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The Density Condition in Subspaces and Quotients of Frechet Spaces

By

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Abstract. The aim of this note is to investigate the topological structure (in particular the density condition) of subspaces and separated quotients of Fréchet spaces. Our main result is the following one: Let E be a Fréchet space which is neither Montel nor isomorphic to a closed subspace of $X \times \omega$, with X a Banach space, also assume that E can be written as $F \oplus G$ with F and G infinite dimensional closed subspaces of E not isomorphic to ω, then E contains a closed subspace with basis and not satisfying the density condition. We also prove that every Köthe echelon space of order p, $1 < p < \infty$, which is not quasinormable has a separated quotient with basis which does not satisfy the density condition.

One of the directions of research in the theory of Fréchet spaces has been the study of the structure of subspaces and quotients of (certain classes of) Fréchet spaces. There are some results which characterize certain properties of a Fréchet space in terms of properties of its (separable) subspaces or quotients. It is well-known that a Fréchet space is reflexive (or Montel) if and only if every closed separable subspace is of the same type. BELLENOT [3] proved that a Fréchet space is Schwartz if and only if all its quotients are Montel, and VALDIVIA [21] has shown that a Fréchet space is the reduced projective limit of a sequence of reflexive Banach spaces if and only if every quotient is reflexive. Our original aim is to clarify the role of the distinguishedness and the density condition in this respect. In [8] we started our investigations of the topological structure of subspaces and quotients of Fréchet spaces. We characterized the Köthe echelon spaces of order 1, $\lambda_1(A)$, such that every closed subspace or every separated quotient is distinguished. Distinguished Fréchet spaces were introduced by Dieudonné, Schwartz and Grothendieck. A

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Fréchet space E is distinguished if its strong dual E'_b is an inductive limit of Banach spaces. We refer to [9] for a recent survey article on distinguished Fréchet spaces.

If a Fréchet space is reflexive, all its subspaces are also reflexive, hence distinguished; and if it is the projective limit of reflexive Banach spaces, every quotient is reflexive and hence distinguished. Accordingly, if we want to deal with subspaces and quotients with a "bad topological structure" in the reflexive case, non-distinguishedness is not appropriate. The density condition, introduced by HEINRICH in his study of ultrapowers of locally convex spaces [15], is the most suitable concept for our study. A locally convex space E with a basis $\mathscr{U}_0(E)$ of closed absolutely convex 0-neighbourhoods and a fundamental system $\mathscr{B}(E)$ of closed bounded subsets satisfies the density condition if the following holds: Given any function $\lambda: \mathscr{U}_0(E) \to \mathbb{R}_+ \setminus \{0\}$ and an arbitrary element $V \in \mathscr{U}(E)$, there always exist a finite subset $\mathscr{U}$ of $\mathscr{U}_0(E)$ and an element B of $\mathscr{B}(E)$ such that

$$\bigcap_{u \in \mathscr{U}} \lambda(U)U \subset B + V.$$

The density condition was thoroughly studied for Fréchet and Köthe spaces by BIERSTEDT and BONET [5]. It was proved there that a Fréchet space E has the density condition if and only if the bounded subsets of its strong dual are metrizable, and that a Köthe echelon space of order one, $\lambda_1(I, A)$, is distinguished if and only if it has the density condition. Every quasinormable and every Montel Fréchet space has the density condition.

Our main result in this paper is Theorem 1.2 where we show that every Fréchet space E which is neither Montel nor isomorphic to a subspace of $X \times \omega$, with X a Banach space, must contain a closed subspace with basis and not satisfying the density condition, provided that E decomposes as $F \oplus G$ where F and G are infinite dimensional subspaces not isomorphic to ω. (It is an open problem if and when a non-Banach Fréchet space can be written as above. A Banach space which does not decompose has been recently constructed by Gowers and Maurey, The unconditional basic sequence problem, preprint.)

Observe that every closed subspace of a Fréchet space which is Montel or a subspace of $X \times \omega$ must be either Montel or normable or isomorphic to $Y \times \omega$ for some Banach Y (cf. [4]) hence it is Montel or quasinormable, in particular, it has the density condition. It follows from Theorem 1.2 that having all closed separable subspaces with the

density condition is a very strong condition for a Fréchet space. Accordingly, we prove that a Fréchet space is quasinormable (resp. has the density condition) if every separable subspace is contained in a separable quasinormable subspace (resp. in a separable subspace with the density condition), see Proposition 2.1. An example shows that this result does not hold for distinguishedness. To treat the corresponding questions for quotient spaces we restrict our attention to Köthe echelon spaces: Every Fréchet space with a total bounded set appears as a quotient of a suitable echelon space $\lambda_1(I, A)$ with the density condition (see 3.3) and a Köthe echelon space $\lambda_p(A)$, $1 < p < \infty$, is quasinormable if and only if all its quotients have the density condition by 3.2.

