations of Operators

In this section we investigate the effect of infinite application of the operators T_l and T_r (respectively, K_l and K_r) to an arbitrary e-variety of regular semigroups. We first have the following observation.

Lemma 8.1. *Let* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$.

(i) $\bigvee_{n \geq 0} \mathscr{U}^{(T_l T_r)^n} \subseteq \mathscr{CS} \circ \mathscr{U} = \mathscr{U}^M$.

(ii) $\bigvee_{n \geq 0} \mathscr{U}^{(K_l K_r)^n} \subseteq \mathscr{RB} \circ \mathscr{U} = \mathscr{U}^K$.

Proof. This follows easily by induction from Lemma 2.2 and that the operators

$$\mathscr{U} \to \mathscr{U}^M \quad \text{and} \quad \mathscr{U} \to \mathscr{U}^K$$

are closure operators. □

In order to establish some partial results regarding the opposite inclusions in Lemma 8.1, we require some preparation. The next result is a generalization of [34, Lemma 7.9].

(†)Added in proof: For an affirmative answer to this question see: K. Auinger and T. E. Hall, "Representations of semigroups by transformations and the congruence lattice of an eventually regular semigroup," Internat. J. Algebra Comput. 6 (1996), 655–685.

Lemma 8.2. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$, $S \in \mathscr{C}\mathscr{S} \circ \mathscr{U}$ and let S be completely semisimple with finitely many $\mathscr{D}$-classes. Then $S \in \bigcup_{n \geq 0} \mathscr{U}^{(T_1 T_r)^n}$.*

Proof. For the sake of notational convenience, let $\gamma = \mathscr{M}^0$. Since $S \in \mathscr{C}\mathscr{S} \circ \mathscr{U}$ then $S/\gamma \in \mathscr{U}$ and γ is over $\mathscr{C}\mathscr{S}$ so that $\gamma \subseteq \mathscr{D}$. We establish the claim by induction on

$$n = |\{D \in S/\mathscr{D} \mid \gamma|_D \neq \iota\}|.$$

If $n = 0$ then clearly we have $S \in \mathscr{U}$. So assume that $n > 0$ and the claim holds for such situations where there are fewer than n $\mathscr{D}$-classes D with $\gamma|_D \neq \iota$.

Let D be a $\mathscr{D}$-class with $\gamma|_D \neq \iota$ and which is minimal in the partial order of the $\mathscr{D}$-classes ($= \mathscr{J}$-classes). Define λ on S by

$$a \lambda b \Leftrightarrow \begin{cases} \text{either } a = b; \\ \text{or } a, b \in D \text{ and } a\ (\gamma \cap \mathscr{L})\ b. \end{cases}$$

Note that $\lambda \subseteq \mathscr{L}$. To see that λ is a congruence, let $a, b, x \in S$ and $a \lambda b$. Since $\mathscr{L}$ is a right congruence, it is evident that $ax \lambda bx$. Likewise, if $a = b$ then we have $xa \lambda xb$. So suppose that $a, b \in D$, $a \neq b$ and $a\ (\gamma \cap \mathscr{L})\ b$. If $xa \notin D$, then we have $xa \gamma xb$ so that $xa \mathscr{D} xb$, but also $D_{xa} < D_a$. By the minimality of D in the partial order of $S/\mathscr{D}$, this means that $\gamma|_{D_{xa}} = \iota$. Hence $xa = xb$ and, in particular, $xa \lambda xb$.

So now let us assume that $xa, xb \in D$. Let

$$I(a) = \{y \in SaS \mid a \notin SyS\}.$$

Then $SaS/I(a)$ is completely 0-simple. In fact, $SaS/I(a) \cong D^0$ where $D^0 = D \cup \{0\}$ with multiplication such that

$$ab = \begin{cases} ab & \text{if } a, b, ab \in D; \\ 0 & \text{otherwise.} \end{cases}$$

Since $(xaa')a = xa \in D$, it follows that $xaa' \in D$ and therefore, since D^0 is completely 0-simple, that $xa = (xaa')a\ \mathscr{L}\ a$ in D^0. Hence $xa \mathscr{L} a$ in S. Similarly, $xb \mathscr{L} b$. Consequently $xa \mathscr{L} xb$. Thus $xa \lambda xb$ and λ is a congruence on S.

