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The Martin Boundary for Harmonic Functions on Groups of Automorphisms of a Homogeneous Tree

By

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Abstract. Consider a closed subgroup Γ of the automorphism group of a homogeneous tree T, and assume that Γ acts transitively on the vertex set. Suppose that μ is a probability measure on Γ which has continuous density with respect to Haar measure and whose support is compact open and generates Γ as a closed semigroup. It is shown that the Martin boundary of Γ with respect to the random walk with law μ coincides with the space of ends of T. This extends known results for free groups and applies, for example, to the affine group over a non archimedean local field.

1. Introduction

For an irreducible, transient Markov chain on a denumerable set, the Martin boundary is the ideal boundary in an abstract compactification of the state space. Its construction goes back to DOOB [9] and HUNT [14], whose approach originates in the work of MARTIN [15] regarding positive harmonic functions on Euclidean domains. The significance of the Martin boundary of a Markov chain relies on the fact that (a) the Markov chain converges almost surely to a random variable taking values in the boundary, and (b) every positive harmonic function with respect to the transition operator has an integral representation with respect to the Martin kernel over the boundary.

Given concrete classes of transition operators which are adapted to a geometric or algebraic structure of the state space ("random walks"), one would like to realize the Martin compactification, that is, to describe its topology in terms of the underlying structure. Several

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methods are available for carrying out this program. For random walks on the abelian groups $\mathbb{Z}^d$, NEY and SPITZER [16] use refined asymptotic estimates of transition probabilities. For "nearest neighbour" random walks on trees and free groups, the situation is particularly simple, as the Green kernel has a strong multiplicativity property, see CARTIER [4] or CARTWRIGHT, SOARDI and WOESS [6]. Extending this to random walks with bounded range on trees is considerably more difficult: DERRIENNIC [8] and PICARDELLO and WOESS [17] use contractivity properties of nonnegative matrices (Birkhoff's ergodic coefficient). For graphs which are hyperbolic in the sense of Gromov, ANCONA [1] has a more potential theoretical method, using balayage and "phi-chains". For further references concerning boundaries of random walks on denumerable state spaces, see the survey by WOESS [22].

For Markov chains on non-discrete state spaces, already the abstract construction of the Martin boundary becomes more complicated, see REVUZ [18] and DERRIENNIC [7]. This includes random walks on locally compact groups, see AZENCOTT and CARTIER [2].

In the present paper, we consider subgroups Γ of the automorphism group $\mathrm{AUT}(T)$ of a homogeneous tree T which act transitively on T. These groups are locally compact, compactly generated and totally disconnected. They comprise free groups, but also nondiscrete groups such as $\mathrm{PGL}(2, \mathfrak{F})$ or $\mathrm{AFF}(\mathfrak{F})$, the group of affine transformations over a non-archimedean local field. Indeed, the main motivation for this note came from the study of random walks on $\mathrm{AFF}(\mathfrak{F})$ and related subgroups of $\mathrm{AUT}(T)$ which are amenable and non-unimodular, see CARTWRIGHT, KAIMANOVICH and WOESS [5]. We consider random walks whose law is some probability measure μ which has continuous density with respect to Haar measure, and whose support is compact and open and generates Γ as a semigroup. Note that openness of the support is a rather natural condition, as our groups are totally disconnected. Also note that so far compactness of the support (or bounded range in the non group-invariant case), or at least strong moment conditions, have been necessary for almost all known realizations of the Martin boundary.

The method used here consists in extending the method of [8] (and its generalizations [17]) to the non-discrete situation. We use a Martin kernel which is defined in terms of the density of μ and prove

that it extends continuously to the space of ends Ω of T. Different ends correspond to different boundary functions (extended kernels), and all boundary functions are minimal μ-harmonic functions. Thus, using the construction of [7] of the Martin boundary, we see that every μ-harmonic function has a unique integral representation over the boundary. With some obvious, but space-consuming modifications, these results will also apply to random walks on groups of automorphisms which act with a finite number of orbits on a locally finite tree (not necessarily homogeneous).

This paper is organized as follows. In §2, we describe the geometry of the homogeneous tree, its end compactification, and its group of automorphisms. We state the main result and compare it with the general construction of [7], and we briefly indicate their application to cases such as $\mathrm{AFF}(\mathfrak{F})$ and $\mathrm{PGL}(2,\mathfrak{F})$, where $\mathfrak{F}$ is a non-archimedean local field. §3 is dedicated to the proofs.

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2. Homogeneous Tree, Automorphism Groups and the Martin Boundary

The *homogeneous tree* $T = T_q$ $(q \geqslant 2)$ is the unique connected graph without cycles in which every vertex has $q+1$ neighbours. For every pair of vertices $x, y \in T$ there is a unique *geodesic segment* $\overline{xy}$ of successive neighbours $x = x_0, x_1, \ldots, x_k = y$ in T which are all different; the *distance* between x and y is $d(x,y) = k$. A *ray* is an infinite sequence $x_0, x_1, x_2, \ldots$ of successive neighbours without repetitions. Two rays which differ only by finitely many vertices are called *equivalent*. An *end* of T is an equivalence class of rays. The set of all ends is denoted Ω, and we write $\hat{T} = T \cup \Omega$. For every $\omega \in \Omega$ and $x \in T$ there is a unique ray $\overline{x\omega}$ starting at x which represents ω.