1. Density Condition in Subspaces of Fréchet Spaces

We first prove the following technical lemma which gives sufficient conditions to ensure that a Fréchet space E does not satisfy the density condition. The conditions (i), (ii) and (iii) in part (b) appeared in [23] and [2] to study Köthe echelon spaces of order one which are (non-)distinguished.

1. Lemma. (a) *Let E be a Fréchet space with an increasing fundamental sequence of seminorms $(p_n)_{n\in\mathbb{N}}$. Assume we can find $(z_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}} \subset E$ and $(u_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}} \subset E'$ satisfying the following conditions:* (1) $u_{ij}(z_{ij}) = 1$, $\forall i, j\in\mathbb{N}$, (2) $p_n(u_{ij}(x)z_{ij}) \leqslant p_n(x)$, $\forall i, j, n\in\mathbb{N}$, $\forall x\in E$, (3) $p_1(z_{ij}) = 1$, $\forall i, j\in\mathbb{N}$, (4) $M_n := \sup\{p_n(z_{ij});\ i \geqslant n, j\in\mathbb{N}\} < \infty$, $\forall n\in\mathbb{N}$, (5) $\sup\{p_{n+1}(z_{nj}); j\in\mathbb{N}\} = \infty$, $\forall n\in\mathbb{N}$, *then E does not satisfy the density condition.*

(b) *Let E be a Fréchet space with an increasing fundamental sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ and with a basis $(z_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ satisfying the following conditions:* (i) $\|z_{ij}\|_1 = 1$, $\forall i, j\in\mathbb{N}$, (ii) $\sup\{\|z_{ij}\|_n;\ i \geqslant n, j\in\mathbb{N}\} < \infty$, $\forall n\in\mathbb{N}$, (iii) $\sup\{\|z_{nj}\|_{n+1}; j\in\mathbb{N}\} = \infty$, $\forall n\in\mathbb{N}$. *Then E does not satisfy the density condition.*

Proof. (a) We set $U_n := \{x\in E;\ p_n(x) \leqslant 1\}$, $n\in\mathbb{N}$. It is enough to show that

$$\forall m\in\mathbb{N},\quad \forall \lambda > 0\colon \bigcap_{n=1}^{m} M_n U_n \not\subset 2^{-1}U_1 + \lambda U_{m+1},$$

to conclude that E does not satisfy the density condition (see e.g. [10, proof of 4]). Given $m\in\mathbb{N}$, $\lambda > 0$, we apply (5) to select $j(m)\in\mathbb{N}$ such

that $p_{m+1}(z_{mj(m)}) > 3\lambda$. For $1 \leqslant n \leqslant m$ we apply (4) to obtain $p_n(z_{mj(m)}) \leqslant$ $\leqslant M_n$, hence $z_{mj(m)} \in \bigcap_{n=1}^{m} M_n U_n$.

Suppose $z_{mj(m)} = 2^{-1}x + \lambda y$ with $x \in U_1$, $y \in U_{m+1}$. From (1) we get

$$1 = u_{mj(m)}(z_{mj(m)}) = 2^{-1}u_{mj(m)}(x) + \lambda u_{mj(m)}(y).$$

By (3) and (2) we have

$$|u_{mj(m)}(x)| = p_1(u_{mj(m)}(x)z_{mj(m)}) \leqslant p_1(x) \leqslant 1$$

and

$$|\lambda u_{mj(m)}(y)| = \lambda p_{m+1}(z_{mj(m)})^{-1} p_{m+1}(u_{mj(m)}(y)z_{mj(m)}) \leqslant 1/3.$$

This is a contradiction.

(b) Let $\sigma: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be the bijection determining the order of the basis and let $(u_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ be the functionals associated to the basis. For $n \in \mathbb{N}$ we define

$$p_n(x) := \sup\left\{ \left\| \sum_{i=h}^{h+k} x_{\sigma(i)} z_{\sigma(i)} \right\|_n ; h, k \in \mathbb{N} \right\}, \quad \forall x = \sum_{i=1}^{\infty} x_{\sigma(i)} z_{\sigma(i)} \in E.$$

The seminorms $(p_n)_{n\in\mathbb{N}}$ define the topology of E and the conditions (1) to (5) of part (a) are satisfied. ■

If a Fréchet space E is Montel or isomorphic to a subspace of $X \times \omega$ with X a Banach space, then every closed subspace of E satisfies the density condition because it is Montel in the first case and normable, isomorphic to ω or isomorphic to $Y \times \omega$ with Y Banach (hence always quasinormable) in the second case, by a result of [4]. Our main result in this section will show that, under mild assumptions, every Fréchet space which is not of this type must contain closed subspaces with basis not satisfying the density condition.