Clearly, if $\gamma_{(S/\lambda)}$ is not the identity relation on D/λ, then D/λ is a minimal such $\mathscr{D}$-class in $(S/\lambda)/\mathscr{D}$.

On S/λ define ρ by

$$a\lambda \rho b\lambda \Leftrightarrow \begin{cases} \text{either } a\lambda = b\lambda; \\ \text{or } a\lambda,\ b\lambda \in D/\lambda \text{ and } a\lambda\ (\gamma_{(S/\lambda)} \cap \mathscr{R})\ b\lambda. \end{cases}$$

Clearly $\rho \subseteq \mathscr{R}$ and by the dual argument above, ρ is a congruence on S/λ. Let

$$\eta = \{(a, b) \in S \times S \mid (a\lambda)\rho(b\lambda)\}$$

so that η is a congruence on S and $\rho = \eta/\lambda$.

It is evident that η is the identity relation except on D. We wish to show that $\gamma|_D \subseteq \eta$.

First let $e \in E(D)$, $d \in D$ and $e \gamma d$. Since $e\gamma$ is a completely simple subsemigroup, then $e\ (\gamma \cap \mathscr{L})\ de$ and $de\ (\gamma \cap \mathscr{R})\ d$. Thus $e \lambda de$ and $(de)\lambda\ \rho\ d\lambda$ so that $e \eta d$. Hence

$$e \in E(D), d \in D \quad \text{and} \quad e \gamma d \Rightarrow e \eta d. \tag{9}$$

Now consider $a, b \in D$ with $a\gamma b$. Let $a' \in V(a)$ and $b' \in V(b)$. Then we have

$$aa'\gamma ba' \quad \text{and} \quad a'a\gamma a'b. \tag{10}$$

Also

$$ba'bb'\gamma aa'ab' = ab'\gamma bb'. \tag{11}$$

Applying (9) to (10) and (11), we obtain

$$aa'\eta ba', a'a\eta a'b \quad \text{and} \quad ba'bb'\eta bb',$$

whence

$$a = aa'a\eta ba'a\eta ba'b = ba'bb'b\eta bb'b = b.$$

Thus $\gamma|_D \subseteq \eta$.

Obviously, γ/η is a congruence on S/η that is over $\mathscr{C}\mathscr{S}$ and such that $(S/\eta)/(\gamma/\eta) \in \mathscr{U}$. Since $\gamma|_D = \eta|_D$, it follows that $\gamma_{(S/\eta)}$ is the identity relation on D/η. Thus the number of $\mathscr{D}$-classes Δ in S/η for which $\gamma_{(S/\eta)}|_\Delta \neq \iota$ is less than n. Hence, by the induction hypothesis, there exists an integer k such that $S/\eta \in \mathscr{U}^{(T_l T_r)^k}$. But $S/\eta \cong (S/\lambda)/\rho$ where $\lambda \subseteq \mathscr{L}$ and $\rho \subseteq \mathscr{R}$ so that $S/\lambda \in \mathscr{U}(T_l T_r)^k T_r$ and $S \in \mathscr{U}^{(T_l T_r)^k T_r T_l} \subseteq \mathscr{U}^{(T_l T_r)^{k+2}}$. Therefore the lemma holds. □

A completely semisimple regular semigroup S is called *upper $\mathscr{D}$-finite* if for every $a \in S$, $\{D_b | D_b > D_a\}$ is finite. We denote by $\mathscr{U}\mathscr{D}\mathscr{F}$ the class of all upper $\mathscr{D}$-finite completely semisimple regular semigroups.

