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Positivity of Cotes Numbers at More Jacobi Abscissas

By

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Abstract. The positivity of certain finite sums of even ultraspherical polynomials has been identified by Askey as a specially interesting case of a more general problem concerning positivity of Cotes numbers at Jacobi abscissas. The authors establish several new inequalities of this type.

1. Introduction

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomials defined by the Rodrigues formula:

$$(1-x)^\alpha(1+x)^\beta P_x^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] \quad (1.1)$$

or, equivalently, by the hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \quad (1.2)$$

The shifted factorial $(a)_k$ is defined by

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}.$$

These polynomials are orthogonal on $[-1,1]$ with respect to $(1-x)^\alpha(1+x)^\beta$ when $\alpha, \beta > -1$. Expand

$$(1-x)^{-\gamma}(1+x)^{-\delta} \sim \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x) \quad (1.3)$$

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and ask when

$$\sum_{k=0}^{n} a_k P_k^{(\alpha,\beta)}(x) \geqslant 0, \quad -1 \leqslant x \leqslant 1, \quad n = 0, 1, \ldots ? \tag{1.4}$$

The values of the left side of (1.4) at the zeros of $P_n^{(\alpha,\beta)}(x)$ are (positive multiples of) the Cotes numbers for integration with respect to

$$(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx$$

(the case $\gamma = \delta = 0$ being Gaussian quadrature).

In his SIAM conference lectures, ASKEY [2], draws attention to these questions noticing that they are more than mere test problems and warrant further work as the source of new and interesting inequalities. He notes moreover that the case $\beta = -\frac{1}{2}$ is especially interesting. This is the case of even ultraspherical polynomials and we concentrate on it here.

We establish several new instances of (1.4) with strict inequality and hence of positive Cotes numbers following in the footsteps of [1]. Our results do indeed correspond to new inequalities. Some of these, concerning even Legendre polynomials (the case $\alpha = 0, \beta = -\frac{1}{2}$), were established in [5] viz.

$$\frac{1}{2} + \sum_{k=1}^{n} P_{2k}(\cos\theta) \geqslant 0, \quad 0 \leqslant \theta \leqslant \pi, \tag{1.5}$$

$$\frac{1}{2} + \sum_{k=1}^{n} P_{4k}(\cos\theta) > 0, \quad 0 \leqslant \theta \leqslant \pi. \tag{1.6}$$

In [5] we found also an analogous result for sums of odd Legendre polynomials;

$$\frac{\sqrt{15}}{45} + \sum_{k=1}^{n} P_{2k-1}(\cos\theta) \geqslant 0, \quad 0 \leqslant \theta \leqslant \pi/2. \tag{1.7}$$

Of course FEJÉR proved in [6] that

$$\sum_{k=0}^{n} P_k(\cos\theta) \geqslant 0, \quad 0 \leqslant \theta \leqslant \pi, \tag{1.8}$$

so it is easy to obtain the weak version of (1.5) in which the leading constant $\frac{1}{2}$ is replaced by 1. Similarly using (1.8) together with (5.8) of GASPER'S paper, [10], we can obtain the weak version of (1.6) in which

the leading constant $\frac{1}{2}$ is replaced by 1. It is further confirmation of the predictive value of (1.4) that the sharper results are needed here. We note, in passing, that (1.8) itself is the special case of (1.4) in which $\alpha = \beta = \delta = 0$, $\gamma = \frac{1}{2}$.

Known cases of (1.4) for $\beta = -\frac{1}{2}$ are $\alpha = \frac{1}{2}$, $\gamma = 1$, $\delta = 0$ (positivity of the Fejér kernel), $\alpha = -\frac{1}{2}$, $\gamma = \frac{1}{4}$, $\delta = -\frac{1}{4}$ (a cosine sum of VIETORIS [13], set in context and given an elegant proof in [3] then sharpened in [4]), $\alpha = \frac{1}{2} = \gamma$, $\beta = \delta$ (contained in results of FEJÉR [7] and SZEGÖ [11]) and $\alpha > \frac{1}{2}$, $\delta = 0$, $\gamma = \frac{1}{2}\alpha + \frac{3}{4}$ (which follows from work of GASPER in [9]).

Our main results are the following:

Theorem 1. *Let $\delta = 0$ and $\beta = -\frac{1}{2}$. All the partial sums in* (1.4) *are strictly positive when*

$$\alpha \geqslant 0, \quad 0 < \gamma \leqslant -\tfrac{1}{2}.$$

Theorem 2. *All the partial sums in* (1.4) *are strictly positive when* $\alpha = 0$, $\beta = -\frac{1}{2}$, $\delta = \frac{1}{4}$, $\gamma = \frac{1}{2}$.