2. Theorem. *Let E be a Fréchet space which is neither Montel nor isomorphic to a closed subspace of $X \times \omega$, with X a Banach space. If $E = F \oplus G$ with F and G infinite dimensional closed subspaces of E not isomorphic to ω, then E contains a closed subspace with basis and not satisfying the density condition.*

Proof. Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be an increasing fundamental sequence of seminorms that defines the topology of E. We proceed in two steps. First we prove that, with minor changes in the seminorms, F and G

satisfy the following conditions:

(a) F has a basic sequence $(e_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ such that $\|e_{ij}\|_1 = 1$, $\forall i,j\in\mathbb{N}$ and $\lim_{j\to\infty}\|e_{nj}\|_n\|e_{nj}\|_{n+1}^{-1} = 0$, $\forall n\in\mathbb{N}$.

(b) G has a bounded basic sequence $(\bar{e}_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ (i.e. $1 \leqslant \|\bar{e}_{ij}\|_n \leqslant$ $\leqslant M_n$, $\forall i,j,n\in\mathbb{N}$).

In fact, first assume that both F and G are not Montel. One of them, say F, is not isomorphic to a subspace of $X\times\omega$. By [4], F contains a nuclear Köthe space with a continuous norm. The basis of this Köthe space is the one of (a). On the other hand, since G is not Montel, it contains a bounded basic sequence (cf. [3]). Suppose now that F is Montel. Since it is not isomorphic to ω it contains a nuclear Köthe space with a continuous norm (cf. [4]). The space G can not be Montel and we proceed as above.

To conclude, we construct a basic sequence in E to which part (b) of Lemma 1 can be directly applied. Without loss of generality we may assume that both bases are ordered in the same way and that $\|x+y\|_n = \|x\|_n + \|y\|_n$, $\forall x\in F$, $y\in G$, $\forall n\in\mathbb{N}$. If the sequence $\{e_{ij}; i,j\in\mathbb{N}\}\cup\{\bar{e}_{ij}; i,j\in\mathbb{N}\}$ is ordered with e_{ij} in the odd positions and $\bar{e}_{ij}$ in the even ones, keeping the order already given in each of them, we get a basic sequence in E. Defining $z_{ij} := \lambda_{ij}e_{ij} + \mu_{ij}\bar{e}_{ij}$, $i,j\in\mathbb{N}$ (always keeping the order) we form a block basis from a basic sequence, which, in turn, must be a basic sequence. Now we select the scalars λ_{ij}, μ_{ij} in order to apply Lemma 1(b) to obtain that $H := \overline{\text{span}}\,\{z_{ij}; i,j\in\mathbb{N}\}$ does not satisfy the density condition. To do this we put $\lambda_{ij} := \|e_{ij}\|_i^{-1}$ and we choose $0 \leqslant \mu_{ij} < 1$ such that $\lambda_{ij}\|e_{ij}\|_1 + \mu_{ij}\|\bar{e}_{ij}\|_1 = 1$ (only a finite number of μ_{ij}'s could be equal to 0). Accordingly condition (i) of Lemma 1(b) is satisfied.

To verify the other two conditions, we fix $n\in\mathbb{N}$. If $i \geqslant n$, $j\in\mathbb{N}$, we have

$$\|\lambda_{ij}e_{ij} + \mu_{ij}\bar{e}_{ij}\|_n = \|e_{ij}\|_n^{-1}\|e_{ij}\|_n + \mu_{ij}\|\bar{e}_{ij}\|_n \leqslant 1 + M_n,$$

and condition (ii) is satisfied. Finally we have,

$$\|\lambda_{nj}e_{nj} + \mu_{nj}\bar{e}_{nj}\|_{n+1} = \|e_{nj}\|_{n+1}\|e_{nj}\|_n^{-1} + \mu_{nj}\|\bar{e}_{nj}\|_{n+1},$$

which tends to ∞ as j goes to ∞. ■

3. Corollary. *Let E be a Fréchet space which is neither Montel nor isomorphic to a closed subspace of $X\times\omega$, with X a Banach space. Then E contains a closed subspace with basis and not satisfying the density*

condition in the following cases:

(1) *E contains an infinite dimensional Banach space Z,*

(2) *E is reflexive and not separable,*

(3) *E has an unconditional basis, in particular $E = \lambda_p(A)$, for a Köthe matrix A on $\mathbb{N}$, $1 \leqslant p < \infty$ or $p = 0$, or $E = \bigcap_{n\in\mathbb{N}} l_{p+(1/n)}$, $1 \leqslant p < \infty$.*

Proof. (1) Since E is not isomorphic to a subspace of $X \times \omega$ with X normable, we can apply [4] to find a subspace F of E which is a nuclear Köthe space with a continuous norm. (Without loss of generality we may assume that $F \cap Z = \{0\}$.) We may conclude from Theorem 2 if we show that $F + X$ is a closed subspace of E. This follows from a more general result due to DREWNOWSKI [14], which in turn is an extension of a result of ROSENTHAL [19] for Banach spaces. The proof in our present case is very easy and will be given for the sake of completeness in the following observation.