Theorem 8.3. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$ be such that $\mathscr{C}\mathscr{S} \circ \mathscr{U} = \langle (\mathscr{C}\mathscr{S} \circ \mathscr{U}) \cap \mathscr{U}\mathscr{D}\mathscr{F} \rangle_{ev}$. Then*

$$\mathscr{C}\mathscr{S} \circ \mathscr{U} = \bigvee_{n \geq 0} \mathscr{U}^{(T_l T_r)^n}.$$

Proof. Let $S \in \mathscr{C}\mathscr{S} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F}$. Clearly S is a subdirect product of the Rees quotients $S/I(a)$, $a \in S$, where $I(a) = \{x \in S | a \notin SxS\}$. Since S is upper $\mathscr{D}$-finite, each quotient $S/I(a)$ has finitely many $\mathscr{D}$-classes. Also, $S/I(a) \in \mathscr{C}\mathscr{S} \circ \mathscr{U} \cap \mathscr{C}\mathscr{S}\mathscr{S}$. By Lemma 8.2, each $S/I(a)$ lies in $\mathscr{U}^{(T_l T_r)^{k_a}}$ for some k_a and hence $S \in \bigvee_{n \geq 0} \mathscr{U}^{(T_l T_r)^n}$. Therefore by the hypothesis, $\mathscr{C}\mathscr{S} \circ \mathscr{U} \subseteq \bigvee_{n \geq 0} \mathscr{U}^{(T_l T_r)^n}$.

The reverse containment was established in Lemma 8.1. □

Corollary 8.4. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$ be such that $\mathscr{R}\mathscr{B} \circ \mathscr{U} = \langle (\mathscr{R}\mathscr{B} \circ \mathscr{U}) \cap \mathscr{U}\mathscr{D}\mathscr{F} \rangle_{ev}$. Then*

$$\mathscr{R}\mathscr{B} \circ \mathscr{U} = \bigvee_{n \geq 0} \mathscr{U}^{(K_l K_r)^n}.$$

Proof. By Lemma 8.1, it remains to show that $\mathscr{R}\mathscr{B} \circ \mathscr{U} \subseteq \bigvee_{n \geq 0} \mathscr{U}^{(K_l K_r)^n}$. Note that for any $\mathscr{V}, \mathscr{W} \in \mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$, we have

$$\mathscr{V}^{T_l} \cap \mathscr{W}^K \subseteq (\mathscr{V} \vee \mathscr{W})^{K_l} \quad \text{and} \quad \mathscr{V}^{T_r} \cap \mathscr{W}^K \subseteq (\mathscr{V} \vee \mathscr{W})^{K_r}.$$

From this, by simple induction and using the fact that $\mathscr{V} \to \mathscr{V}^{K_l}$ and $\mathscr{V} \to \mathscr{V}^{K_r}$ are $\cap$-homomorphisms, we have

$$\mathscr{U}^{(T_l T_r)^n} \cap \mathscr{U}^K = \mathscr{U}^{(K_l K_r)^n} \tag{12}$$

for any positive integer n.

Let $S \in \mathscr{R}\mathscr{B} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F}$. As in the proof of Theorem 8.3, S is a subdirect product of completely semisimple semigroups with finitely many $\mathscr{D}$-classes. There will, therefore, be no loss of generality in assuming that S itself has only finitely many $\mathscr{D}$-classes. Then by Lemma 8.2, $S \in \mathscr{U}^{(T_l T_r)^k}$ for some integer $k \geqslant 1$, and it follows from (12) that

$$S \in \mathscr{U}^{(T_l T_r)^k} \cap \mathscr{U}^K = \mathscr{U}^{(K_l K_r)^k} \subseteq \bigvee_{n \geqslant 0} \mathscr{U}^{(K_l K_r)^n}.$$

Combined with the hypothesis, this yields $\mathscr{R}\mathscr{B} \circ \mathscr{U} \subseteq \bigvee_{n \geqslant 0} \mathscr{U}^{(K_l K_r)^n}$ and the proof of the corollary is complete. □