We choose and fix a reference vertex $o \in T$. If $\xi, \eta \in \hat{T}$, then the *confluent* $c = c(\xi, \eta)$ is the last common vertex on $\overline{o\xi}$ and $\overline{o\eta}$, unless $\xi = \eta \in \Omega$, in which case we set $c(\xi, \eta) = \xi$. We set

$$(\xi|\eta) = \begin{cases} d(o,c), & c \in T; \\ \infty, & \text{otherwise.} \end{cases}$$

Note that $(\xi|\eta) \geqslant \min\{(\xi|\zeta), (\zeta|\eta)\}$ for all $\xi, \eta, \zeta \in \hat{T}$ (*ultrametric in-*

equality). Consequently,

$$\rho(\xi,\eta)=\begin{cases} e^{-(\xi|\eta)}, & \text{if } \xi\neq\eta, \\ 0, & \text{if } \xi=\eta \end{cases}$$

defines an ultrametric. Thus $\hat{T}$ becomes a totally disconnected, compact space with T open, dense and discrete.

An *automorphism* of the tree T is a self-isometry of the metric space (T,d). Equipped with the topology of pointwise convergence, the group $\mathrm{AUT}(T)$ of all automorphisms of T is a locally compact Hausdorff group. A neighbourhood base at the identity is given by the family of all pointwise stabilizers of finite sets of vertices. These are open and compact subgroups, so that $\mathrm{AUT}(T)$ is totally disconnected. Every automorphism extends naturally to a homeomorphism of $\hat{T}$ onto itself; the mapping $(\gamma,\omega)\mapsto\gamma\omega$ of $\mathrm{AUT}(T)\times\Omega$ onto Ω is jointly continuous.

The central topic of our investigations is a closed subgroup Γ of $\mathrm{AUT}(T)$ which acts *transitively* on T, together with a regular probability measure μ on Γ. In this context, a *harmonic function* is a function $h:\Gamma\to\mathbb{R}$ satisfying

$$h(\alpha)=\int_\Gamma h(\alpha\gamma)\mu(d\gamma)\;\forall\alpha\in\Gamma.$$

We intend to describe the cone of positive harmonic functions on Γ with respect to μ via a realization of the *Martin (exit) boundary*. It will turn out that the Martin boundary coincides with Ω. Observe that Ω can also be viewed as the boundary in a compactification $\hat{\Gamma}$ of Γ: a sequence $(\gamma_n)_n$ in Γ converges to $\omega\in\Omega$ if $\gamma_n x\to\omega$ in the topology of $\hat{T}$ for some (and hence every) $x\in T$.

We need the following additional hypotheses concerning μ.

(*) μ is an irreducible probability measure on Γ which has compact open support and continuous density with respect to left Haar measure on Γ:

$$\mu(d\gamma)=p(\gamma)d\gamma.$$

(*Irreducible* means that supp μ generates Γ as a closed semigroup.) In particular, harmonic functions satisfy the convolution equation $\check{h}=p*\check{h}$, where $\check{h}(\alpha)=h(\alpha^{-1})$, and must be continuous (see HEWITT and ROSS [13, 20.16], and recall that supp p is compact).

We consider a sequence (X_n) of i.i.d. Γ-valued random variables with distribution μ. The symbols Pr and **E** will refer to probability and expectation in the probability space where the X_n are defined. The *right random walk with law* μ is the sequence

$$R_0 = \iota \text{ (the group identity)}, \quad R_n = X_1 X_2 \cdots X_n \ (n \geqslant 1).$$

More generally, the right random walk starting at $\alpha \in \Gamma$ is $(\alpha R_n)_{n \geqslant 0}$. The distribution of R_n is $\mu^{(n)}$, the n-th convolution power of μ. By [21], Γ is either nonamenable or nonunimodular, so that (R_n) is transient, that is, $\sum_n \mu^{(n)}$ is a Radon measure, see GUIVARC'H, KEANE and ROYNETTE [12, Thms. 47 and 51]. Thus, the *Green kernel*

$$G(\alpha, f) = \sum_{n=0}^{\infty} \mathbf{E}(f(\alpha R_n))$$

is finite for all $\alpha \in \Gamma$ and for all $f \in \mathscr{K}(\Gamma)$, the space of compactly supported continuous functions on Γ. Observe that

$$G(\alpha, f) = f(\alpha) + \int_{\Gamma} g(\alpha, \gamma) f(\gamma) d\gamma,$$

where

$$g(\alpha, \gamma) = \sum_{n=1}^{\infty} p^{(n)}(\alpha^{-1}\gamma).$$

Here, $p^{(n)}$ is the (continuous) density of $\mu^{(n)}$; by assumptions (*), the function g is continuous and strictly positive. Our *Martin kernel*, or more precisely, the density of the Martin kernel, is

$$k(\alpha, \gamma) = \frac{g(\alpha, \gamma)}{g(\iota, \gamma)}, \quad \alpha, \gamma \in \Gamma.$$

Our main result is the following.

Theorem. (I) *For every* $\alpha \in \Gamma$, *the function* $k(\alpha, \cdot)$ *extends continuously to* $\hat{\Gamma} = \Gamma \cup \Omega$.

(II) *For every* $\omega \in \Omega$, *the function* $k(\cdot, \omega)$ *is continuous and minimal harmonic with respect to* μ *on* Γ.

(III) *If* $\omega_1 \neq \omega_2$ *then* $k(\cdot, \omega_1) \neq k(\cdot, \omega_2)$.

In (II) we have tacitly used the convention to maintain the notation $k(\cdot, \cdot)$ for the extension of the Martin kernel to $\Gamma \times \hat{\Gamma}$. Also,

recall that *minimal* means that $k(\cdot, \omega)$ is an extreme point in the cone of positive harmonic functions.

Remark. In the proof of (I), we shall see that the following holds: if (γ_n) is a sequence in Γ such that $\gamma_n o \to \omega \in \Omega$, then the sequence of functions $k(\cdot, \gamma_n)$ converges to $k(\cdot, \omega)$ uniformly on compact subsets of Γ.

