Werk

Titel: Rigidity and the Alexandrov-Fenchel Inequality.

Autor: Filliman, P.

Jahr: 1992

PURL: https://resolver.sub.uni-goettingen.de/purl?362162050_0113|log4

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Mh. Math. 113, 1—22 (1992)

Monatshefte für
Mathematik
© by Springer-Verlag 1992
Printed in Austria

Rigidity and the Alexandrov-Fenchel Inequality

By

P. Filliman, Bellingham, WA

(Received 29 October 1990; in revised form 9 October 1991)

Abstract. New proofs are given for Cauchy's and Alexandrov's classical theorems on the rigidity of polyhedral frameworks, as well as their higher dimensional generalizations. Through duality, the rigidity of these frameworks follows from characterizations of the case of equality in Minkowski's quadratic inequality.

1. Introduction

A connection between the theories of rigidity and mixed volumes was first noticed by ALEXANDROV in his classical text on convex polytopes ([1], chapter XI, § 3). He proves Minkowski's theorem for 3-polytopes by induction and the Brunn-Minkowski theorem in the plane. Minkowski's theorem states that a convex polytope is uniquely determined (up to translation) by the normals and volumes of its facets, and can thus be interpreted as a kind of rigidity result. Using this interpretation, he then shows that a 3-polytope with rigid faces is rigid (for definitions of various types of rigidity, see Section 2).

LEE [25] has recently discovered that rigidity and mixed volumes can also be linked through the notion of polar duality. Using this approach, we can extend Alexandrov's result in two ways. First, in Section 3 we prove that the 1-skeleton of a simplicial convex polytope A^* in $\mathbb{R}^n$, $n \geqslant 3$, is infinitesimally rigid. Our proof also relies on the Brunn-Minkowski theorem, but it differs from Alexandrov's method in that we work with volumes of faces of the polar dual A instead of those of A^*.

Section 4 contains the second generalization, which concerns non-simplicial polytopes. We prove that a framework obtained by trian-

gulating the 2-faces of an arbitrary n-dimensional polytope, $n \geqslant 3$, is infinitesimally rigid. The proof of this result for $n = 3$ is also due to ALEXANDROV (see [1, chapter X] or [4]), but he uses techniques similar to Cauchy's original arguments instead of mixed volumes. Although the results in Sections 3 and 4 are known (see [42]), our methods may prove useful in some unsolved problems in combinatorics, mixed volumes and rigidity (see Section 6).

Our work was motivated by a result of KALAI [22] which gave a new proof of BARNETTE's lower bound theorem ([5]) using the theorems above on rigidity. The results in Sections 3 and 4 prove these rigidity theorems with techniques from mixed volumes, and therefore give yet another proof of the lower bound theorem.

Instead of trying to summarize the details of our duality approach, we describe a simple example in $\mathbb{R}^3$ which will illustrate the basic idea. Let A^* be a simplicial polytope in $\mathbb{R}^3$ with $0 \in \operatorname{int} A^*$, and let $A = \{q \in \mathbb{R}^3 \mid \langle p, q \rangle \leqslant 1, \forall p \in A^*\}$ be its polar dual. Cauchy's theorem (see [28], chapter 3) implies that the 1-skeleton of A^* is infinitesimally rigid. However, if we add a point p in the interior of a 2-face σ of A, and join p to the vertices of σ, the resulting framework F is not infinitesimally rigid. This follows because the motion of p along an outer normal to σ does not produce any first-order change in the lengths of bars of F.

Let H_p denote the plane orthogonal to p which supports A at the vertex dual to σ. Then the motion of p outward corresponds to an inward motion of H_p which slices off a corner of A. The resulting polytope has A as a "cap body" (see [7], § 3.12). In $\mathbb{R}^3$, cap bodies are precisely the exceptional (i.e., non-homothetic) cases of equality in Minkowski's quadratic inequality for mixed volumes ([17]).

I am deeply indebted to Carl Lee for pointing out the connection between mixed volumes and rigidity, and to Bob Connelly for his patience during my struggles to grasp the seemingly endless types of rigidity. I would also like to thank the referee for his comments, especially in regard to Section 5.

2. Rigidity

This section contains an outline of the basic notions of rigidity (see [3], [4], [9], [19] and [34]). A framework F in $\mathbb{R}^n$ is an ordered collection of v points $\{p_1, \ldots, p_v\}$ which span $\mathbb{R}^n$, together with a set E of pairs of

these points called bars. We are interested primarily in the case when the points and bars correspond to the vertices and edges of a convex polytope A^*, which we denote by $\mathrm{skel}_1 A^*$. The framework is said to be rigid if every continuous motion of the points keeping all the bars of fixed length is the restriction of an isometry of $\mathbb{R}^n$.