Corollary 8.5. (i) [27, Theorem 4.6 (iii)] $\mathscr{C}\mathscr{R} = \mathscr{C}\mathscr{S} \circ \mathscr{S} = \bigvee_{n \geqslant 0} \mathscr{S}^{(T_l T_r)^n}$.
(ii) [34, Proposition 8.4] $\mathscr{O} = \mathscr{R}\mathscr{B} \circ \mathscr{I} = \bigvee_{n \geqslant 0} \mathscr{I}^{(K_l K_r)^n}$.

Proof. (i) By [24, Theorem 3.12], $\mathscr{C}\mathscr{R} = \mathscr{C}\mathscr{S} \circ \mathscr{S}$ is generated by its finite members. Since finite regular semigroups are completely semisimple and upper $\mathscr{D}$-finite, the assertion is now an immediate consequence of Theorem 8.3.

(ii) By [6, Lemma 3.7], $\mathscr{O} = \mathscr{R}\mathscr{B} \circ \mathscr{I}$ is generated by its finite members. It follows from Corollary 8.4 that the equality holds. □

In certain circumstances we can improve the above results slightly by imposing conditions on $\mathscr{U}$ instead of $\mathscr{C}\mathscr{S} \circ \mathscr{U}$ (or $\mathscr{R}\mathscr{B} \circ \mathscr{U}$).

Lemma 8.6. *Let $S \in \mathscr{R}\mathscr{S}$ and ρ be a congruence on S over $\mathscr{C}\mathscr{S}$, that is $\rho \subseteq \mathscr{M}$.*

(i) *Then S is completely semisimple if and only if $T = S/\rho$ is completely semisimple.*
(ii) *If S is completely semisimple, then S is upper $\mathscr{D}$-finite if and only if $T = S/\rho$ is upper $\mathscr{D}$-finite.*

Proof. (i) First let $S \in \mathscr{C}\mathscr{S}\mathscr{S}$. Now let $e\rho$, $f\rho \in E(T)$ and $x\rho \in T$ be such that $e\rho \mathscr{R} x\rho \mathscr{L} f\rho$ and $e\rho \leqslant f\rho$, where $e, f \in E(S)$ and $x \in S$. Let $x', x^* \in S$ be such that $x' \in V(x)$ and $(x\rho)(x^*\rho) = e\rho$. Then

$$\begin{aligned}
(e\rho)(x\rho) = x\rho &\Rightarrow exρx \\
&\Rightarrow exx'ρxx' \\
&\Rightarrow exx' \mathscr{L} xx' \quad \text{since } (xx')\rho \in \mathscr{C}\mathscr{S} \\
&\Rightarrow ex \mathscr{L} x \quad \text{since } \mathscr{L} \text{ is a right congruence.}
\end{aligned}$$

Also

$$\begin{aligned}
(xx^*)\rho = e\rho &\Rightarrow xx^* \in e\rho \in \mathscr{C}\mathscr{S} \\
&\Rightarrow exx^* \in e\rho \\
&\Rightarrow exx^* \mathscr{R} e \\
&\Rightarrow ex \mathscr{R} e.
\end{aligned}$$

Thus $e \mathscr{R} ex \mathscr{L} x$. Dually, $f \mathscr{L} xf \mathscr{R} x$, so that $e \mathscr{D} f$.

Let $g \in V((efe)^2)$ and $h = (efe)g(efe)$. It is easily verified that $h \in E(S)$. Also since $e\rho \leqslant f\rho$ we have $efe\rho e$ so that $efe\rho(efe)^2$. Hence

$$h\rho(efe)^2 g(efe)^2 = (efe)^2 \rho efe\rho f.$$

Thus $h \in f\rho \in \mathscr{C}\mathscr{S}$, whence $h \mathscr{D} f \mathscr{D} e$. But $h \leqslant e$. Since S is completely semisimple, it follows that $h = e$. Hence $f\rho h = e$, and therefore by [9, Result 6], T is completely semisimple.