Let us relate our results with DERRIENNIC's [7, §II] construction of the Martin compactification. This construction corresponds to considering the Green kernel

$$G(f, \gamma) = f(\gamma) + \int_\Gamma f(\alpha) g(\alpha, \gamma) d\alpha$$

and the Martin kernel

$$K(f, \gamma) = \frac{G(f, \gamma)}{G(f_0, \gamma)}, \quad f \in \mathscr{H}(\Gamma), \gamma \in \Gamma,$$

where f_0 is the density of the initial distribution of the random walk. This means that one considers the right random walk $X_0 R_n$, where X_0 is a random variable independent of the X_n, $n \geqslant 1$, having density f_0 with respect to left Haar measure (while above, in the definition of $k(\cdot, \cdot)$, we chose the point mass at ι for the distribution of X_0). The Martin compactification is then the minimal compactification of Γ (with the usual requirements concerning a compactification) which allows continuous extension of all $K(f, \cdot)$, $f \in \mathscr{H}(\Gamma)$. If we choose f_0 in $\mathscr{H}(\Gamma)$, and if $\gamma \in \Gamma \setminus (\operatorname{supp} f_0 \cup \operatorname{supp} f)$, then

$$K(f, \gamma) = \frac{\int_\Gamma f(\alpha) k(\alpha, \gamma) d\alpha}{\int_\Gamma f_0(\alpha) k(\alpha, \gamma) d\alpha}.$$

Using the above remark concerning uniform convergence on compact sets, we see from the Theorem that the Martin compactification in the sense of [7, §II] is just $\Gamma \cup \Omega$, and the extended kernel in the sense of [7] is

$$K(f, \omega) = \frac{\int_\Gamma f(\alpha) k(\alpha, \omega) d\alpha}{F_0(\omega)},$$

where

$$F_0(\omega) = \int_\Gamma f_0(\alpha) k(\alpha, \omega) d\alpha.$$

From [7] it then follows that for every positive μ-harmonic function h there is a positive Borel measure ν_0^h on Ω such that

$$\int_\Gamma f(\gamma) h(\gamma) d\gamma = \int_\Omega K(f, \omega) \nu_0^h(d\omega)$$

for all $f \in \mathscr{K}(\Gamma)$. But the last term is equal to

$$\int_\Gamma f(\gamma) \left(\int_\Omega k(\gamma, \omega) \frac{1}{F_0(\omega)} \nu_0^h(d\omega) \right) d\gamma.$$

Using continuity of the involved functions, we see that $h(\gamma) = \int_\Omega k(\gamma, \omega) \nu^h(d\omega)$, where $\nu^h(d\omega) = \frac{1}{F_0(\omega)} \nu_0^h(d\omega)$. As all functions $k(\cdot, \omega)$ are minimal, the integral representation is unique. We summarize

Corollary. *Under assumption* (*), *the Martin boundary of the random walk is* Ω, *and every positive harmonic function has a unique integral representation*

$$h(\gamma) = \int_\Omega k(\gamma, \omega) \nu^h(d\omega),$$

where ν^h *is a positive Borel measure on* Ω.

Obviously, our results apply to finitely supported, irreducible probability measures on *free groups* as a particular case. This is the case treated in [8]. There are other interesting cases.

Let $\mathfrak{F}$ be a non archimedean local field whose residual field has order q (a prime power). The group $GL(2, \mathfrak{F})$ is the group of non-singular two by two matrices over $\mathfrak{F}$, and $PGL(2, \mathfrak{F})$ is the quotient group of $GL(2, \mathfrak{F})$ with respect to its center. (The latter consists of all multiples of the identity matrix). It is well known that $PGL(2, \mathfrak{F})$ acts transitively on $T = T_q$ as a closed subgroup of $AUT(T)$. Besides the books of SERRE [19], [20], a short and readable account of this topic can be found in the volume by FIGÀ-TALAMANCA and NEBBIA [11]. In particular, the results presented here apply to harmonic

functions with respect to some probability measure μ on $\mathrm{PGL}(2, \mathfrak{F})$ satisfying assumptions (*).

Another example is the *affine group* over $\mathfrak{F}$, that is, the group $\mathrm{AFF}(\mathfrak{F})$ of all transformations $\mathfrak{z} \mapsto \mathfrak{a}\mathfrak{z} + \mathfrak{b}$ ($\mathfrak{z} \in \mathfrak{F}$), where $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$, $\mathfrak{a} \neq 0$. This group may be viewed as a subgroup of $\mathrm{PGL}(2, \mathfrak{F})$, so that it acts on $T = T_q$. However, its action on T can be described in a much simpler way, see [5]. As a closed subgroup of $\mathrm{AUT}(T)$, it acts transitively on T and on $\Omega \backslash \{\omega\} \cong \mathfrak{F}$, where ω is a fixed element of Ω, left invariant by the group. Once more, the results presented here apply to $\mathrm{AFF}(\mathfrak{F})$. There is a natural analogy between the action of $\mathrm{AFF}(\mathfrak{F})$ on T and the action of $\mathrm{AFF}(\mathbb{R})$ (the affine group over $\mathbb{R}$) on the hyperbolic upper half plane (see also the comments in [5]). In this context, we mention the work of ÉLIE [10] concerning the Martin boundary of $\mathrm{AFF}(\mathbb{R})$.

3. Proofs

For the proof of the Theorem, we extend the method of [8] and its generalizations [17]. We need several preparatory steps; after that, the proof will be clear, following these two references.

