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On Averaging Multisets

By

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Abstract. We give sufficient conditions to ensure that, given a set $M \subseteq \mathbb{R}^s$, every $x \in \operatorname{int} \operatorname{conv} M$ can be represented as a convex combination, $x = \sum_{i=1}^{n} \lambda_i x_i$, where $x_i \in M$, λ_i rational, and $n = 2s$ or $n = 2s - 1$, respectively.

0. Introduction

Let x be a point in the convex hull of some set M in s-dimensional Euclidean space $\mathbb{R}^s$. By Carathéodory's well-known theorem, x can be written as a *convex combination*,

$$x = \sum_{i=1}^{n} \lambda_i x_i, \tag{1}$$

where $x_i \in M$, the nonnegative weights λ_i satisfy $\sum_{i=1}^{n} \lambda_i = 1$, and $n \leqslant s + 1$.

Difficulties arise if we want all the weights to be rational numbers. The corresponding set $\{x_1, \ldots, x_n\}$ will then be called an *averaging multiset* for x. Granting the existence of averaging multisets, it is natural to ask how "small" they may be chosen. To be more specific, the following two questions come into mind:

(a) What is the minimal Q such that (1) can be satisfied with Q as the least common denominator of the rationals λ_i?

This problem can be considered as a special case of the more general notion of averaging sets, introduced by SEYMOUR and ZASLAVSKY [3]. It turns out that there exists no upper bound on Q depending only on the dimension s. WAGNER [4] and the author [2] obtained quanti-

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tative results which involve essentially some geometric properties of the set M. See also [5] where an application is given.

(b) What is the minimum number n of different points in (1)?

This will be discussed in the present paper. Interestingly, the geometric shape of M plays only a minor role here. Instead we can prove a result "highly in Carathéodory's spirit": for a wide class of sets, (1) holds with $n = 2s$ or even $n = 2s - 1$ points and rational weights.

1. Definitions

1.1. Let conv M, aff M, lin M, int M, rel int M, bd M denote the convex hull, affine hull, linear hull, interior, relative interior (as a subset of aff M) and boundary of a set $M \subseteq \mathbb{R}^s$. Denote the dimension of a (linear or affine) space V by dim V and the cardinality of a set I by $|I|$.

1.2. We use the notation $A + B = \{a + b | a \in A, b \in B\}$.

1.3. Let

$$\operatorname{rconv}_n M := \left\{ \sum_{i=1}^{n} \lambda_i x_i \,|\, x_i \in M, \lambda_i \in \mathbb{Q}, \lambda_i \geqslant 0, \sum_{i=1}^{n} \lambda_i = 1 \right\},$$

i.e., the set of all points which possess an averaging multiset consisting of at most n points of M.

1.4. For any $x \in \operatorname{int} \operatorname{conv} M$ a tuple $(\mathfrak{A}; \mathfrak{X}) = (A_1, \ldots, A_l; X_1, \ldots, X_l)$ satisfying (1) through (4) below is called an *M-configuration.*

(1) $A_1, \ldots, A_l$ are affine subspaces of $\mathbb{R}^s$ whose intersection is the singleton $\{x\}$.

(2) $\mathbb{R}^s = \sum_{i=1}^{l} A_i$ and $s = \sum_{i=1}^{l} \dim A_i$.

(3) $X_i \subseteq M$ with $|X_i| = \dim A_i + 1$ for each $i = 1, \ldots, l$.

(4) For $i = 2, \ldots, l$, let π_i denote the affine projection of $\mathbb{R}^s$ onto $A_i + \cdots + A_l$ parallel to $A_1 + \cdots + A_{i-1}$; and let π_1 denote the identity on $\mathbb{R}^s$. Then for $i = 1, \ldots, l$ we have

$$A_i = \operatorname{aff} \pi_i(X_i) \quad \text{and} \quad x \in \operatorname{rel\,int} \operatorname{conv} \pi_i(X_i).$$

The tuple $(\dim A_1, \ldots, \dim A_l)$ is called the *type* of $(\mathfrak{A}; \mathfrak{X})$.

An M-configuration is said to be *maximal* if its type is (lexicographically) maximal among all M-configurations representing the same point x.

Examples.

- Type (s): X_1 is the set of vertices of a nondegenerate simplex.
- Type $(1,\dots,1)$: $X_1 \cup \cdots \cup X_l$ is the set of vertices of a cross-polytope.