Now assume that T is completely semisimple. Let $e, f \in E(S)$ be such that $e \leqslant f$ and $e \mathscr{D} f$. By [14, Proposition II.3.6], there exists $a \in S$ and $a' \in V(a)$ such that $e = a'a$ and $f = aa'$. Then $a'\rho \in V(a\rho)$, $e\rho = (a'\rho)(a\rho)$, $f\rho = (a\rho)(a'\rho)$ and $e\rho \leqslant f\rho$, where $e\rho, f\rho \in E(T)$. Again by [14, Proposition II.3.6], $e\rho \mathscr{D} f\rho$ in T, and hence $e\rho = f\rho$, since T is completely semisimple. But $e\rho \in \mathscr{C}\mathscr{S}$. Therefore $e = f$ and S is completely semisimple.

(ii) It suffices to show that ρ induces an order isomorphism of $S/\mathscr{J}$ onto $T/\mathscr{J}$. Clearly ρ induces an order preserving mapping of $S/\mathscr{J}$ onto $T/\mathscr{J}$. Let $a, b \in S$ be such that $J_{a\rho} \leqslant J_{b\rho}$. From [5, Exercise 3, Section 8.4], there exist idempotents $e\rho \in J_{b\rho}$, $f\rho \in J_{a\rho}$ with $(e\rho)(f\rho) = f\rho$. By LALLEMENT's Lemma [14, Lemma II.4.6], we may assume that $e, f \in E(S)$. By the hypothesis and (i), we have $\mathscr{D} = \mathscr{J}$ on both S and $T = S/\rho$. Thus $e\rho \mathscr{J} b\rho$ implies that $e\rho \mathscr{D} b\rho$. Since $\rho \subseteq \mathscr{M} \subseteq \mathscr{D}$ on S, it follows easily that $e \mathscr{D} b$. Similarly, $f \mathscr{D} a$ and $ef \mathscr{D} f$. Thus, $J_a = J_f = J_{ef} \leqslant J_e = J_b$. Hence ρ induces an order isomorphism. □

Lemma 8.7. *Let $S, T \in \mathscr{R}\mathscr{S}$, $\varphi: S \to T$ be a surjective homomorphism and $\bar{\varphi}$ be over $\mathscr{C}\mathscr{S}$. Let $s \in S$ and $t \in V(s\varphi)$. Then $V(s) \cap t\varphi^{-1} \neq \varnothing$.*

Proof. Let $x \in t\varphi^{-1}$ and $s' \in V(s)$. Then

$$(sxss')\varphi = (s\varphi)(x\varphi)(s\varphi)(s'\varphi) = (s\varphi)t(s\varphi)(s'\varphi) = (s\varphi)(s'\varphi) = (ss')\varphi,$$

so that $sxss' \in (ss')\bar{\varphi} \in \mathscr{C}\mathscr{S}$. It follows that

$$sxss' = ss'(sxss') \mathscr{R} ss' \mathscr{R} s,$$

thus $sx \mathscr{R} s$. Now $(sx)\varphi = (s\varphi)t \in E(T)$ so that $(sx)\bar{\varphi} \in \mathscr{C}\mathscr{S}$. Hence H_{sx} is a subgroup of S. Let $g = sx$ and g^{-1} be the inverse of g in the group H_{sx}. Then $sxg^{-1} \in$ $\in E(R_{sx}) = E(R_s)$, so that

$$s(xg^{-1})s = (sxg^{-1})s = s.$$

Since sxg^{-1} is the identity of $H_{sx} = H_g$, we also have

$$g^{-1}(sxg^{-1}) = g^{-1},$$

whence $xg^{-1} \in V(s)$. On the other hand,

$$g^{-1}\varphi = ((sx)^{-1})\varphi = ((sx)\varphi)^{-1} = ((s\varphi)t)^{-1} = (s\varphi)t,$$

since homomorphisms respect inverses within subgroups and $(s\varphi)t \in E(T)$. Hence

$$(xg^{-1})\varphi = (x\varphi)(g^{-1}\varphi) = t(s\varphi)t = t.$$

Therefore, $xg^{-1} \in V(s) \cap t\varphi^{-1}$, as required. □

Theorem 8.8. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{R}\mathscr{S})$ be such that $\mathscr{U} = \mathbf{H}(\mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$. Then*