In the sequel, if $A \subset \Gamma$, we denote by $\mathscr{C}(A)$ the space of continuous functions on A, by $\mathscr{C}^+(A)$ the nonnegative cone, and by $\mathscr{C}^{+\,+}(A)$ the strictly positive cone. Subsets of Γ appearing in integrals, etc., are tacitly assumed to be Borel sets. Write $S = \operatorname{supp} \mu$ and define

$$s = \max\{d(\gamma o, o) \mid \gamma \in S\}$$

and, for $x \in T$,

$$\Gamma^x = \{\gamma \in \Gamma \mid \gamma o = x\}, \quad A_x = \{\gamma \in \Gamma \mid d(x, \gamma o) \leqslant s/2\}.$$

Observe that $\alpha A_x = A_{\alpha x}$ for $\alpha \in \Gamma$.

Lemma 1. *Let x, y, z be vertices of T such that $x \in \overline{yz}$. If $R_k o = y$ and $R_m o = z$ for $m > k$, then almost surely there is an $\ell \in \{k, k+1, \ldots, m\}$ such that $R_\ell \in A_x$.*

Proof. (Compare with [17, Lemma 5]). We have $d(R_n o, R_{n+1} o) \leqslant s$ almost surely. Hence, A_x is a "barrier" when passing from y to z. □

By assumption, $\{\gamma \in \Gamma \mid p^{(n)}(\gamma) > 0\} = S^n$ is both compact and open.

Let $\alpha \in S$. Then $\alpha^{-1}S$ is an open neighbourhood of the identity, and

$$\Gamma = \left(\bigcup_{n \geqslant 1} S^n \right)^- \subset \alpha^{-1} S \cdot \bigcup_{n \geqslant 1} S^n \subset \alpha^{-1} \cdot \bigcup_{n \geqslant 1} S^n.$$

Multiplying on the left by α, we find that $\Gamma = \bigcup_{n \geqslant 1} S^n$. Consequently, we can find K such that

$$\{\gamma \in \Gamma \mid d(\gamma o, o) \leqslant s\} \subset \bigcup_{n=1}^{K-1} S^n.$$

Write

$$D = Ks + 1.$$

For $x \in T$ and $\alpha \in \Gamma$, define the stopping time

$$t^x(\alpha) = \inf\{n \geqslant 0 \mid \alpha R_n \in A_x\}.$$

Lemma 2. (a) *If $d(y, x) \geqslant D$ and $\alpha \in A_y$ then $t^x(\alpha) \geqslant K$ almost surely.*
(b) *If $\alpha_n \to \alpha$ then there is n_0 such that $t^x(\alpha_n) = t^x(\alpha)$ for all $n \geqslant n_0$.*

Proof. Statement (a) is immediate from the inequality $d(R_n o, R_{n+1} o) \leqslant s$. To see (b), recall that convergence of α_n to α implies $\alpha_n^{-1} x = \alpha^{-1} x$ for all but finitely many n. □

Next, we introduce a family of "first hitting" or "balayage" operators: for $x \in T$,

$$\mathbf{s}^x(\alpha, f) = \mathbf{E}(f(\alpha R_{t^x(\alpha)}) \cdot \mathbf{1}_{[t^x(\alpha) < \infty]}),$$

where $f \in \mathscr{C}(A_x)$ and $\alpha \in \Gamma$. It arises from integration with respect to the hitting distribution of (αR_n) on A_x. Alternatively, we shall consider $\mathbf{s}^x(\alpha, \cdot)$ as a Borel measure on A_x. We shall often write $\mathbf{s}^x f(\alpha)$ for $\mathbf{s}^x(\alpha, f)$, also allowing that $f \in \mathscr{C}(\Gamma)$, in which case we intend $\mathbf{s}^x(\alpha, \mathbf{1}_{A_x} \cdot f)$.

Lemma 3. *If $\alpha \notin A_x$ then $\mathbf{s}^x(\alpha, \cdot)$ has density $u^x(\alpha, \cdot)$, its support is a compact open subset of A_x, and $u^x(\cdot, \cdot)$ is jointly continuous on $(\Gamma \backslash A_x) \times A_x$.*

Proof. We first assume that $\alpha = \iota$, the identity, and that $\iota \notin A_x$. Define recursively a sequence of functions on Γ:

$$\bar{g}_1(\gamma) = p(\gamma), \quad \bar{g}_{n+1}(\gamma) = \int_{\Gamma \backslash A_x} \bar{g}_n(\beta) p(\beta^{-1}\gamma) d\beta.$$

Note that $\bar{g}_n \leqslant p^{(n)}$, so that $\bar{g}_n$ is integrable. The function $\mathbf{1}_{\Gamma \backslash A_x} \cdot \bar{g}_n$ has compact support, and

$$\bar{g}_{n+1} = (\mathbf{1}_{\Gamma \backslash A_x} \cdot \bar{g}_n) * p.$$

Hence, the $\bar{g}_n$ are continuous [13, 20.16]. For $U \subset \Gamma$,

$$\int_U \bar{g}_n(\gamma)d\gamma = \Pr[R_i \notin A_x \,\forall i < n, R_n \in U].$$

Thus, $\mathbf{1}_{A_x} \cdot \bar{g}_n$ is the density of the distribution which describes the first entrance into A_x at time n. (We shall also use this observation in the proof of Lemma 5.) Set

$$\bar{g} = \sum_{n=1}^{\infty} \bar{g}_n.$$

By transience, $\bar{g}$ is integrable on every relatively compact set. For $U \subset A_x$, we have

$$\mathbf{s}^x(\iota, U) = \sum_{n=1}^{\infty} \int_U \bar{g}_n(\gamma)d\gamma = \int_U \bar{g}(\gamma)d\gamma,$$

so that $\mathbf{s}^x(\iota, \cdot)$ has density $u^x(\iota, \cdot) = \mathbf{1}_{A_x} \cdot \bar{g}$. Let $W = \{\gamma \in \Gamma \backslash A_x | \; |\mu(\gamma^{-1} A_x) > 0\} = A_x \cdot S^{-1} \backslash A_x$ be the set of all γ outside of A_x from which A_x can be reached in one step. This is compact and open, and

$$u^x(\iota, \gamma) = p(\gamma) + (\mathbf{1}_W \cdot \bar{g}) * p(\gamma),$$

which is a continuous function. The support of $\mathbf{s}^x(\iota, \cdot)$ is

$$(S \cup (\operatorname{supp} \mathbf{1}_W \cdot \bar{g}) \cdot S) \cap A_x.$$

This is open and compact [13, 4.4].