The existence of M-configurations will be shown in Lemma 3. Note that, even with given A_i, the sets X_i need not be unique.

1.5. By an *M-path* we mean a continuous mapping $\varphi: I \to M$ where $I = [0,1]$. The vector $\Delta\varphi := \varphi(1) - \varphi(0)$ is called the *principal chord* of φ.

Paths are said to be *independent* if their principal chords are linearly independent. We write prk M for the *path rank* of M, which is defined as the maximum number of independent M-paths.

2. Lemmas

2.1. Given $x_0,\dots,x_s$ as the vertices of a nondegenerate simplex in $\mathbb{R}^s$, each point x has a unique representation in barycentric coordinates,

$$x = \sum_{i=0}^{s} \rho_i x_i, \quad \sum \rho_i = 1. \tag{2}$$

Given arbitrary $x_0,\dots,x_n$, $n > s$, such that aff $\{x_0,\dots,x_n\} = \mathbb{R}^s$, the solution to (2) is not unique. We will describe a procedure which will serve as a replacement for barycentric coordinates in the special situation that the x_i are the "vertices" of a configuration.

2.2. First let us collect some elementary facts.

Lemma 1. *Let* $x_0,\dots,x_s \in \mathbb{R}^s$ *and* $x = \sum_{i=0}^{s} \lambda_i x_i \in \operatorname{int}\operatorname{conv}\{x_0,\dots,x_s\}$ *be given with* $\lambda_i > 0$, $\sum \lambda_i = 1$. *For any* $y = \sum_{i=0}^{s} \mu_i x_i$, $\sum \mu_i = 1$, *put* $\mu^* := \max_{i=0,\dots,s} (\mu_i/\lambda_i)$. *Then:*

(a) $$x = \frac{1}{\mu^*}\, y + \sum_{i=0}^{s} \left(\lambda_i - \frac{\mu_i}{\mu^*}\right) x_i$$

is a convex combination where y and at most s of the x_i are given positive weights.

(b) $X_0 = \{x_i \mid \lambda_i - \mu_i/\mu^* > 0\}$ *is the (unique) proper subset of* $\{x_0,\dots,x_s\}$ *such that* $x \in \operatorname{rel}\operatorname{int}\operatorname{conv}(X_0 \cup \{y\})$; *and* $\operatorname{conv}(X_0 \cup \{y\})$ *is a nondegenerate simplex of some dimension* $\leqslant s$.

(c) *The mapping* $y \mapsto \left(\frac{1}{\mu^*}, \lambda_0 - \frac{\mu_0}{\mu^*}, \ldots, \lambda_s - \frac{\mu_s}{\mu^*}\right)$ *is continuous with respect to the Euclidean topology.*

The proof is left to the reader.

We will use the following version of Carathéodory's theorem.

Lemma 2. *Let* $\dim \operatorname{aff} M \geqslant 1$ *and* $x \in \operatorname{rel\,int} \operatorname{conv} M$. *Then there exists* $X \subseteq M$ *such that* $x \in \operatorname{rel\,int} \operatorname{conv} X$ *and* $1 \leqslant \dim \operatorname{aff} X = |X| - 1$.

2.3. It is not difficult to show the existence of configurations.

Lemma 3. *Let* $M \subseteq \mathbb{R}^s$. *Then every* $x \in \operatorname{int} \operatorname{conv} M$ *has an* M*-configuration.*

Proof. Step 1. Put $i = 1$ and $\pi_1 = \text{identity on } \mathbb{R}^s$.

Step 2. Since $x = \pi_i(x) \in \operatorname{rel\,int} \operatorname{conv} \pi_i(M)$ and $\dim \operatorname{aff} \pi_i(M) = \dim \pi_i(\mathbb{R}^s) \geqslant 1$, by Lemma 2 there exists a set $\tilde{X}_i \subseteq \pi_i(M)$ such that $x \in \operatorname{rel\,int} \operatorname{conv} \tilde{X}_i$ and $1 \leqslant n_i := \dim \operatorname{aff} \tilde{X}_i = |\tilde{X}_i| - 1$. Put $A_i := \operatorname{aff} \tilde{X}_i$ and pick a pre-image $X_i \subseteq M$ such that $\pi_i(X_i) = \tilde{X}_i$ and $|X_i| = |\tilde{X}_i|$.