(i) $\mathscr{C}\mathscr{S} \circ \mathscr{U} = \mathbf{H}(\mathscr{C}\mathscr{S} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$.

(ii) $\mathscr{R}\mathscr{B} \circ \mathscr{U} = \mathbf{H}(\mathscr{R}\mathscr{B} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$.

Proof. (i) It suffices to show that $\mathscr{CS} \circ \mathscr{U} \subseteq \mathbf{H}(\mathscr{CS} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$. So let $S \in \mathscr{CS} \circ \mathscr{U}$. Then there exists a congruence ρ on S over $\mathscr{CS}$ with $T = S/\rho \in \mathscr{U}$. Let $\varphi: S \to T$ be defined by $a\varphi = a\rho$ $(a \in S)$. Then φ is an epimorphism of S onto T such that $\bar{\varphi}$ is over $\mathscr{CS}$. Since $T \in \mathscr{U} = \mathbf{H}(\mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$, there exists $R \in \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F}$ and a surjective homomorphism $\theta: R \to T$. Let $\tau = \varphi\theta^{-1}$. Then τ is a relational morphism of S onto R. We wish to show that τ is regular (see [35]), that is, $g(\tau) = \{(s, r) \in S \times R \mid s\varphi = r\theta\}$ is a regular subsemigroup of $S \times R$. Let $(s, r) \in g(\tau)$, and let $r' \in V(r)$ and $t = r'\theta$. Then $t \in V(r\theta) = V(s\varphi)$, so that by Lemma 8.7, there exists $x \in V(s) \cap t\varphi^{-1}$. Thus $(x, r') \in V((s, r)) \cap g(\tau)$. Hence $g(\tau)$ is a regular subsemigroup of $S \times R$, and hence τ is regular. Clearly S is a homomorphic image of $g(\tau)$.

It remains to show that $g(\tau) \in \mathscr{CS} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F}$. Since $S \times R \in \mathscr{CS} \circ \mathscr{U}$, it follows that $g(\tau) \in \mathscr{CS} \circ \mathscr{U}$. Let $\pi: g(\tau) \to R$ be the projection of $g(\tau)$ into R. Clearly π is a surjective homomorphism. Let $\alpha = \bar{\pi}$. Thus $g(\tau)/\alpha \cong R$. Let (e_1, f_1), $(e_2, f_2) \in E(g(\tau))$ be such that

$$(e_1, f_1)(\alpha \cap \leqslant)(e_2, f_2).$$

Then $e_1 \leqslant e_2$ and $f_1 = f_2$. On the other hand, (e_1, f_1), $(e_2, f_2) \in g(\tau)$ imply that $e_1\varphi = f_1\theta = f_2\theta = e_2\varphi$, so that $e_1(\leqslant \cap \bar{\varphi})e_2$. Since $\bar{\varphi}$ is over $\mathscr{CS}$, it follows from [22, Theorem 2.8] that $e_1 = e_2$. Hence $(e_1, f_1) = (e_2, f_2)$, and $\alpha \cap \leqslant = \iota$. Again by [22, Theorem 2.8], α is over $\mathscr{CS}$. Since $R \in \mathscr{U}\mathscr{D}\mathscr{F}$, it follows from Lemma 8.6 that $g(\tau) \in \mathscr{U}\mathscr{D}\mathscr{F}$. Hence $g(\tau) \in \mathscr{CS} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F}$ and $S \in \mathbf{H}(\mathscr{CS} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$, and therefore $\mathscr{CS} \circ \mathscr{U} \subseteq \mathbf{H}(\mathscr{CS} \circ \mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$, as required.