4. Observation. *Let E be a Fréchet space, X and F subspaces of E such that X is normable (let p denote the norm defining the induced topology) and F is Montel. Then $X + F$ is closed in E.* Indeed, we may assume $F \cap X = \{0\}$. We take $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset F$ with $\lim_{n\to\infty}(x_n + y_n) = z \in E$. Assume that $(x_n)_{n\in\mathbb{N}}$ is not bounded. By passing to a subsequence (which will be still denoted $(x_n)_{n\in\mathbb{N}}$), we have $\lim_{n\to\infty} p(x_n) = \infty$. This implies $\lim_{n\to\infty} p(x_n)^{-1}(x_n + y_n) = 0$, hence $(p(x_n)^{-1}y_n)_{n\in\mathbb{N}}$ is a bounded sequence in F, whence it has a convergent subsequence $(p(x_{\sigma(n)})^{-1}y_{\sigma(n)})_{n\in\mathbb{N}}$ with a limit $y \in F$. This implies that $(p(x_{\sigma(n)})^{-1}x_{\sigma(n)})_{n\in\mathbb{N}}$ also converges to some $x \in X$ and $x + y = 0$. Since $F \cap X = \{0\}$, this yields $y = 0 = x = \lim_{n\to\infty} p(x_{\sigma(n)})^{-1}x_{\sigma(n)}$, a contradiction.

Consequently $(x_n)_{n\in\mathbb{N}}$ must be bounded, which implies that $(y_n)_{n\in\mathbb{N}}$ is bounded, and using that F is Montel and proceeding as before we find $y \in F$, $x \in X$ such that $z = x + y \in X + F$. ■

(2) Let E be reflexive and not separable. Since a Fréchet space is Montel if and only if all its separable subspaces are Montel, E contains a separable subspace H which is not Montel. By a result of VALDIVIA [22], there is a complemented separable subspace G of E such that $H \subset G$. We denote by F a complement of G in E. Now G is not isomorphic to ω, since it contains H and F is not separable. The conclusion follows by Theorem 2. The condition of E being reflexive may be relaxed using results of [22].

(3) If E has an unconditional basis we can decompose $E = F \oplus G$ in a way that Theorem 2 can be directly applied. Observe that the space $l_{p+} := \bigcap_{n\in\mathbb{N}} l_{p+1/n}$ is a quasinormable Fréchet space which is not Montel but does not contain any infinite dimensional Banach space by [11]. ■

We close this section with a result on distinguished spaces which extends the main part of [8, Th. 3(ii)].

5. Proposition. *Let E be a Fréchet space containing l_1 but not isomorphic to a subspace of $X \times \omega$ with X a Banach space. Then E contains a non-distinguished closed subspace with basis.*

Proof. By [4] E contains a nuclear Köthe space F with a continuous norm. Without loss of generality we may assume $l_1 \cap F = \{0\}$. According to observation 4, $F + l_1$ is a closed subspace of E and the conclusion follows with the argument given in the proof of [8, Th. 3, part (ii.a)]. ■

Recall that it is an open problem due to Valdivia if every non-distinguished Fréchet space contains l_1.

2. Density Condition, Quasinormability and Distinguishedness in Non-Separable Fréchet Spaces

According to the results in Sect. 1, if E is a Fréchet space such that every separable subspace has the density condition then one could hope for a stronger property on E. (We conjecture that E must be Montel or normable or isomorphic to $X \times \omega$ for some Banach X.) Accordingly we think that it is interesting to study the following weaker problem: *Let E be a Fréchet space such that every separable subspace is contained in a separable subspace with a property (P). Does E satisfy (P)?* It is well known that the answer is "yes" for (P) being reflexivity or Montel property. We prove in this section that the answer is also positive if (P) is the density condition or quasinormability and negative whenever (P) is being distinguished. This should be compared with the following question of Valdivia: Must a Fréchet space be distinguished if every separable subspace is distinguished?