Step 3. If $n_1 + \cdots + n_i = s$, then an M-configuration $(A_1, \ldots, A_i; X_1, \ldots, X_i)$ has been constructed. Otherwise let $A_i' \ni x$ be an affine subspace complementary to $A_1 + \cdots + A_i$. Let π_{i+1} denote the affine projection onto A_i' parallel to $A_1 + \cdots + A_i$, replace i by $i + 1$ and go to Step 2. □

2.4. Now we introduce the above-mentioned substitute for barycentric coordinates.

Lemma 4. *Let* $M \subseteq \mathbb{R}^s$ *and an* M*-configuration* $(\mathfrak{A}; \mathfrak{X})$ *of type* $(n_1, \ldots, n_l)$ *for some point* x *be given. Then:*

(a) *For any* $y \in \mathbb{R}^s$ *there is a convex combination,*

$$x = \sigma y + \sum_{z \in X_1 \cup \cdots \cup X_l} \rho(z) z, \quad \sigma > 0, \quad \rho(z) \geqslant 0, \quad \sigma + \sum \rho(z) = 1,$$

where at most n_i *points* $z \in X_i$ *are given a nonzero weight* $\rho(z)$.

(b) $\operatorname{conv}(\{z \in X_1 \cup \cdots \cup X_l \mid \rho(z) > 0\} \cup \{y\})$
is a nondegenerate simplex of some dimension $\leqslant s$.

(c) *The weights* σ *and* $\rho(z)$ *depend continuously on* y.

Proof. Step 1. Put $k = l$ and $y_l = y$.

Step 2. With the affine projection π_k as in 1.4, $x = \pi_k(x) \in \operatorname{rel\,int} \operatorname{conv} \pi_k(X_k)$ and $\pi_k(y_k) \in A_k$ hold. By Lemma 1 (a) there exists a convex

combination,

$$x = \sigma_k \pi_k(y_k) + \sum_{z \in X_k} \rho_k(z) \pi_k(z), \tag{3}$$

where $\sigma_k > 0$ and $\rho_k(z) > 0$ for at most n_k values of z.

Step 3. If $k = 1$ then go to Step 4. Otherwise put

$$y_{k-1} = \sigma_k y_k + \sum_{z \in X_k} \rho_k(z) z. \tag{4}$$

Relation (3) implies $\pi_k(y_{k-1}) = x$, hence $y_{k-1} \in A_1 + \cdots + A_{k-1}$, which establishes $\pi_{k-1}(y_{k-1}) \in A_{k-1}$. Replace k by $k-1$ and go to Step 2.

Step 4. Iteratively substituting equation (4) one gets the desired representation,

$$\begin{aligned} x &= \sigma_1 \pi_1(y_1) + \sum_{z \in X_1} \rho_1(z) \pi_1(z) = \\ &= \sigma_1 y_1 + \sum_{z \in X_1} \rho_1(z) z = \\ &= \sigma_1 \sigma_2 y_2 + \sum_{z \in X_1} \rho_1(z) z + \sum_{z \in X_2} \sigma_1 \rho_2(z) z = \\ &\cdots \\ &= \sigma_1 \cdots \sigma_l y_l + \sum_{z \in X_1} \rho_1(z) z + \cdots + \sum_{z \in X_l} \sigma_1 \cdots \sigma_{l-1} \rho_l(z) z = \\ &=: \sigma y + \sum_{z \in X_1 \cup \cdots \cup X_l} \rho(z) z. \end{aligned}$$

Assumptions (b) and (c) follow immediately from the construction and Lemma 1. □

2.5. The following result will be useful.

Lemma 5. *Let $\varphi_1, \ldots, \varphi_s$ be independent $\mathbb{R}^s$-paths. Then the set $\sum_{i=1}^{s} \varphi_i(I)$ has nonempty interior.*

Proof. Cf. [3], Lemma 5.2.

3. Results

3.1 Let us start with a trivial observation: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, Lemma 2 implies that $\operatorname{rconv}_{\dim \operatorname{aff} M + 1} M$ is dense in $\operatorname{conv} M$.

We will use the following paraphrase of this fact.