Theorem 3. *Let $\delta = 0$. We suppose that $\alpha + \beta + 1 > 0$. The strict inequality in* (1.4) *holds for $\alpha \geqslant \beta$, $-1 < \gamma < 0$. Moreover the partial sums in* (1.4) *form a decreasing function of x for all $n \geqslant 1$, when*

(i) $\alpha \geqslant \beta + 1, \quad -1 < \gamma < 0,$
(ii) $|\alpha| < \beta + 1, \quad -1 < \gamma \leqslant \frac{1}{2}(\alpha - \beta - 1).$

An immediate consequence of (ii) is

Corollary. *Let $\delta = 0$. The strict inequality in* (1.4) *holds for $-\beta - 1 < \alpha < \beta$ and $-1 < \gamma \leqslant \frac{1}{2}(\alpha - \beta - 1)$.*

2. Preliminary Reductions

Writing

$$h_k^{(\alpha,\beta)} = \frac{(2k + \alpha + \beta + 1)\Gamma(k+1)\Gamma(k + \alpha + \beta + 1)}{2^{\alpha+\beta+1}\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}, \tag{2.1}$$

and bearing in mind that

$$\int_{-1}^{1} (P_k^{(\alpha,\beta)}(x))^2 (1-x)^\alpha (1+x)^\beta dx = (h_k^{(\alpha,\beta)})^{-1}, \tag{2.2}$$

it is natural to rewrite a_k in (1.4) as

$$a_k = h_k^{(\alpha,\beta)} \hat{f}_k, \tag{2.3}$$

where

$$\hat{f}_k = \int_{-1}^{1} (1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta} P_k^{(\alpha,\beta)}(x)dx \tag{2.4}$$

is the Fourier-Jacobi coefficient in the expansion of $f(x) = (1-x)^{-\gamma}(1+x)^{-\delta}$. In summary

$$f \sim \sum h_k^{(\alpha,\beta)} \hat{f}_k P_k^{(\alpha,\beta)}$$

and we are concerned with proving

$$\sum_{k=0}^{n} h_k^{(\alpha,\beta)} \hat{f}_k P_k^{(\alpha,\beta)}(x) > 0, \quad -1 \leqslant x \leqslant 1, \quad n = 0, 1, 2, \ldots . \tag{2.5}$$

Of course we can give an explicit formula for the right hand side of (2.4). In fact, using (1.2), we have, for $\alpha - \gamma > -1, \beta - \delta > -1$

$$\hat{f}_k = \frac{2^{\alpha-\gamma+\beta-\delta+1}\Gamma(\alpha-\gamma+1)\Gamma(\beta-\delta+1)}{\Gamma(\alpha-\gamma+\beta-\delta+2)} \cdot \frac{(\alpha+1)_k}{k!} \times$$
$$\times {}_3F_2(-k, k+\alpha+\beta+1, \alpha-\gamma+1; \alpha+1, \alpha-\gamma+\beta-\delta+2; 1). \tag{2.6}$$

Note that, in the special case $\delta = 0$, the terminating hypergeometric series is Saalschützian and hence (2.6) simplifies to give

$$\hat{f}_k = \frac{2^{\alpha-\gamma+\beta+1}\Gamma(\alpha-\gamma+1)\Gamma(\beta+1)}{\Gamma(\alpha-\gamma+\beta+2)} \frac{(\beta+1)_k(\gamma)_k}{k!(\alpha+\beta-\gamma+2)_k}. \tag{2.7}$$

In the other special case of interest to us ($\alpha = 0, \beta = -\frac{1}{2}, \gamma = \frac{1}{2}, \delta = \frac{1}{4}$) observe that the terminating hypergeometric series takes the form

$$s_k = {}_3F_2(-k, k+\tfrac{1}{2}, \tfrac{1}{2}; 1, \tfrac{3}{4}; 1). \tag{2.8}$$

Using a standard formula which is due to WATSON [14], we find that

$$s_k = \begin{cases} 0 & k = 2j+1; \\ \dfrac{(\frac{1}{2})_j(\frac{1}{4})_j}{j!(\frac{3}{4})_j}, & k = 2j; \quad j = 0, 1, \ldots . \end{cases} \tag{2.9}$$

Let us recall also from [12] the quadratic transformation

$$\frac{P_k^{(\alpha,-1/2)}(2x^2-1)}{P_k^{(\alpha,-1/2)}(1)}=\frac{C_{2k}^{\alpha+1/2}(x)}{C_{2k}^{\alpha+1/2}(1)}=\frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)}, \tag{2.10}$$

where $C_n^{\lambda}(x)$ is the Gegenbauer polynomial of degree n, order λ.