Obviously, if $\alpha \in \Gamma \backslash A_x$ is arbitrary, then

$$u^x(\alpha, \gamma) = u^{\alpha^{-1}x}(\iota, \alpha^{-1}\gamma).$$

If $\alpha_n \to \alpha$ in $\Gamma \backslash A_x$, then there is n_0 such that $\alpha_n^{-1} x = \alpha^{-1} x$ and hence

$$u^x(\alpha_n, \gamma) = u^{\alpha^{-1}x}(\iota, \alpha_n^{-1}\gamma) \quad \text{for all } n \geqslant n_0.$$

Consequently, $u^x(\cdot, \cdot)$ is continuous on $(\Gamma \backslash A_x) \times A_x$. □

Lemma 4. *Let x, y, z be vertices of T such that $x \in \overline{yz}$ and $\min\{d(x, y), d(x, z)\} > s/2$. Then*

$$g(\alpha, \gamma) = \mathbf{s}^x(\alpha, g(\cdot, \gamma))$$

for every $\alpha \in \Gamma^y$ and $\gamma \in \Gamma^z$.

Proof. By transitivity of Γ and Γ-invariance of g, we may assume without loss of generality that $y = o$ and $\alpha = \iota$. The sets A_x, Γ^o and

Γ^z are mutually disjoint. By Lemmas 1 and 3, if $U \subset \Gamma^z$, then

$$\int_U g(\iota,\gamma)d\gamma = \sum_{n=1}^{\infty} \Pr[R_n \in U] =$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \Pr[R_n \in U, t^x(\iota) = k] =$$

$$= \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \int_{A_x} \Pr[R_n \in U \mid R_k = \beta] \bar{g}_k(\beta) d\beta =$$

$$= \int_{A_x} \left(\int_U g(\beta,\gamma)d\gamma \right) u^x(\iota,\beta)d\beta.$$

Thus,

$$g(\iota,\gamma) = \mathbf{s}^x(\iota, g(\cdot,\gamma))$$

for almost every $\gamma \in \Gamma^z$. As both sides are continuous, equality follows. □

Lemma 5. *If* $d(y,x) \geqslant D$, *then* $\operatorname{supp} \mathbf{s}^x(\alpha,\cdot) = B_{y,x}$ *is the same for all* $\alpha \in A_y$.

Proof. By Lemma 2, for every $\alpha \in A_y$, (αR_n) does not hit A_x for $n < K$. Consequently, if $\alpha \in A_y$ and $U \subset A_x$ then

$$\mathbf{s}^x(\alpha, U) \geqslant \int_{A_y} \mathbf{s}^x(\beta, U)\delta_\alpha * \mu^{(n)}(d\beta) = \int_{A_y} \mathbf{s}^x(\beta, U)p^{(n)}(\alpha^{-1}\beta)d\beta$$

for all $n < K$. We get

$$\mathbf{s}^x(\alpha, U) \geqslant \int_{A_y} \mathbf{s}^x(\beta, U)\left(\frac{1}{K-1} \sum_{n=1}^{K-1} p^{(n)}(\alpha^{-1}\beta) \right) d\beta$$

for all $\alpha \in A_y$, $U \subset A_x$. If $\alpha, \beta \in A_y$, then $d(\alpha^{-1}\beta o, o) \leqslant s$, and by our choice of K,

$$\frac{1}{K-1} \sum_{n=1}^{K-1} p^{(n)}(\alpha^{-1}\beta) > 0.$$

Hence, if $\mathbf{s}^x(\beta, U) > 0$ for some $\beta \in A_y$, then $\mathbf{s}^x(\alpha, U) > 0$: that is,

$$\operatorname{supp} \mathbf{s}^x(\beta,\cdot) \subset \operatorname{supp} \mathbf{s}^x(\alpha,\cdot)$$

for all $\alpha, \beta \in A_y$. □

The last lemma, in combination with Lemma 3, tells us that for $d(x, y) \geqslant D$,

$$\{(\alpha, \gamma) \in A_y \times A_x \mid u^x(\alpha, \gamma) > 0\} = A_y \times B_{y,x}.$$

We define

$$m_{y,x} = \min_{A_y \times B_{y,x}} u^x(\alpha, \gamma), \quad M_{y,x} = \max_{A_y \times B_{y,x}} u^x(\alpha, \gamma).$$

Then $0 < m_{y,x} \leqslant M_{y,x} < \infty$. (It is here that we have used the hypothesis that $\operatorname{supp} \mu$, besides being compact, is also open.) If $\gamma \in \Gamma$ with $\gamma o = y$ then $m_{y,x} = m_{o,\gamma^{-1}x}$ and $M_{y,x} = M_{o,\gamma^{-1}x}$. The number of vertices of T at distance D from o is finite. Hence

$$m_0 = \min_{d(x,y)=D} m_{y,x} > 0 \quad \text{and} \quad M_0 = \max_{d(x,y)=D} M_{y,x} < \infty.$$

Given $x, y \in T$, we write $\mathbf{s}_y^x$ for the restriction (in the first variable) of $\mathbf{s}^x$ to A_y. This is an operator which maps $\mathscr{C}(A_x)$ to $\mathscr{C}(A_y)$. It may be useful to have in mind that all the spaces $\mathscr{C}(A_x)$, $x \in T$, may be identified with $\mathscr{C}(A_o)$ via some element of Γ sending x to o. Thus one may think of $\mathbf{s}_y^x$ as an operator of $\mathscr{C}(A_o)$ onto itself.