Lemma 6. *Let $M \subseteq \mathbb{R}^s$, $x \in \operatorname{int} M$, $y_1, \ldots, y_k \in \mathbb{R}^s$ such that*

$$0 = \sigma x + \rho_1 y_1 + \cdots + \rho_k y_k,$$

where $\sigma, \rho_i > 0$. *Then, for any* $\varepsilon > 0$, *there exist* $x' \in \operatorname{int} M$ *and positive rationals* $\sigma', \rho'_1, \ldots, \rho'_k$ *such that*

$$|\sigma - \sigma'| < \varepsilon, \quad |\rho_i - \rho'_i| < \varepsilon \quad \text{for } i = 1, \ldots, k,$$

and

$$0 = \sigma' x' + \rho'_1 y_1 + \cdots + \rho'_k y_k.$$

3.2. In view of the *Example*,

$$M \subset \mathbb{R}^s = \mathbb{R} \oplus \mathbb{R}^{s-1},$$
$$M = M_0 \cup M_1, M_0 \subseteq \{0\} \times \mathbb{R}^{s-1}, M_1 \subseteq \{1\} \times \mathbb{R}^{s-1},$$

it is clear that in general $\operatorname{rconv}_n M$ need not have interior points at all. However, a further assumption on the path rank of M will allow us to apply Lemma 5.

Theorem 1. *Let* $M \subseteq \mathbb{R}^s$ *with* $\operatorname{prk} M = s$. *Then* $\operatorname{int} \operatorname{conv} M \subseteq$ $\subseteq \operatorname{rconv}_{2s} M$.

Proof. Let $x \in \operatorname{int} \operatorname{conv} M$. By a translation we may assume $x = 0$. Then we have to prove a relation

$$0 = \sum_{i=1}^{n} \rho_i z_i, \quad z_i \in M, \quad \rho_i \in \mathbb{Q}, \quad \rho_i > 0,$$

where $n \leqslant 2s$.

By assumption there are independent M-paths $\varphi_1, \ldots, \varphi_s$. Lemma 5 establishes the existence of some $q \in \operatorname{int} \sum_{i=1}^{s} \varphi_i(I)$. If $q = 0$ then we are done. Otherwise choose $\sigma > 0$ such that $-\sigma q \in \operatorname{conv} M$. By Lemma 2 there is a set $N \subseteq M$ such that $-\sigma q \in \operatorname{rel\,int} \operatorname{conv} N$ and $\dim \operatorname{aff} N = |N| - 1$. Here we may assume $|N| \leqslant s$: If $|N| = s + 1$ (i.e., $S = \operatorname{conv} N$ is a nondegenerate simplex), we choose σ as to satisfy $\{-\sigma q\} = \{-tq \mid t > 0\} \cap \operatorname{bd} S$ and replace N by the set of vertices of the corresponding facet of S. Now the assertion follows immediately from Lemma 6. □

The bound $n = 2s$ is sharp:

Theorem 1'. *There exists* $M \subset \mathbb{R}^s$ *such that* $\operatorname{prk} M = s$ *but* $\operatorname{int} \operatorname{conv} M \nsubseteq \operatorname{rconv}_{2s-1} M$.

Proof. Put $M = \{x_1, \ldots, x_s\} \cup \bigcup_{i=1}^{s} Y_i \subset \mathbb{R}^s$, where the points x_i and the sets Y_i, $i = 1, \ldots, s$, are defined as follows.

$$x_i = (0, \ldots, 0, 1_i, 0, \ldots, 0), \quad Y_i = \{-x_i - \varepsilon a_i | 0 < \varepsilon < 1\},$$

where $A = (a_1, \ldots, a_s)$ is some positive regular matrix with the property that the elements of each row of A^{-1} are linearly independent over the field $\mathbb{Q}$.

Clearly $0 \in \operatorname{int} \operatorname{conv} M$. We show $0 \notin \operatorname{rconv}_{2s-1} M$. Since each Y_i is a convex set, any representation of 0 as a convex combination of M with rational weights may be written as

$$0 = \sum_{i=1}^{s} \lambda_i x_i + \sum_{i=1}^{s} \mu_i y_i, \quad y_i = -x_i - \varepsilon_i a_i \in Y_i, \varepsilon_i > 0, \tag{5}$$

where obviously all the λ_i must be positive. Equivalently,

$$\begin{pmatrix} \varepsilon_1 \mu_1 \\ \vdots \\ \varepsilon_s \mu_s \end{pmatrix} = A^{-1} \begin{pmatrix} \lambda_1 - \mu_1 \\ \vdots \\ \lambda_s - \mu_s \end{pmatrix}.$$