In particular, for $\alpha=0$ we have

$$C_{2k}^{1/2}(x)=P_{2k}(x)$$

the Legendre polynomial of degree $2k$.

3. Proof of Theorem 1

We define

$$B_k(\alpha,\gamma)=\left(2k+\alpha+\frac{1}{2}\right)\frac{(\alpha+\frac{1}{2})_k(\gamma)_k}{k!(\alpha-\gamma+\frac{3}{2})_k},\quad k=0,1,\ldots.$$

Taking into account (2.1), (2.7) and (2.10) inequality (2.5) reduces to

$$\sum_{k=0}^{n} B_k(\alpha,\gamma)\frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)}>0,\quad 0\leqslant x\leqslant 1. \tag{3.1}$$

It can be easily checked that

$$\frac{B_{k+1}(\alpha,\gamma)}{B_{k+1}(\alpha,\frac{1}{2})}<\frac{B_k(\alpha,\gamma)}{B_k(\alpha,\frac{1}{2})},\quad k=0,1,\ldots$$

when $\alpha\geqslant 0, 0<\gamma<\frac{1}{2}$.

A summation by parts shows that inequality (3.1) is an immediate consequence of

$$\sum_{k=0}^{n} b_k(\alpha)\frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)}>0 \text{ for all } n \quad \text{and } 0\leqslant x\leqslant 1, \tag{3.2}$$

where

$$b_k(\alpha)=B_k\left(\alpha,\frac{1}{2}\right)=\left(2k+\alpha+\frac{1}{2}\right)\frac{(\alpha+\frac{1}{2})_k(\frac{1}{2})_k}{k!(\alpha+1)_k},\quad k=0,1,\ldots.$$

In order to prove (3.2), we observe that

$$b_{k+1}(\alpha)-b_k(\alpha)=\frac{1-4\alpha^2}{8(\alpha+1)}\frac{(\alpha+\frac{1}{2})_k(\frac{1}{2})_k}{(k+1)!(\alpha+2)_k},\quad k=0,1,\ldots \tag{3.3}$$

and

$$\lim_{k\to\infty} b_k(\alpha) = \frac{2\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}. \tag{3.4}$$

Then we set

$$S_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)}, \quad n = 0, 1, \ldots.$$

It is well-known that

$$S_n^{\alpha}(x) > 0, \quad \text{for all } n, \quad 0 \leqslant x \leqslant 1, \quad \alpha \geqslant 0. \tag{3.5}$$

(This is the even part of (3) in [8].)

When $\alpha > \frac{1}{2}$ the sequence $b_k(\alpha)$, $k = 0, 1, \ldots$ is strictly decreasing hence (3.5) implies (3.2) by a summation by parts. Note that for $\alpha = \frac{1}{2}$, (3.2) is the classical inequality

$$\sum_{k=0}^{n} \frac{\sin(2k+1)\theta}{(2k+1)\sin\theta} > 0, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

Next we assume that $0 \leqslant \alpha < \frac{1}{2}$.

It follows from (1.5) and FELDHEIM's integral [8]

$$\frac{P_n^{(a,a)}(\cos\theta)}{P_n^{(a,a)}(1)} = \frac{2\Gamma(a+1)}{\Gamma(a-b)\Gamma(b+1)} \times$$

$$\times \int_0^{\pi/2} \sin^{2b+1} t \cos^{2a-2b-1} t (1 - \sin^2\theta\cos^2 t)^{n/2} \times$$

$$\times \frac{P_n^{(b,b)}(\cos\theta(1-\sin^2\theta\cos^2 t)^{-1/2})}{P_n^{(b,b)}(1)}\, dt,$$

$$a > b > -1, \quad 0 \leqslant \theta \leqslant \pi, \tag{3.6}$$

that

$$S_n^{\alpha}(x) \geqslant -\tfrac{1}{2}, \quad \text{for all } n \quad \text{and } 0 \leqslant x \leqslant 1. \tag{3.7}$$