The first-order constraint on a bar $\{p_i, p_j\}$ under such motions is

$$\langle p_j - p_i, p_j' - p_i' \rangle = 0. \tag{2.1}$$

We say that $p' = \{p_1', \ldots, p_v'\}$ is an infinitesimal flex of F if (2.1) holds for each bar of F. The infinitesimal flexes are the kernel of an $|E| \times vn$ matrix $\mathbf{R}(p)$ whose $(ij)^{\text{th}}$ row is

$$[0, \ldots, 0, \overbrace{p_i - p_j}^{i}, 0, \ldots, 0, \overbrace{p_j - p_i}^{j}, 0, \ldots, 0]. \tag{2.2}$$

The framework F is said to be infinitesimally rigid if the only infinitesimal flexes are trivial, i.e., first derivatives of direct (or continuous) isometries. If the affine span of F has dimension n, then the space of trivial infinitesimal flexes is $\binom{n+1}{2}$. Thus, F is infinitesimally rigid if and only if rank $\mathbf{R}$ is $vn - \binom{n+1}{2}$. In this setting, the inverse function applies to show that the infinitesimal rigidity of F implies the rigidity of F.

An alternate way to establish the infinitesimal rigidity of F is to show dim coker $\mathbf{R}$ is $|E| - vn + \binom{n+1}{2}$. An element ω in coker $\mathbf{R}$ is called a self-stress on F since it satisfies the equilibrium condition

$$\sum_{j=1}^{v} \omega_{ij}(p_j - p_i) = 0 \tag{2.3}$$

at each point p_i.

Suppose now that $F = \mathrm{skel}_1 A^*$, where A^* is a simplicial polytope in $\mathbb{R}^n$. In order to introduce mixed volumes into rigidity, it will be necessary to define some auxiliary spaces (see [25]):

$$\mathscr{C}^1 := \left\{(c_i) \in \mathbb{R}^v \mid \sum_{i=1}^{v} c_i p_i = 0\right\} \tag{2.4}$$

$$\mathscr{C}^2 := \left\{(c_{ij}) \in \mathbb{R}^{v^2} \mid \sum_{j=1}^{v} c_{ij} p_j = 0,\ c_{ij} = c_{ji}, \text{ and } c_{ij} = 0 \text{ if } \{i, j\} \notin E\right\}. \tag{2.5}$$

The space $\mathscr{C}^1$ is the set of linear relations on $\{p_1, \ldots, p_v\}$. The second space $\mathscr{C}^2$ is isomorphic to the space of self-stresses on the cone over F.

To see this, we rewrite the sum in (2.5) as

$$0 = \sum_{j=1}^{v} c_{ij}\, p_j = \sum_{j \neq i} c_{ij}\, p_j + c_{ii}\, p_i = \sum_{j \neq i} c_{ij}(p_j - p_i) + \left(c_{ii} + \sum_{j \neq i} c_{ij}\right) p_i. \tag{2.6}$$

The last part of (2.6) is precisely the equilibrium condition for the framework F^0 consisting of the edges of A^*, together with bars joining each vertex to the origin. We shall call F^0 simply the cone over F.

The framework F^0 is easily seen to be infinitesimally rigid since it is a collection of non-degenerate, and therefore rigid, n-simplices joined along $(n-1)$-faces (see [9]). It follows that

$$\dim \mathscr{C}^2 = (|E| + v) - (v+1)n + \binom{n+1}{2}. \tag{2.7}$$

This dimension can also be obtained using techniques of commutative algebra (see [37]).

Inside $\mathscr{C}^2$ is the subspace of self-stresses on the original 1-skeleton F of A^*. We can identify this subspace as the kernel of a linear map

$$\cdot\mathbf{e}\colon \mathscr{C}^2 \to \mathscr{C}^1 \tag{2.8}$$

$$\mathbf{C} \mapsto \mathbf{C}\mathbf{e},$$

where $\mathbf{C} := (c_{ij})$ is the $v \times v$ symmetric matrix associated to a stress in $\mathscr{C}^2$, and $\mathbf{e} = (1, 1, \ldots, 1)^t$. The matrix product $\mathbf{C}\mathbf{e} = 0$ if and only if

$$\forall i, \quad \sum_{j=1}^{v} c_{ij} = 0. \tag{2.9}$$

If (2.9) holds, then the coefficient on the bar joining p_i to the origin in (2.6) is 0. This shows that the stress $\mathbf{C}$ is supported on F as claimed.

Proposition 1. *The 1-skeleton of A^* is infinitesimally rigid if and only if $\cdot\mathbf{e}\colon \mathscr{C}^2 \to \mathscr{C}^1$ is onto.*

Proof. From the discussion above, A^* is infinitesimally rigid if and only if the dimension of its stress space, $\ker \cdot\mathbf{e}$, is $|E| - vn + \binom{n+1}{2}$. By (2.7), this number is

$$\dim \mathscr{C}^2 - (v - n) = \dim \mathscr{C}^2 - \dim \mathscr{C}^1,$$

and the proposition follows. ∎

3. Cauchy's Theorem

Suppose P and Q are combinatorially isomorphic 3-polytopes. Cauchy's theorem states that if a combinatorial isomorphism between P and Q defines a congruence between corresponding faces, then the isomorphism extends to a congruence of P and Q. In case P is simplicial, Cauchy's methods work much more easily to show that P is infinitesimally rigid (see [15], [24], [28] and [39]). The infinitesimal rigidity of simplicial n-polytopes follows from the 3-dimensional case by standard induction argument (see [32]). The goal of this section is to give a direct proof of the infinitesimal rigidity of simplicial n-polytopes, $n \geqslant 3$, based on the Brunn-Minkowski theorem.