Now suppose we had $\mu_i = 0$ for *some* i. Then the ith row of the above equation would impose a linear homogeneous condition on the rationals $\lambda_j - \mu_j$, $j = 1, \ldots, s$, which (by our assumption on A) would enforce $\lambda_j - \mu_j = 0$ for all j, hence $\mu_i = 0$ *for all* i, which is impossible. Thus all the μ_i must be positive too, which means that we need $2s$ nonzero weights to satisfy (5). □

3.3. The following lemma on maximal configurations will serve us as a tool to reduce the bound from $n = 2s$ to $n = 2s - 1$ for sets with some particular property.

Lemma 7. *Let* $(\mathfrak{A}; \mathfrak{X})$ *be a maximal* M*-configuration for some point* x*. If* φ *is an* M*-path such that* $\varphi(0) \in X_1$*, then* $\varphi[0, \varepsilon] \subseteq A_1$ *for some* $\varepsilon > 0$.

Proof. Let $(n_1, \ldots, n_l)$ denote the type of $(\mathfrak{A}; \mathfrak{X})$. Putting $z_0 := \varphi(0)$, we may apply Lemma 4 to $y = \varphi(t)$, $t \in [0, 1]$ arbitrary, to get a representation

$$x = \sigma(t)\varphi(t) + \rho(z_0, t) z_0 + \sum_{z \in X_1 \setminus \{z_0\}} \rho(z, t) z + \sum_{z \in X_2 \cup \cdots \cup X_l} \rho(z, t) z, \tag{6}$$

where at most $s + 1$ weights $\sigma(t)$, $\rho(z, t)$ are positive.

For $t=0$, these nonvanishing weights are exactly $\sigma(0)$ and all $\rho(z,0)$, $z\in X_1\setminus\{z_0\}$. By Lemma 4(c) there exists $\varepsilon>0$ such that these coefficients remain positive for all $t\in[0,\varepsilon]$. Suppose we had some other positive coefficient $\rho(z,t)$. Then, by Lemma 4(b), x would be contained in the relative interior of some simplex with vertices in M and dimension $n_1'\geqslant|X_1|=n_1+1$. We could use this simplex as the starting point to construct a configuration of type $(n_1',\ldots)$ as described in the proof of Lemma 3 which would contradict maximality of $(\mathfrak{A};\mathfrak{X})$. Hence, for each $t\in[0,\varepsilon]$, (6) becomes

$$x\in\operatorname{rel\,int}\operatorname{conv}((X_1\setminus\{z_0\})\cup\{\varphi(t)\}),$$

which implies the assertion

$$\varphi(t)\in\operatorname{aff}(X_1\cup\{x\})=X_1.$$

□

Theorem 2. *Let $s>1$ and $M\subseteq\mathbb{R}^s$ with* prk $M=s$. *Assume that for each $m\in M$ there exists a nonconstant M-path φ such that $\varphi(0)=m$. Then* int conv $M\subseteq\operatorname{rconv}_{2s-1}M$.

Proof. As in the proof of Theorem 1 we may assume $x=0\in\operatorname{int}\operatorname{conv}M$. Let $(\mathfrak{A};\mathfrak{X})$ be a maximal M-configuration for x and $X_1=\{z_0,\ldots,z_{n_1}\}$. By assumption there are nonconstant M-paths φ_i such that $\varphi_i(0)=z_i$, $i=0,\ldots,n_1$. By Lemma 7, we may shrink these paths if necessary to assume $\varphi_i(I)\subseteq A_1$ and

$$0\in\operatorname{rel\,int}\operatorname{conv}(\varphi_0(t_0),\ldots,\varphi_{n_1}(t_{n_1}))\quad\text{for all } t_0,\ldots,t_{n_1}\in I. \tag{7}$$

First let us deal with the special case $n_1=1$. Then $A_1=\operatorname{aff}(z_0,z_1)$ is a line, $\varphi_0(I)$, $\varphi_1(I)$ are nondegenerate line segments, and $x\in\operatorname{conv}(z_0,z_1)$ is contained in rconv_2M. Assume $n_1>1$ in the sequel.