1. Proposition. (a) *Let E be a Fréchet space such that every separable subspace is contained in a separable subspace with the density condition. Then E satisfies the density condition.*

(b) *Let E be a Fréchet space such that every separable subspace is contained in a separable and quasinormable subspace. Then E is quasinormable.*

Proof. (a) If E does not have the density condition we can apply [18, Prop. 1] to conclude

$$\exists(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}(\lambda_k > 0, \forall k\in\mathbb{N})\ \exists n(0)\ \forall m\in\mathbb{N}\ \forall(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}, \exists p\in\mathbb{N},$$

$$\bigcap_{k=1}^{m} \lambda_k U_k \not\subset U_{n(0)} + \bigcap_{j=1}^{p} n_j U_j,$$

where $(U_n)_{n\in\mathbb{N}}$ is a basis of 0-neighbourhoods in E. Now the set

$$\mathscr{A} := \left\{(m, n_1, \ldots, n_p);\ \bigcap_{k=1}^{m} \lambda_k U_k \Big\backslash \left(U_{n(0)} + \bigcap_{j=1}^{p} n_j U_j \right) \neq \varnothing \right\},$$

is countable and for each $a = (m, n_1, \ldots, n_p)\in\mathscr{A}$ there is $x(a)\in E$ with

$$x(a)\in \bigcap_{k=1}^{m} \lambda_k U_k \Big\backslash \left(U_{n(0)} + \bigcap_{j=1}^{p} n_j U_j \right). \tag{1}$$

By (1) and [18, Prop. 1], any separable subspace F that contains $\overline{\operatorname{span}}\{x(a); a\in\mathscr{A}\}$ does not satisfy the density condition. A contradiction.

The same proof works in case (b). One just need to use the characterization of quasinormability given in [17] (see also [7]) to construct a suitable countable family $\mathscr{A}$. ■

The result above does not hold for distinguishedness. This follows from the next Proposition and the fact that there are Köthe echelon spaces $\lambda_\infty(\mathbb{N}, A)$ which are non-distinguished (see [1]).

2. Proposition. *Let S be a separable subspace of a Köthe echelon space $\lambda_\infty(I, A)$. There is a separable and distinguished subspace of $\lambda_\infty(I, A)$ that contains S.*

Proof. Since S is separable it has a total bounded set B (i.e. $S = \overline{\bigcup_{n\in\mathbb{N}} nB}$). Given B there is a sequence of pairwise disjoint sets $(I_n)_{n\in\mathbb{N}}$, $I_n \subset I$ $(n\in\mathbb{N})$ such that the sectional subspace $\lambda_\infty(I_n, A) := \{f\cdot\chi_{I_n}; f\in\lambda_\infty(I, A)\}$ is normable, $\forall n\in\mathbb{N}$, and

$$x = \lim_{n\to\infty} \sum_{j=1}^{n} x\cdot\chi_{I_j}, \quad \forall x\in B, \tag{1}$$

(see [12] or [13, Th. 5]). The space $Z_n := \overline{\text{span}}\{x \cdot \chi_{I_n}; x \in B\} \subset \subset \lambda_\infty(I_n, A)$ is separable and normable, for all $n \in \mathbb{N}$. We now construct

$$c_0(Z_n) := \left\{ (x_n) \in \prod_{n \in \mathbb{N}} Z_n; \lim_{n \to \infty} \| x_n \|_k = 0, \forall k \in \mathbb{N} \right\}.$$

With the sup-norm topology, $c_0(Z_n)$ is a separable and distinguished (cf. [13]) Fréchet space which can be identified with a subspace of $\lambda_\infty(I, A)$ as follows:

$$\psi : c_0(Z_n) \to \lambda_\infty(I, A), \quad (x_n) \to \sum_{n=1}^{\infty} x_n \cdot \chi_{I_n}.$$

Moreover, on account of (1), $\psi(c_0(Z_n))$ contains B, therefore it also contains S. ∎

Observe that, by our Proposition 5 in Sect. 1, $\lambda_\infty(I, A)$ always contains a non-distinguished subspace with basis, unless it is Montel or normable or isomorphic to $l_\infty \times \omega$.

3. Density Condition in Quotients of Köthe Spaces $\lambda_p(A)$, $1 < p < \infty$, $p = 0$.

In this section we restrict our attention to Köthe echelon spaces $\lambda_p(A)$ of order $1 < p < \infty$ or $p = 0$, for a Köthe matrix $A = (a_n)_{n \in \mathbb{N}}$ on the set of natural numbers $\mathbb{N}$. (The case $p = 1$ was completely clarified in [8].) Our notation for echelon and co-echelon spaces is as in [6]. We recall that A satisfies (a) $0 \leqslant a_n(i) \leqslant a_{n+1}(i)$, $\forall i, n \in \mathbb{N}$, and (b) $\forall i \in \mathbb{N} \exists n \in \mathbb{N}$: $a_n(i) > 0$.

We start with a preliminary lemma which is the version for $1 < p < \infty$ or $p = 0$ of a useful lemma due to WAGNER (see e.g. [24]). This lemma should also be compared with [16, II. p. 22].