Following BIRKHOFF [3], we equip each of the cones $\mathscr{C}^+(A_x)$, $x \in T$, with its projective metric:

$$\theta(f_1, f_2) = \log\left(\max_{A_x} \frac{f_1}{f_2} \cdot \max_{A_x} \frac{f_2}{f_1} \right).$$

This is finite if and only if both functions vanish in no point of A_x. Observe that no strictly positive functions with $\| f \| = \max_{A_x} f = 1$, this metric induces the topology of uniform convergence [3, Cor. 6.1]. We now study contractivity of the operator $\mathbf{s}_y^x$ in this metric, assuming that $d(y, x) = D$.

If $f \in \mathscr{C}^+(A_x)$, then $\mathbf{s}_y^x f$ either vanishes (if $f \equiv 0$ on $B_{y,x}$) or otherwise is strictly positive on A_y (Lemma 5). Hence the image of $\mathscr{C}^+(A_x)$ under $\mathbf{s}_y^x$ is the cone with vertex 0 spanned by the functions $\mathbf{s}_y^x f$, where $f \neq 0$ on $B_{y,x}$. Let f be such a function. Then

$$m_0 \int_{B_{y,x}} f(\beta) d\beta \leqslant \mathbf{s}_y^x f \leqslant M_0 \int_{B_{y,x}} f(\beta) d\beta,$$

and

$$\frac{\max \mathbf{s}_y^x f}{\min \mathbf{s}_y^x f} \leqslant \frac{M_0}{m_0}.$$

This leads us to the following.

Lemma 6. *If $d(y, x) = D$, then the projective diameter of the cone $\mathbf{s}_y^x(\mathscr{C}^+(A_x))$ is bounded by $2\log(M_0/m_0)$. Furthermore, if $f_1, f_2 \in \mathscr{C}^{+\,+}(A_x)$ then*

$$\theta(\mathbf{s}_y^x f_1, \mathbf{s}_y^x f_2) \leqslant \frac{M_0 - m_0}{M_0 + m_0}\theta(f_1, f_2).$$

Proof. The above considerations show that for $f_1, f_2 \in \mathscr{C}^+(A_x)$ which do not vanish on $B_{y,x}$,

$$\theta(\mathbf{s}_y^x f_1, \mathbf{s}_y^x f_2) \leqslant 2\log(M_0/m_0).$$

Besides the vertex 0, all elements of $\mathbf{s}_y^x(\mathscr{C}^+(A_x))$ are strictly positive on A_y. Thus, $\mathbf{s}_y^x(\mathscr{C}^+(A_x))$ has diameter bounded by $\Delta = 2\log(M_0/m_0)$. [3, Lemma 5.1] now yields that the contraction coefficient is $\tanh(\Delta/4) = (M_0 - m_0)/(M_0 + m_0)$. □

We say that $f \in \mathscr{C}^+(A_x)$ is *normalized*, if $\|f\| = 1$. For arbitrary $f \in \mathscr{C}^+(A_x)$, $f \neq 0$, we write

$$\mathscr{P}_x f = \frac{1}{\|f\|} f.$$

In the sequel, writing $\mathscr{P}_x g(\alpha, \gamma)$ or $\mathbf{s}_y^x g(\alpha, \gamma)$, we intend that the respective operator acts on the first of the two variables.

Proposition 1. (a) *For every $x \in T$ and $\omega \in \Omega$ there is a normalized function $g_x(\cdot, \omega) \in \mathscr{C}^{+\,+}(A_x)$ such that*

$$\lim_{n \to \infty} \mathscr{P}_x \mathbf{s}_x^{y_n} f_n(\alpha) = g_x(\alpha, \omega)$$

uniformly for $\alpha \in A_x$, whenever (y_n) is a sequence in T converging to ω and $f_n \in \mathscr{C}^{+\,+}(A_{y_n})$. In particular

$$\lim_{n \to \infty} \mathscr{P}_x g(\alpha, \gamma_n) = g_x(\alpha, \omega)$$

uniformly for $\alpha \in A_x$, whenever (γ_n) is a sequence in Γ such that $\gamma_n o \to \omega$.

(b) *If* $x, z \in \overline{y\omega}$ *are such that* $\min\{d(x,y), d(x,z)\} > s$ *and* $z \in \overline{x\omega}$ *then*

$$\mathbf{s}_y^z \mathbf{g}_z(\cdot,\omega) = \mathbf{s}_y^x \mathbf{s}_x^z g_z(\cdot,\omega) \quad \text{and} \quad g_y(\cdot,\omega) = \mathscr{P}_y \mathbf{s}_y^x g_x(\cdot,\omega).$$

(c) *For fixed* x, *the function* $g_x(\cdot,\cdot)$ *is continuous on* $A_x \times \Omega$.