We cannot apply Lemma 5 directly since there need not be $n_1=\dim A_1$ independent paths among the φ_i. Putting

$$d(\xi):=\max\{\dim\operatorname{lin}\{\varphi_i(t_i')-\varphi_i(t_i)\,|\,i=0,\ldots,n_1\}:|t_i'-t_i|<\\ <\xi\text{ for }i=0,\ldots,n_1\}$$

for any $\xi>0$ defines an integer-valued monotonously increasing function. Since the φ_i are nonconstant, there is $d_0\geqslant1$ such that $d(\xi)=d_0=$ const. for ξ sufficiently small. Let ξ descend to 0 through a sequence, choose t_i,t_i' such that the maximum $d(\xi)$ is attained, and pass to a subsequence to ensure the limits $t_i\to t_i^*$ exist. By (7) the

configuration property remains intact if we replace each z_i by $z_i^* := \varphi_i(t_i^*)$ (and drop the asterisks again).

Now two cases must be distinguished:

Case A. $d_0 = 1$. Thus there exist $\varepsilon > 0$ and a unit vector v such that

$$z + (-\varepsilon, \varepsilon)v \subseteq M \quad \text{for } z \in X_1. \tag{8}$$

Because $\operatorname{prk} M = s$, we can find M-paths $\psi_1, \ldots, \psi_{s-1}$ such that $v, \Delta\psi_1, \ldots, \Delta\psi_{s-1}$ are linearly independent vectors. Let π denote the orthogonal projection onto the hyperplane which contains the origin and is orthogonal to v. By Lemma 5 the set $\sum_{i=1}^{s-1} \pi(\psi_i(I))$ has (relative) interior points, say $\pi(w)$, where $w \in \sum \psi_i(I)$. An application of Lemma 4 yields

$$0 = \sigma w + \sum_{z \in X_1 \cup \cdots \cup X_l} \rho(z) z, \tag{9}$$

where at most s of the weights $\rho(z)$ are positive. By varying $\pi(w)$ if necessary we may assume $\sum_{z \in X_1} \rho(z) > 0$ (here we need $n_1 > 1$). Now shrink the ψ_i, keeping linear independence of their principal chords, to obtain

$$\frac{\sum_{z \in X_1} \rho(z)}{(s+1)\sigma} > \delta \tag{10}$$

for all $w \in \sum \psi_i(I)$ and some $\delta > 0$; then shrink them further until the diameter of $\sum \psi_i(I)$ is less than $\delta\varepsilon$. Hence, we may choose w in the interior of

$$\sum_{i=1}^{s-1} \psi_i(I) + (-\delta\varepsilon, \delta\varepsilon)v.$$

Now use Lemma 6 to pass from (9) to rational weights,

$$0 = \sigma' w' + \sum_{z \in X_1 \cup \cdots \cup X_l} \rho'(z) z, \tag{11}$$

where $\sigma', \rho'(z) \in \mathbb{Q}^+$, $w' = \sum_{i=1}^{s-1} \psi_i(t_i) + \eta v$ for some $t_1, \ldots, t_{s-1} \in I$, and $|\eta| < \delta\varepsilon$. Note that at most s weights $\rho'(z) > 0$ are positive, and that we may take $\sum_{z \in X_1} \rho'(z) > \delta\sigma' s$ using (10). Pick $z^* \in X_1$ satisfying $\rho'(z^*) = \max_{z \in X_1} \rho'(z) > \delta\sigma'$ and use

$$\sigma'\eta v + \rho'(z^*) z^* = \rho'(z^*)\left(z^* + \frac{\sigma'\eta}{\rho'(z^*)} v \right),$$

where $|\sigma'\eta/\rho'(z^*)| < |\eta|/\delta < \varepsilon$ and (8) ensure that

$$z^{**} := z^* + \frac{\sigma'\eta}{\rho'(z^*)} v \in M,$$

in order to rewrite (11) as

$$0 = \sum_{i=1}^{s-1} \sigma'\psi_i(t_i) + \rho'(z^*)z^{**} + \sum_{\substack{z \in X_1 \cup \cdots \cup X_l \\ z \neq z^*}} \rho'(z)z,$$

from which $0 \in \operatorname{rconv}_{2s-1} M$ follows immediately.