1. Lemma. *Let $A = (a_n)_{n \in \mathbb{N}}$ be a Köthe matrix on $\mathbb{N} \times \mathbb{N}$ such that* $\forall n \in \mathbb{N} \exists i(n) \in \mathbb{N}$:

$$b_k(j) := \inf\{a_k(i,j); i \in \mathbb{N}\} = \min\{a_k(i,j); 1 \leqslant i \leqslant i(k)\}.$$

Then the Köthe echelon space $\lambda_p(B)$, where $B = (b_n(j))_{n \in \mathbb{N}}$, is a quotient of $\lambda_p(A)$, $1 < p < \infty$ or $p = 0$.

Proof. We only treat the case $1 < p < \infty$. The case $p = 0$ needs minor modifications. We may assume $a_1(i,j) = 1$, $\forall i, j \in \mathbb{N}$. This implies the convergence of all the series $\sum_{i=1}^{\infty} 2^{-i} x_{ij}$ for every

$x = (x_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in \lambda_p(A)$. We define

$$\phi:\lambda_p(A)\to\lambda_p(B),\quad x=(x_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}\to\left(\sum_{i=1}^{\infty}2^{-i}x_{ij}\right)_{j\in\mathbb{N}},$$

$$\tilde{\phi}:\lambda_p(B)'\to\lambda_p(A)',\quad u=(u_j)_{j\in\mathbb{N}}\to(2^{-i}u_j)_{(i,j)\in\mathbb{N}\times\mathbb{N}}.$$

It follows that $\tilde{\phi}$ is well-defined and hence ϕ is also well-defined and $\tilde{\phi}=\phi^t$. It is easy to see that ϕ is continuous and has dense range. For the continuity one may use the closed graph theorem.

By the homomorphism theorem, to conclude that ϕ is a quotient map it is enough to show that $\tilde{\phi}^{-1}(C)$ is a bounded set of $\lambda_p(B)'_b$ for every bounded subset C of $\lambda_p(A)'_b$. If C is a bounded subset of $\lambda_p(A)'_b$, we can find $n(0)\in\mathbb{N}$, $\lambda>0$, such that

$$\forall v=(v_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}\in C:\ \sum_{i,j=1}^{\infty}|v_{ij}a_{n(0)}(i,j)^{-1}|^q\leqslant\lambda,$$

(see [6]). If $u=(u_j)_{j\in\mathbb{N}}\in\tilde{\phi}^{-1}(C)$ we conclude, for $1\leqslant i\leqslant i(n(0))$,

$$\sum_{j=1}^{\infty}|u_j a_{n(0)}(i,j)^{-1}|^q\leqslant\lambda 2^{iq},$$

which implies

$$\sum_{j=1}^{\infty}|u_j b_{n(0)}(j)|^q\leqslant\lambda 2^{i(n(0))q},$$

and $\tilde{\phi}^{-1}(C)$ is bounded in $\lambda_p(B)'_b$. ■

Every separated quotient of a quasinormable (Fréchet) space is again quasinormable. We show that, within the class of $\lambda_p(A)$ spaces, $1<p<\infty$ or $p=0$, quasinormable spaces are the only ones whose quotients have the density condition.

2. Proposition. *Every space $\lambda_p(A)$, $1<p<\infty$ or $p=0$, which is not quasinormable has a quotient with basis and without the density condition.*

Proof. Proceeding exactly as we did in the proof of [8, Th. 3(i)], using that the matrix A is not regularly decreasing (cf. [6]), $\lambda_p(A)$ has a sectional subspace isomorphic to $\lambda_p(\mathbb{N}^3,\bar{A})$ for a Köthe matrix $\bar{A}=(\bar{a}_n)_{n\in\mathbb{N}}$ satisfying:

(1) $\bar{a}_1(i,j,k)=1$, $\forall i,j,k\in\mathbb{N}$, (2) $\lim_{j\to\infty}\bar{a}_n(n-1,j,k)=\infty$, $\forall n,k\in\mathbb{N}$,

$\forall n \geqslant 2$, and (3) $\forall i \geqslant n$, $\exists \alpha_{n,i} \geqslant 0$: $\bar{a}_n(i,j,k) \leqslant \alpha_{n,i}$, $\forall j,k \in \mathbb{N}$. To simplify we denote $\alpha_i := \alpha_{i,i}$, $i \in \mathbb{N}$, and we assume without loss of generality $\alpha_1 = 1$, $\alpha_i^2 \leqslant \alpha_{i+1}$, $\forall i \in \mathbb{N}$. Now we define

$$c(i,j,k) := \begin{cases} \min\{\bar{a}_{k+1}(s,j,k): \quad 1 \leqslant s \leqslant k\}, & i > k \\ 1, & i \leqslant k \end{cases}$$

and we construct a new matrix $A^* = (a_n^*)_{n\in\mathbb{N}}$ by setting

$$a_n^*(i,j,k) := \alpha_i \bar{a}_n(i,j,k) c(i,j,k), \quad i,j,k \in \mathbb{N}.$$