Proof. (a) Let $x = x_0, x_1, x_2, \ldots$ be successive elements at distance $d(x_{i-1}, x_i) = D$ on the geodesic $\overline{x\omega}$. If f_n, f'_n are two functions in $\mathscr{C}^{++}(A_{x_n})$, $n = 0, 1, 2, \ldots$, then by Lemmas 4 and 6

$$\theta(\mathbf{s}_x^{x_n} f_n, \mathbf{s}_x^{x_n} f'_n) = \theta(\mathbf{s}_{x_0}^{x_1} \mathbf{s}_{x_1}^{x_2} \cdots \mathbf{s}_{x_{n-1}}^{x_n} f_n, \mathbf{s}_{x_0}^{x_1} \mathbf{s}_{x_1}^{x_2} \cdots \mathbf{s}_{x_{n-1}}^{x_n} f'_n) \leqslant$$

$$\leqslant \left(\frac{M_0 - m_0}{M_0 + m_0}\right)^{n-1} \cdot 2 \log \frac{M_0}{m_0}.$$

In particular, setting $f'_n = \mathbf{s}_{x_n}^{x_{n+1}} f_{n+1}$, we see that $(\mathbf{s}_x^{x_n} f_n)$ is a Cauchy sequence with respect to θ in $\mathscr{C}^{++}(A_x)$ and converges in direction ($\equiv$ projectively). The above shows that for arbitrary f'_n, the sequence $(\mathbf{s}_x^{x_n} f'_n)$ has the same projective limit. This yields the first part of (a).

Now let (γ_n) be as proposed, and let $y_n = \gamma_n o$. Among the vertices $x_0, x_1, \ldots$ defined above, let $x_{m(n)}$ be the last one which lies on $\overline{xy_n}$. Then $m(n) \to \infty$. Let g_n be the restriction of $g(\cdot, \gamma_n)$ to $A_{x_{m(n)}}$. Then Lemma 4 yields

$$g(\alpha, \gamma_n) = \mathbf{s}_x^{x_{m(n)}} g_n(\alpha)$$

for all $\alpha \in A_x$. Hence, by the above,

$$\lim_{n \to \infty} \mathscr{P}_x g(\alpha, \gamma_n) = g_x(\alpha, \omega)$$

uniformly for $\alpha \in A_x$.

(b) Changing notation, this time we write g_n for the restriction of $g(\cdot, \gamma_n)$ to A_z. By Lemma 4, if n is sufficiently large, $\mathbf{s}_y^z g_n = \mathbf{s}_y^x \mathbf{s}_x^z g_n$ and hence

$$\mathbf{s}_y^z \mathscr{P}_z g_n = \mathbf{s}_y^x \mathbf{s}_x^z \mathscr{P}_z g_n.$$

Letting $n \to \infty$, (a) yields the first identity. The second identity follows immediately from the fact that by Lemma 4

$$g(\alpha, \gamma_n) = \mathbf{s}_y^x g(\alpha, \gamma_n)$$

for all $\alpha \in A_y$, if n is sufficiently large.

(c) Let $\omega_n \to \omega$, and assume $\omega_n \neq \omega$ for all n. This time, let $x_{m(n)}$ be the last element in the above sequence which lies on $\overline{x\omega_n}$. Then

(b) yields

$$g_x(\cdot,\omega_n)=\mathscr{P}_x\mathbf{s}_x^{x_{m(n)}}g_{x_{m(n)}}(\cdot,\omega_n) \quad \text{and} \quad g_x(\cdot,\omega)=\mathscr{P}_x\mathbf{s}_x^{x_{m(n)}}g_{x_{m(n)}}(\cdot,\omega).$$

Hence

$$\theta(g_x(\cdot,\omega_n),g_x(\cdot,\omega))\to 0. \quad \square$$

We can now conclude the proof of our main result.

Proof of the Theorem. (I) Let $\omega\in\Omega$, $\alpha\in\Gamma$, $\alpha o=y$. Write $c=c(y,\omega)$ and let x be the first point on $\overline{c\omega}$ which satisfies $\min\{d(x,o),d(x,y)\}>$ $>s/2$. Define

$$k(\alpha,\omega)=\frac{\mathbf{s}_y^x g_x(\alpha,\omega)}{\mathbf{s}_o^x g_x(\iota,\omega)}.$$

(We remark that from Proposition 1(b) one gets that $k(\alpha,\omega)$ does not change when x is replaced with any vertex on $\overline{x\omega}$.) In the second variable, this function is continuous on Ω by Proposition 1(c). If (γ_n) is a sequence in Γ such that $\gamma_n o\to\omega$, then for all but finitely many n, the confluent $c(y,\gamma_n o)$ is c and $d(x,\gamma_n o)>s/2$. Thus (Lemma 4)

$$g(\alpha,\gamma_n)=\mathbf{s}_y^x g(\alpha,\gamma_n), \quad g(\iota,\gamma_n)=\mathbf{s}_o^x g(\iota,\gamma_n).$$

Hence, if $\mathscr{P}_x$ is the projection on $\mathscr{C}^{++}(A_x)$, then Proposition 1(a) yields

$$k(\alpha,\gamma_n)=\frac{\mathbf{s}_y^x\mathscr{P}_x g(\alpha,\gamma_n)}{\mathbf{s}_o^x\mathscr{P}_x g(\iota,\gamma_n)}\to k(\alpha,\omega)$$

as $n\to\infty$. (Indeed, inserting $\mathscr{P}_x$ is dividing denominator and enumerator by $\max_{A_x} g(\cdot,\gamma_n)$.) Note that convergence is uniform on compact sets: it is uniform on $\{\alpha|\alpha o=y\}$ by Proposition 1(a), and every compact set is contained in a finite union of sets of this form. This settles the remark after the Theorem in §2.

Proposition 1(b) yields that for $x,y\in T$ with $x\in\overline{y\omega}$ and $d(x,y)>s$,

$$k(\alpha,\omega)=\mathbf{s}_y^x k(\alpha,\omega) \quad \text{for all } \alpha\in A_y.$$

This will be crucial in (II) and (III).