(ii) This follows by a similar argument as in (i). □

In conjunction with Theorem 8.3 and Corollary 8.4, from the previous theorem we immediately obtain the desired result.

Theorem 8.9. *Let $\mathscr{U} \in \mathscr{L}_{\mathrm{ev}}(\mathscr{RS})$ be such that $\mathscr{U} = \mathbf{H}(\mathscr{U} \cap \mathscr{U}\mathscr{D}\mathscr{F})$. Then*

(i) $\mathscr{CS} \circ \mathscr{U} = \bigvee_{n \geqslant 0} \mathscr{U}^{(T_l T_r)^n}$.
(ii) $\mathscr{RB} \circ \mathscr{U} = \bigvee_{n \geqslant 0} \mathscr{U}^{(K_l K_r)^n}$.

Let X be a nonempty set and $X' = \{x' \mid x \in X\}$ be a disjoint copy of X, $x \to x'$ being a bijection. Let S be a regular semigroup. A mapping $\theta: X \cup X' \to S$ is *matched* if $x'\theta$ is an inverse of $x\theta$ in S for each $x \in X$. Let $\mathscr{U}$ be a class of regular semigroups, X and X' be as above. A semigroup $F \in \mathscr{U}$ together with a matched mapping $\iota: X \cup X' \to F$ is a *bifree object* on X in $\mathscr{U}$ if for any $S \in \mathscr{U}$ and any matched mapping $\theta: X \cup X' \to S$ there is a unique homomorphism $\varphi: F \to S$ extending θ, that is, satisfying $t\varphi = \theta$ (see [18]).

If a bifree object exists then it is unique (up to isomorphic copies and up to cardinality of X) and will be denoted by $BF_{\mathscr{U}}(X)$.

In [39], Y. T. YEH defined the notion of "e-*free object*" formally in the same fashion as the notion of bifree object. The two definitions are slightly different, however it turns out that they are equivalent.

Result 8.10 [39, Theorem 4.12]. *In any* e*-variety $\mathscr{U}$ of E-solid* [*locally inverse*] *regular simigroups, $BF_{\mathscr{U}}(X)$ exists on any nonempty set X.*

Corollary 8.11. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{E}\mathscr{S})$ or $\mathscr{L}_{ev}(\mathscr{I}L)$ be such that all bifree objects in $\mathscr{U}$ are upper $\mathscr{D}$-finite. Then*

(i) $\mathscr{C}\mathscr{S} \circ \mathscr{U} = \bigvee_{n \geq 0} \mathscr{U}^{(T_l T_r)^n}$.
(ii) $\mathscr{R}\mathscr{B} \circ \mathscr{U} = \bigvee_{n \geq 0} \mathscr{U}^{(K_l K_r)^n}$.

Proof. This follows immediately from Theorem 8.9. □

We denote by $\mathscr{B}_2$ the class of all combinational strict inverse semigroups. From [25], $\mathscr{B}_2$ is a variety of inverse semigroups and generated by the five element combinatorial Brandt semigroup. Clearly $\mathscr{B}_2 = \mathscr{I} \cap \mathscr{C}\mathscr{S}\mathscr{R}$, where $\mathscr{C}\mathscr{S}\mathscr{R}$ is the e-variety of all combinatorial strict regular semigroups.

Some particular cases, based on the e-varieties $\mathscr{B}_2$, $\mathscr{S}L$ and $\mathscr{I}L$ are worthy of note.

Corollary 8.12.