It is easy to see that $\lambda_p(A^*) = \lambda_p(\bar{A})$. Finally we put $B = (b_n)_{n\in\mathbb{N}}$,

$$b_n(j,k) := \inf\{a_n^*(i,j,k);\ i \in \mathbb{N}\},\ i,j,n \in \mathbb{N}.$$

We check that the conditions of Lemma 1 are satisfied. First it is easy to see that $b_1(j,k) = 1 = \min\{a_1^*(i,j,k);\ i = 1\}$. For $b_{n+1}(j,k)$, $j,k \in \mathbb{N}$, we distinguish three cases:

(a) $k \geqslant n+1$. For $r \geqslant n+2$, $j \in \mathbb{N}$, we have

$$\min\{a_{n+1}^*(i,j,k);\ 1 \leqslant i \leqslant n+1\} \leqslant \alpha_{n+1}^2 \leqslant \alpha_{n+2} \leqslant \alpha_r \leqslant a_{n+1}^*(r,j,k).$$

(b) $k < n$. Since $c(r,j,k) = c(s,j,k)$ $\forall r,s > k$, we have for $r \geqslant n+2$, $j \in \mathbb{N}$,

$$\begin{aligned} \min\{a_{n+1}^*(i,j,k);\ 1 \leqslant i \leqslant n+1\} &\leqslant \min\{a_{n+1}^*(i,j,k);\ k < i \leqslant n+1\} = \\ &= c(i,j,k)\alpha_{n+1}^2 \leqslant c(r,j,k)\alpha_r \leqslant \\ &\leqslant a_{n+1}^*(r,j,k). \end{aligned}$$

(c) $k = n$. For $r > n$ and $j \in \mathbb{N}$ we have,

$$\begin{aligned} &\min\{a_{n+1}^*(i,j,n);\ 1 \leqslant i \leqslant n\} = \min\{\alpha_i \bar{a}_{n+1}(i,j,n);\ 1 \leqslant i \leqslant n\} \leqslant \\ &\leqslant \alpha_n \min\{\bar{a}_{n+1}(i,j,n);\ 1 \leqslant i \leqslant n\} \leqslant a_{n+1}^*(r,j,n). \end{aligned}$$

These three cases imply

$$b_{n+1}(j,k) = \min\{a_{n+1}^*(i,j,k);\ 1 \leqslant i \leqslant n+1\}, \quad \forall j,k \in \mathbb{N},$$

which yields that $\lambda_p(\mathbb{N} \times \mathbb{N}, B)$ is a quotient of $\lambda_p(\mathbb{N}^3, A^*)$.

To conclude it remains to show that $\lambda_p(\mathbb{N} \times \mathbb{N}, B)$ does not satisfy the density condition. This follows directly from Lemma 1(b) and the following estimates that show that the conditions there are satisfied (also compare with [2, 5] and [23]).

(i) $b_n(j,k) \leqslant a_n^*(n,j,k) \leqslant \alpha_n^2$, $\forall k \geqslant n$, $\forall j \in \mathbb{N}$.

(ii) By case (c) above

$$b_{n+1}(j,n) = \min\{\alpha_i \bar{a}_{n+1}(i,j,n);\ 1 \leqslant i \leqslant n\},$$

and $\bar{a}_{n+1}(i,j,n) \geqslant \bar{a}_{i+1}(i,j,n)$ which converges to ∞ as j does, for $1 \leqslant i \leqslant n$. Thus $\lim_{j\to\infty} b_{n+1}(j,n) = \infty$.

The proof is complete. ■

We close this section with a result of universal type which characterizes the Fréchet spaces that appear as a quotient of a Köthe echelon space of order one that satisfies the density condition (or equivalently that is distinguished, [5]). Our result is motivated by the following two (classical) ones: (i) It is well-known that every Banach space is a quotient of $l_1(I)$ for a suitable index set I, (ii) every separable Fréchet space is a quotient of a Köthe echelon space $\lambda_1(A)$ which is Montel and has a continuous norm (this is a result of Valdivia and Wagner, see e.g. [20]).

3. Proposition. *A Fréchet space F is a quotient of a suitable Köthe echelon space $\lambda_1(I,A)$ satisfying the density condition if and only if F has a total bounded set.*

Proof. Every Fréchet space with the density condition has a total bounded set and this property is inherited by separated quotients (see [13]). Thus the necessity of the condition follows. We recall that there are examples of reflexive Fréchet spaces which do not have a total bounded set (cf. [16, I. p. 404]).

To complete the proof we fix a Fréchet space F with an increasing fundamental system of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ and a total bounded set B and proceed in several steps.