Now summing by parts the left hand side of (3.2) and taking into consideration (3.3), (3.7) and the standard estimate

$$S_k^{\alpha}(x) \leqslant k+1, \quad \text{for all } k,$$

we obtain

$$\sum_{k=0}^{n} b_k(\alpha)\frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)} =$$

$$= \sum_{k=0}^{n-1} (b_k(\alpha) - b_{k+1}(\alpha))S_k^{\alpha}(x) + b_n(\alpha)S_n^{\alpha}(x) \geqslant$$

$$\geqslant \sum_{k=0}^{n-1} (k+1)(b_k(\alpha) - b_{k+1}(\alpha)) + \frac{b_n(\alpha)}{2} =$$

$$= \frac{b_n(\alpha)}{2} - \frac{1-4\alpha^2}{8(\alpha+1)} \sum_{k=0}^{n-1} \frac{(\frac{1}{2})_k(\alpha+\frac{1}{2})_k}{(\alpha+2)_k k!}. \tag{3.8}$$

Since the sequence $b_n(\alpha)$ is strictly increasing, it is easy to see that the sequence

$$\ell_n^{\alpha} = \frac{b_n(\alpha)}{2} - \sum_{k=0}^{n-1} (k+1)(b_{k+1}(\alpha) - b_k(\alpha))$$

is strictly decreasing.

Now taking into account (3.4) and the fact that

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k(\alpha+\frac{1}{2})_k}{(\alpha+2)_k k!} = {}_2F_1(\tfrac{1}{2}, \alpha+\tfrac{1}{2}; \alpha+2; 1) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\frac{3}{2})\Gamma(\frac{3}{2})},$$

we get

$$\ell_n^{\alpha} \geqslant \lim_{n\to\infty} \ell_n^{\alpha} = \frac{\frac{1}{2}+\alpha}{B(\frac{1}{2}, \frac{1}{2}+\alpha)}. \tag{3.9}$$

The combination of (3.8) with (3.9) gives

$$\sum_{k=0}^{n} b_k(\alpha)\frac{P_{2k}^{(\alpha,\alpha)}(x)}{P_{2k}^{(\alpha,\alpha)}(1)} > \frac{\frac{1}{2}+\alpha}{B(\frac{1}{2}, \frac{1}{2}+\alpha)}$$

which establishes (3.2). The proof of Theorem 1 is then complete.

□

4. Proof of Theorem 2

We follow an argument similar to that in the proof of Theorem 1 by making use of (1.6). More specifically, by virtue of Watson's formula (2.9), inequality (2.5) reduces to

$$\sum_{k=0}^{n} b_k P_{4k}(x) > 0, \quad 0 \leqslant x \leqslant 1, \tag{4.1}$$

where

$$b_k = (8k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k}{k!(\frac{3}{4})_k}, \quad k = 0, 1, \ldots.$$

A simple calculation shows that (b_k) is an increasing sequence, such that

$$b_{k+1} - b_k = \frac{1}{2}\frac{(\frac{1}{2})_k(\frac{1}{4})_k}{(k+1)!(\frac{7}{4})_k}, \quad k = 0, 1, \ldots. \tag{4.2}$$

On the other hand it can be easily verified that

$$\lim_{n\to\infty} b_n = \frac{8\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}. \tag{4.3}$$

Let

$$\sigma_k(x) = \sum_{j=0}^{k} P_{4j}(x), \quad k = 0, 1, \ldots.$$

It follows from (1.6) that

$$\sigma_n(x) > \tfrac{1}{2}, \quad 0 \leqslant x \leqslant 1, \quad n = 0, 1, \ldots. \tag{4.4}$$

Now summing by parts the left hand side of (4.1) using (4.2), (4.4) and the obvious inequality

$$\sigma_k(x) \leqslant k+1, \quad k = 0, 1, \ldots$$

we get

$$\sum_{k=0}^{n} b_k P_{4k}(x) = \sum_{k=0}^{n-1} (b_k - b_{k+1})\sigma_k(x) + b_n\sigma_n(x) >$$

$$> \sum_{k=0}^{n-1} (k+1)(b_k - b_{k+1}) + \frac{b_n}{2}. \tag{4.5}$$

Clearly the sequence

$$q_n = \frac{b_n}{2} - \sum_{k=0}^{n-1} (k+1)(b_{k+1} - b_k) =$$

$$= \frac{b_n}{2} - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(\frac{1}{2})_k(\frac{1}{4})_k}{(\frac{7}{4})_k k!} \tag{4.6}$$

decreases with n.