Case B. $d_0 \geqslant 2$. Renumber the points of X_1 so that there exist independent M-paths φ_0, φ_1 (of arbitrarily small diameter) satisfying $\varphi_i(I) \ni z_i$, $i = 0, 1$, and use $\operatorname{prk} M = s$ to get a system $\varphi_0, \varphi_1, \psi_1, \ldots, \psi_{s-2}$ of independent M-paths. We intend to apply Lemma 5 to some set

$$W = W(\lambda, \lambda_1, \ldots, \lambda_{s-2}) := \lambda\varphi_0(I) + (1-\lambda)\varphi_1(I) + \sum_{i=1}^{s-2} \lambda_i \psi_i(I)$$

where $\lambda, \lambda_i \in \mathbb{Q}$, $0 < \lambda < 1$, $0 < \lambda_i \ll 1$. Unfortunately we cannot control the size and position of $\operatorname{int} W$ as a function of λ and λ_i. Thus our next step consists in finding "good" values for these parameters.

Put $w_0 = \lambda z_0 + (1-\lambda)z_1$ with arbitrary $\lambda \in (0, 1)$. By Lemma 4,

$$0 = \sigma w_0 + \sum_{z \in X_1} \rho(z, w_0)z + 0, \tag{12}$$

where $\rho(z_0, w_0) = 0$ or $\rho(z_1, w_0) = 0$, depending on which side of the affine space (hyperplane in A_1) $K = \operatorname{aff}\{0, z_2, \ldots, z_{n_1}\}$ the point w_0 is located. Note that $\rho(z_0, w_0) = \rho(z_1, w_0) = 0$ holds if and only if $w_0 \in K$, which corresponds to some (unique!) value of λ. Now let the other A_1-paths φ_i, $\varphi_i(0) = z_i$, come into play. For each $i = 2, \ldots, n_1$ and $t \in I$ the set

$$K_i(t) = \operatorname{aff}\{0, z_2, \ldots, \varphi_i(t), \ldots, z_{n_1}\}$$

is a hyperplane in A_1, starting from $K_i(0) = K$. From this definition two subcases arise:

Case B1. $K_i(t)$ is nonconstant for some i, likewise the unique λ satisfying $\lambda z_0 + (1-\lambda)z_1 \in K_i(t)$. Choose $t' < t''$ and (without loss of generality) $\lambda' > \lambda''$ so that

$$\lambda' z_0 + (1-\lambda')z_1 \in K_i(t'), \quad \lambda'' z_0 + (1-\lambda'')z_1 \in K_i(t'').$$

Now fix $\lambda \in (\lambda'', \lambda') \cap \mathbb{Q}$ and $w_0 = \lambda z_0 + (1-\lambda) z_1$. If we denote explicitly the dependence of (12) on t by virtue of $X_1 = X_1(t) = \{z_0, z_1, \ldots, \varphi_i(t), \ldots, z_{n_1}\}$, this amounts to

$$\rho_{t'}(z_0, w_0) > 0, \quad \rho_{t'}(z_1, w_0) = 0,$$
$$\rho_{t''}(z_0, w_0) = 0, \quad \rho_{t''}(z_1, w_0) > 0.$$

For any $w \in W$ as above, Lemma 4 yields

$$0 = \sigma_t w + \sum_{z \in X_1(t)} \rho_t(z, w) z + \sum_{z \in X_2 \cup \cdots \cup X_l} \rho_t(z, w) z. \tag{13}$$

Note that, by Lemma 4(c), we may fix the numbers $\lambda_1, \ldots, \lambda_{s-2} \in \mathbb{Q}^+$ so small (and shrink φ_0, φ_1 if necessary) that

$$\rho_{t'}(z_0, w) > 0, \quad \rho_{t''}(z_1, w) > 0$$

holds for all $w \in W$. Now fix $w \in \operatorname{int} W$ which is nonempty by Lemma 5. Applying the intermediate value theorem to the continuous function $t \mapsto \rho_t(z_0, w) - \rho_t(z_1, w)$, we obtain t^* such that

$$\rho_{t^*}(z_0, w) = \rho_{t^*}(z_1, w) = 0. \tag{14}$$

Fix $t = t^*$, hereby replacing z_i by $\varphi_i(t^*)$. By (14) there are at most $n_1 - 1 + \sum_{i=2}^{l} n_i = s - 1$ nonzero weights $\rho_t(z, w)$ in (13). Again we pass to rational weights by Lemma 6. Substituting the expansion

$$w = \lambda \varphi_0(t_0) + (1-\lambda)\varphi_1(t_1) + \sum_{i=1}^{s-2} \psi_i(t_{i+1}),$$

for some $t_i \in I$, into (13) shows that $0 \in \operatorname{rconv}_{2s-1} M$.