(i) $\mathscr{B}L = \mathscr{R}\mathscr{B} \circ (\mathscr{S}L) = \mathscr{R}\mathscr{B} \circ \langle C_2 \rangle_{ev} = \bigvee_{n \geq 0} \langle C_2 \rangle_{ev}^{(K_l K_r)^n}$.
(ii) $\mathscr{O}L = \mathscr{R}\mathscr{B} \circ (\mathscr{I}L) = \bigvee_{n \geq 0} (\mathscr{I}L)^{(K_l K_r)^n}$.
(iii) $\mathscr{C}\mathscr{S} \circ \mathscr{B}_2 = \bigvee_{n \geq 0} \mathscr{B}_2^{(T_l T_r)^n}$.
(iv) $(\mathscr{C}\mathscr{R})L = \mathscr{C}\mathscr{S} \circ (\mathscr{S}L) = \mathscr{C}\mathscr{S} \circ \langle C_2 \rangle_{ev} = \bigvee_{n \geq 0} \langle C_2 \rangle_{ev}^{(T_l T_r)^n}$.
(v) $(\mathscr{E}\mathscr{S})L = \mathscr{C}\mathscr{S} \circ (\mathscr{I}L) = \bigvee_{n \geq 0} (\mathscr{I}L)^{(T_l T_r)^n}$.

Proof. (i) and (iv). From [1, Section 3], we can see that all bifree objects in $\mathscr{S}L = \langle C_2 \rangle_{ev}$ are completely semisimple and upper $\mathscr{D}$-finite. The assertions are now immediate consequences of Corollaries 6.6, 6.11 and 8.11.

(ii) and (v). It follows easily from [4, Lemma 5.3 and Theorem 5.11] that all bifree objects in $\mathscr{I}L$ are completely semisimple and upper $\mathscr{D}$-finite. Hence by Corollary 6.4, Lemma 6.12 and Corollary 8.11, the equalities hold.

(iii) by [32, Corollary 3.4], all free objects in $\mathscr{B}_2$ are completely semisimple and upper $\mathscr{D}$-finite. The equality now follows from Corollary 8.11. □

More examples can be generated by using the results on bifree objects in AUINGER [2, 3].

For any variety $\mathscr{U}$ of groups, we denote by $\hat{\mathscr{U}}$ the largest variety of inverse semigroups having E-unitary coves over $\mathscr{U}$ (see [25, Definition XIII.9.3]).

Result 8.13 [25, Theorem XII.9.11]. *Let $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{G})$. Then*

$$\hat{\mathscr{U}} = \langle \mathscr{S} \circ \mathscr{U} \rangle_{ev} = [u^2 = u | u = 1 \text{ is a law in } \mathscr{U}].$$

In [19], MARGOLIS and MEAKIN described the free objects in the inverse semigroup variety $\hat{\mathscr{U}}$, $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{G})$, by using the Cayley graphs of the free groups in $\mathscr{U}$. The following result is an immediate consequence of [19, Corollaries 2.10, 3.5 and 3.6].

Result 8.14. *For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{G})$, all free objects in $\hat{\mathscr{U}}$ are upper $\mathscr{D}$-finite.*

An immediate consequence is the following.

Corollary 8.15. *For any* $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{G})$, *we have*

(i) $\mathscr{CS} \circ \hat{\mathscr{U}} = \bigvee_{n \geqslant 0} \hat{\mathscr{U}}^{(T_l T_r)^n}$.
(ii) $\mathscr{RB} \circ \hat{\mathscr{U}} = \bigvee_{n \geqslant 0} \hat{\mathscr{U}}^{(K_l K_r)^n}$.

Since $\mathscr{I} = \hat{\mathscr{G}}$, it follows that Theorem 7.10 of [34] is now a special case of Corollary 8.15. We conclude this section with the following question.

Question 8.16. For any $\mathscr{U} \in \mathscr{L}_{ev}(\mathscr{RS})$, do the following two equalities hold?

(i) $\mathscr{CS} \circ \mathscr{U} = \bigvee_{n \geqslant 0} \mathscr{U}^{(T_l T_r)^n}$.
(ii) $\mathscr{RB} \circ \mathscr{U} = \bigvee_{n \geqslant 0} \mathscr{U}^{(K_l K_r)^n}$.

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