Since

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k(\frac{1}{4})_k}{(\frac{7}{4})_k k!} = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{4}; \tfrac{7}{4}; 1) = 6\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})},$$

it follows from (4.3), (4.6) and the above that

$$q_n \geqslant \lim_{n\to\infty} q_n = \frac{1}{B(\frac{1}{2}, \frac{1}{4})}. \tag{4.7}$$

Finally, combining (4.5) with (4.7) we conclude that

$$\sum_{k=0}^{n} b_k P_{4k}(x) > \frac{1}{B(\frac{1}{2}, \frac{1}{4})} = 0.1906\ldots .$$

This completes the proof of Theorem 2. □

5. Proof of Theorem 3

Let

$$T_n(x) = \sum_{k=0}^{n} (2k+\alpha+\beta+1)\frac{(\alpha+\beta+1)_k(\gamma)_k}{k!(\alpha+\beta-\gamma+2)_k}\cdot\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)}$$

for $-1 \leqslant x \leqslant 1$ and $n = 0, 1, \ldots$.

By virtue of (2.1) and (2.7), (2.5) reduces to

$$T_n(x) > 0 \quad \text{for all } n.$$

The condition $\alpha+\beta+1>0$ implies $\max(\alpha,\beta) \geqslant -\frac{1}{2}$. Then, by [12, 7.32.2.] we have

$$\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \leqslant 1, \quad k = 0, 1, \ldots .$$

From this it follows that

$$T_n(x) \geqslant T_n(1) \quad -1 \leqslant x \leqslant 1,$$

for $\alpha+\beta+1>0$ and $-1<\gamma<0$. So it suffices to prove

$$T_n(1) = \sum_{k=0}^{n} (2k+\alpha+\beta+1)\frac{(\alpha+\beta+1)_k(\gamma)_k}{k!(\alpha+\beta-\gamma+2)_k} > 0 \quad \text{for all } n \geqslant 1.$$

We have

$$T_n(1) > (\alpha+\beta+1) + \sum_{k=1}^{\infty} (2k+\alpha+\beta+1)\frac{(\alpha+\beta+1)_k(\gamma)_k}{k!(\alpha+\beta-\gamma+2)_k} =$$

$$= (\alpha+\beta+1)\Bigg[{}_2F_1(\alpha+\beta+1,\gamma;\alpha+\beta-\gamma+2;1) +$$

$$+\frac{2\gamma}{\alpha+\beta-\gamma+2}\,{}_2F_1(\alpha+\beta+2,\gamma+1;\alpha+\beta-\gamma+3;1)\Bigg] =$$

$$= (\alpha+\beta+1)\Bigg[\frac{\Gamma(\alpha+\beta-\gamma+2)\Gamma(1-2\gamma)}{\Gamma(\alpha+\beta-2\gamma+2)\Gamma(1-\gamma)} +$$

$$+\frac{2\gamma}{\alpha+\beta-\gamma+2}\,\frac{\Gamma(\alpha+\beta-\gamma+3)\Gamma(-2\gamma)}{\Gamma(\alpha+\beta-2\gamma+2)\Gamma(1-\gamma)}\Bigg] =$$

$$= 0.$$

Now using the differentiation formula

$$\frac{d}{dx}P_k^{(\alpha,\beta)}(x) = \frac{1}{2}(k+\alpha+\beta+1)P_{k-1}^{(\alpha+1,\beta+1)}(x)$$

we see that, the inequality

$$\frac{d}{dx}T_n(x) < 0, \quad -1 < x < 1, \quad \text{and } n = 1,2,\ldots$$

is equivalent to

$$\sum_{k=0}^{n} c_k \frac{P_k^{(\alpha+1,\beta+1)}(x)}{P_k^{(\beta+1,\alpha+1)}(1)} > 0 \tag{5.1}$$

for $-1 < x < 1$, where

$$c_k = (2k+\alpha+\beta+3)\frac{(\alpha+\beta+3)_k(\gamma+1)_k(\beta+2)_k}{(\alpha+2)_k(\alpha+\beta-\gamma+3)_k k!}.$$

It is known that

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha+1,\beta+1)}(x)}{P_k^{(\beta+1,\alpha+1)}(1)} > 0, \quad -1 < x \leqslant 1, \quad \text{for all } n$$

(see [10, Theorem 4]).

A summation by parts shows that (5.1) is true for all the values of α, β and γ such that

$$c_{k+1} \leqslant c_k, \quad k = 0, 1, \ldots. \tag{5.2}$$

A routine calculation proves that the above inequality is equivalent to

$$\gamma \leqslant (\alpha - \beta - 1) \frac{k^2 + (\alpha + \beta + 4)k + \alpha + \beta + 3}{2k^2 + (2\alpha + 2\beta + 8)k + (\beta + 3)(\alpha + \beta + 3)}.$$

From this we conclude that (5.2) is valid for the values of α, β and γ stated in assertions (i) and (ii). The proof of Theorem 3 is complete. □

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