(II) By continuity of $g_x(\cdot,\omega)$ and the properties of $\mathbf{s}_y^x$ (Lemma 3), $k(\cdot,\omega)$ is continuous. Let $\omega\in\Omega$, and let (γ_n) in Γ be such that $\gamma_n o\to\omega$. Set $k_n(\alpha)=k(\alpha,\gamma_n)$. Then

$$\int_\Gamma k_n(\alpha\gamma)\mu(d\gamma)=k_n(\alpha)-\frac{p(\alpha^{-1}\gamma_n)}{g(\iota,\gamma_n)}.$$

As $S = \operatorname{supp}\mu$ is compact and (γ_n) leaves every compact subset of Γ, the term on the right is $k_n(\alpha)$ if n is sufficiently large. Furthermore, for fixed α, $k_n(\alpha\gamma) \to k(\alpha\gamma, \omega)$ uniformly for $\gamma \in S$. Hence

$$k(\alpha, \omega) = \lim_{n\to\infty} k_n(\alpha) = \lim_{n\to\infty} \int_S k_n(\alpha\gamma)\mu(d\gamma) = \int_\Gamma k(\alpha\gamma, \omega)\mu(d\gamma).$$

Thus, $k(\cdot, \omega)$ is harmonic. Next, we prove minimality, adapting the corresponding proof (regarding free groups) of [8]. Suppose that h_1, h_2 are positive harmonic functions with $h_1(\iota) = h_2(\iota) = 1$ and such that

$$k(\cdot, \omega) = t \cdot h_1 + (1-t) \cdot h_2$$

for some t, $0 < t < 1$. Recall that every nonzero harmonic function $h \geqslant 0$ is continuous and strictly positive. Furthermore, $h \geqslant \mathbf{s}^x(\cdot, h)$. (Indeed, the latter is the smallest superharmonic function which coincides with h on A_x.) Let $\alpha o = y$, and let $x \in \overline{y\omega}$ with $d(x, y) > s/2$. Then

$$\begin{aligned} k(\alpha, \omega) = \mathbf{s}_y^x k(\alpha, \omega) &= t \cdot \mathbf{s}_y^x h_1(\alpha) + (1-t) \cdot \mathbf{s}_y^x h_2(\alpha) \leqslant \\ &\leqslant t \cdot h_1(\alpha) + (1-t) \cdot h_2(\alpha) = k(\alpha, \omega). \end{aligned}$$

Thus, inequalities cannot be strict, and

$$h_i(\alpha) = \mathbf{s}_y^x h_i(\alpha), \quad i = 1, 2.$$

Now let $c = c(y, \omega)$, and choose $x = x_0, x_1, x_2, \ldots \in \overline{x\omega}$ at distance $d(x_{i-1}, x_i) = D$, with $\min\{d(x, o), d(x, y)\} > s/2$. Then we get for $i = 1, 2$

$$h_i(\alpha) = \mathbf{s}_y^x \mathbf{s}_x^{x_n} h_i(\alpha) \quad \text{and} \quad 1 = h_i(\iota) = \mathbf{s}_o^x \mathbf{s}_x^{x_n} h_i(\iota) \quad \text{for all } n \geqslant 0.$$

Thus, applying Proposition 1(a) to $f_n = \mathbf{1}_{A_{x_n}} \cdot h$,

$$h_i(\alpha) = \lim_{n\to\infty} \frac{\mathbf{s}_y^x \mathbf{s}_x^{x_n} h_i(\alpha)}{\mathbf{s}_o^x \mathbf{s}_x^{x_n} h_i(\iota)} = \frac{\mathbf{s}_y^x g_x(\alpha, \omega)}{\mathbf{s}_o^x g_x(\iota, \omega)} = k(\alpha, \omega).$$

(III) Let ω_1, ω_2 be different ends. Suppose that $k(\cdot, \omega_1) \equiv k(\cdot, \omega_2)$. Let $c = c(\omega_1, \omega_2)$, and choose $x \in \overline{c\omega_1}$, $y \in \overline{c\omega_2}$ such that $d(x, y) = D$. If $\beta \in A_y$, then

$$k(\beta, \omega_1) = \mathbf{s}_y^x k(\beta, \omega_1).$$

If $\alpha \in A_x$ then

$$k(\alpha, \omega_1) = k(\alpha, \omega_2) = \mathbf{s}_x^y k(\alpha, \omega_2) = \mathbf{s}_x^y k(\alpha, \omega_1) = \mathbf{s}_x^y \mathbf{s}_y^x k(\alpha, \omega_1).$$

Let $f(\alpha) = k(\alpha, \omega_1)$, $\alpha \in A_x$. Then $f \in \mathscr{C}^{++}(A_x)$ and $\mathbf{s}_x^y \mathbf{s}_y^x f = f$.

For $\alpha = A_x$, consider the two stopping times

$$t(\alpha) = \inf\{n > 0 | \alpha R_n \in A_x\} \quad \text{and}$$
$$s(\alpha) = \inf\{n > 0 | \alpha R_n \in A_x \text{ and } \alpha R_k \in A_y \text{ for some } k < n\}.$$

Obviously, $t \leqslant s$. Transience yields $\Pr[t(\alpha) = \infty] > 0$ and hence $\Pr[s(\alpha) = \infty] > 0$ for all $\alpha \in A_x$, see [12, Th. 24]. Let

$$u(\alpha, \alpha') = \int_{A_y} u^y(\alpha, \beta) u^x(\beta, \alpha') d\beta$$

be the density of $\mathbf{s}_x^y \mathbf{s}_y^x$ on A_x. This is a continuous function supported by $A_x \times B_{y,x}$ (Lemma 5), and

$$\int_{A_x} u(\alpha, \alpha') d\alpha' = \Pr[s(\alpha) < \infty] < 1.$$

This is a continuous function of $\alpha \in A_x$, and hence

$$\max_{\alpha \in A_x} \int_{A_x} u(\alpha, \alpha') d\alpha' = \lambda < 1.$$

We infer

$$\| \mathbf{s}_x^y \mathbf{s}_y^x f \| \leqslant \lambda \| f \|,$$

a contradiction. □

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