Case B2. $K_i(t) = K$ holds for all i and t. Equivalently,

$$\varphi_i(I) \subseteq \operatorname{aff}\{0, z_2, \ldots, z_{n_1}\}. \tag{15}$$

Pick $j \in \{2, \ldots, n_1\}$. Since $d_0 \geqslant 2$ we may repeat the whole argumentation of Case B but with z_j playing the role of either z_0 or z_1. If the proof cannot be finished with Case B1 then, we get another affine condition similar to (15). If Case B1 never occurs, $n_1 - 1$ (independent) such conditions accumulate whose intersection yields, for each $i = 0, \ldots, n_1$,

$$\varphi_i(I) \subseteq \operatorname{aff}\{0, z_i\}.$$

Knowing that $\varphi_i(I)$ is a line segment, we can rewrite the equation

$$0 = \sum_{i=0}^{n_1} \nu_i z_i$$

with barycentric coordinates v_i, using $v_i z_i = v_i' \varphi_i(t_i)$ for some $v_i' \in \mathbb{Q}^+$ and $t_i \in I$, to obtain $0 \in \operatorname{rconv}_{n_1+1} M$ which completes the proof. □

This bound is sharp as well:

Theorem 2'. *There exists $M \subset \mathbb{R}^s$ satisfying the conditions of Theorem 2 such that* $\operatorname{int conv} M \not\subseteq \operatorname{rconv}_{2s-2} M$.

Proof. Cf. Example 8.1 in [3].

4. Remarks and Open Problems

4.1. The above reasoning does not apply to $x \in \operatorname{bd conv} M$. Of course, we may replace M by $M \cap H$ for some hyperplane $H \ni x$ supporting M (and iterate this process if necessary).

4.2. Under the assumptions of Theorem 2 every $x \in \operatorname{int conv} M$ admits a representation (1) with *exactly* $2s-1$ nonzero rational weights. This is obvious from the above proof: In Cases A and B1, apply Lemma 6 with the side-effect of adding some extra points; in Case B2 replace $v_0 z_0$ by $\sum_{j=1}^{2s-1-n_1} v_j'' \varphi_0(t_j')$, with suitably chosen $v_j'' \in \mathbb{Q}^+$ and $t_j' \in I$.

4.3. There is a proof of $\operatorname{int conv} M \subseteq \operatorname{rconv}_{2s+1} M$ for path-connected M in [3] which relies on the presumption that every $x \in \operatorname{int conv} M$ would possess an M-configuration of type (s). However, this is not true in general. Some related problems were discussed in [1].

4.4. Let M with $\operatorname{prk} M = s$ be given such that there is an M-configuration of type (s) for some x. Need there exist such a configuration with (using the same notation as in the proof of Theorem 2) $d_0 \geqslant 2$? This conjecture is false: there is a counterexample with $s = 4$.

4.5. Let M with $\operatorname{prk} M = s$ be given such that there is no M-configuration of type (s) for some $x \in \operatorname{rel int conv} M$. Can we always reduce the dimension of the problem, i.e., does there exist a proper affine subspace A of $\mathbb{R}^s$ such that

$$x \in \operatorname{rel int conv}(A \cap M) \quad \text{and} \quad \operatorname{prk}(A \cap M) = \dim A?$$

This is true for $s = 3$.

4.6. Let M be as in Theorem 2. Assume further that each φ may be chosen as an *analytic* path such that $\varphi(I)$ is not contained in any

hyperplane. Then Lemma 7 ensures that any maximal configuration has type (s). Using the implicit function theorem one may deduce $\operatorname{int}\operatorname{conv} M \subseteq \operatorname{rconv}_{s+1} M$, cf. [3].

4.7. It has been conjectured that even $\operatorname{int}\operatorname{conv} M \subseteq \operatorname{rconv}_s M$ holds if M itself is an analytic curve. Here is a simple counterexample: Let $s = 2$, $\alpha \in \mathbb{R}$ with $e^{\alpha\pi}$ transcendental, and

$$M = \{e^{(\alpha+i)t} \mid t \in \mathbb{R}\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

Then $0 \in \operatorname{int}\operatorname{conv} M$ but $0 \notin \operatorname{rconv}_2 M$.

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