(a) We construct $\lambda_1(I,A)$ with a total bounded set and a quotient map $Q: \lambda_1(I,A) \to F$. For each $k\in\mathbb{N}$ we put,

$$J_k := \{x \in kB;\ \|x\|_k \geqslant k^{-1}\}, \quad I_k := J_{k+1} \setminus J_k, \quad k\in\mathbb{N},$$

$$I = \bigcup_{k\in\mathbb{N}} I_k = \bigcup_{k\in\mathbb{N}} kB.$$

Clearly I is dense in F. We define for $i\in I$, $n\in\mathbb{N}$ $a_n(i) := \|i\|_n$, and

$$Q: \lambda_1(I,A) \to F,\ Q(\alpha) := \sum_{i\in I} \alpha_i i, \quad \text{for every } \alpha = (\alpha_i)_{i\in I} \in \lambda_1(I,A).$$

It is easy to see that Q is a quotient map onto F. In order to see that $\lambda_1(I,A)$ has a total bounded set it is enough to prove that every sectional subspace $\lambda_1(I_n,A)$, $n\in\mathbb{N}$, is normable (see [13]). This is clear:

if $m\in\mathbb{N}$, $i\in I_n$ we get, with $M_k := \sup\{\|x\|_k; x\in B\}$, $(k\in\mathbb{N})$,

$$a_m(i) = \|i\|_m \leqslant nM_m \leqslant n^2 M_m \|i\|_n = n^2 M_m a_n(i),$$

thus a_n defines the topology induced on $\lambda_1(I_n, A)$.

(b) We construct a Köthe echelon space $\lambda_1(I\times\mathbb{N}, B)$ with the density condition such that $\lambda_1(I, A)$ is a quotient of it.

In order to simplify the notation we write (i,k) for the elements of I_k. And we define $B = (b_n)_{n\in\mathbb{N}}$ on $I\times\mathbb{N}$ by

$$b_n(i,k,j) := \begin{cases} k^n a_n(i,k), & \text{if } j < n \\ n^{j+n} a_n(i,k), & \text{if } j \geqslant n. \end{cases}$$

Since $\inf_{j\in\mathbb{N}} b_n(i,k,j) = \min\{k^n a_n(i,k),\ n^{2n} a_n(i,k)\} = n^{2n} a_n(i,k)$ for k large enough and $i\in I_k$, we can apply the lemma of WAGNER [24, 1.2] to get that $\lambda_1(I\times\mathbb{N}, B)$ has a quotient isomorphic to $\lambda_1(I, A)$.

It remains to show that $\lambda_1(I\times\mathbb{N}, B)$ has the density condition. We use the characterization in [5]. To do this we define an increasing sequence $J = (X_p)_{p\in\mathbb{N}}$ of subsets of $I\times\mathbb{N}$ by

$$X_p := \{(i,k,j);\ i\in I_k,\ k\leqslant p, j\leqslant p\},\ p\in\mathbb{N}.$$

First observe that, since a_p dominates all the a_n's on $I_1\cup\cdots\cup I_p$, the definition of b_n permits to find $\delta(n,p) > 0$ such that

$$b_n(i,k,j) \leqslant \delta(n,p) b_p(i,k,j), \quad \forall (i,k,j)\in X_p.$$

This shows that condition (N, J) in (D) is satisfied.

To complete the proof one checks condition (M, J): We take $I_0 := \{(i,k_j,m_j);\ i\in I_{k_j}, k_j, m_j\in\mathbb{N}, j\in\mathbb{N}\}$ satisfying $I_0\cap(I\setminus X_p)\neq\varnothing, \forall p\in\mathbb{N}$, and fix $n\in\mathbb{N}$. It is easy to see that there is $n' > n$ such that $\inf_{I_0}(b_n/b_{n'}) = 0$. One just need to choose $n' > \max(n, M)$ if $\sup\{m_j; (i,k_j,m_j)\in I_0\} =: M < \infty$, and $n' = n+1$ in other cases. ■

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Note added in proof

(i) By reducing the general case to the one treated in Thm. 1.2, ÖNAL and TERZIOGLU have recently proved that every closed subspace of a Fréchet space E has the density condition if and only if E is either Montel or isomorphic to $\mathbb{K}^J\times$ Banach for some set J. (A characterization of a class of Fréchet spaces and the density condition, Arch. Math. **61**, 257–259 (1993).)

(ii) The second named author has constructed in the paper Two problems of Valdivia on distinguished Fréchet spaces, Manuscripta Math. **79**, 403–410 (1993), the following spaces: (a) a Fréchet space E with basis which is non-distinguished and contains no copy of l_1 (see the end of Sect. 1) and (b) a non-distinguished Fréchet space F whose separable subspaces are distinguished (see the introduction of Sect. 2.)

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