Let A^* be a simplicial polytope in $\mathbb{R}^n$ with vertices $\{p_1, \ldots, p_v\}$ and $0 \in \operatorname{int} A^*$. The motivation for our proof of the infinitesimal rigidity of A^* comes from Minkowski's theorem. Given v real numbers $V_1, \ldots, V_v > 0$ which satisfy

$$\sum_{i=1}^{v} V_i p_i / |p_i| = 0, \tag{3.1}$$

Minkowski's theorem states that there is a unique (up to translation) polytope X for which $\{p_i, i = 1, \ldots, v\}$ are the normals to its facets, and V_i is the volume of the i^{th} facet ([7], § 13). Conversely, any polytope X in the class $\mathscr{P}$ of polytopes in $\mathbb{R}^n$ with facet normals $\{p_i\}$ always satisfies (3.1).

Comparing (3.1) and (2.4), one sees that Minkowski's theorem implies $\mathscr{C}^1 \cap \mathbb{R}^v_+$ is identical to the set of facet volumes obtained from polytopes in $\mathscr{P}$. If $X \in \mathscr{P}$ is a combinatorial dual to A^*, then the easy part of Minkowski's theorem also implies the volumes of the $(n-2)$-faces of X give an element in $\mathscr{C}^2$. A more complicated argument based on shelling (see [26]) shows that these elements form a subvariety whose linear span is $\mathscr{C}^2$.

In order to prove A^* is infinitesimally rigid, it suffices to show that the image of this subvariety under ·e spans $\mathscr{C}^1$. This will be accomplished by appealing to the theory of mixed volumes for polytopes.

For convenience in merging the fields of mixed volumes and rigidity, we slightly modify the $\mathscr{C}$-spaces. We define isomorphic subspaces

$$\mathscr{D}^1 := \left\{(d_i) \in \mathbb{R}^v \mid \sum_{i=1}^{v} d_i u_i = 0\right\} \tag{3.2}$$

$$\mathscr{D}^2 := \left\{(d_{ij}) \in \mathbb{R}^{v^2} \mid \sum_{j=1}^{v} d_{ij} u_j = 0,\ d_{ji} = d_{ji}, \text{ and } d_{ij} = 0 \text{ if } \{i, j\} \notin E\right\}, \tag{3.3}$$

where $u_i := p_i/|p_i|$. These spaces can be thought of as stress spaces on the same underlying graph as A^*, but whose points lie on the unit sphere. The relationships between these spaces can be summarized in the commutative diagram

$$\begin{array}{ccc} \mathscr{C}^2 & \to & \mathscr{D}^2 \\ \cdot e \downarrow & & \downarrow \cdot a \\ \mathscr{C}^1 & \to & \mathscr{D}^1 \end{array}$$

The horizontal mappings are the linear isomorphisms $c \mapsto d$ given by

$$d_i = c_i/a_i \quad \text{and} \quad d_{ij} = c_{ij}/a_i a_j, \quad a_i := 1/|p_i|, \tag{3.4}$$

and $\cdot a$ is matrix multiplication of the $v \times v$ matrix $\mathbf{D} := (d_{ij})$ by $a := (a_1, \ldots, a_v)^t$. Carrying out this multiplication gives

$$\mathbf{D}\, a = (d_{ij})\, a = (c_{ij}/a_i a_j)\, a = \left(\sum_{j=1}^{v} c_{ij} a_j / a_i a_j \right) = \left(\sum_{j=1}^{v} c_{ij} \right) \Big/ a_i,$$

which shows the diagram commutes. Thus the infinitesimal rigidity of A^* is equivalent to $\cdot a$ being onto.

We next describe in more detail the class $\mathscr{P}$ of polytopes with fixed facet normals $\{p_1, \ldots, p_v\}$. By choosing the origin at the centroid of $\{p_i\}$, we may assume

$$0 = \sum_{i=1}^{v} p_i = \sum_{i=1}^{v} \frac{u_i}{a_i}. \tag{3.5}$$

Each unit vector u_i has a family of orthogonal hyperplanes

$$\begin{aligned} H_i(\alpha) &:= \{z \in \mathbb{R}^n \mid \langle u_i, z \rangle = \alpha\}, \\ H_i &:= H_i(a_i). \end{aligned} \tag{3.6}$$

We may associate to any $x \in \mathbb{R}^v$ the set $X = P(x)$ given by

$$P(x) := \{z \in \mathbb{R}^n \mid \langle u_i, z \rangle \leqslant x_i, \ \forall i \in \{1, \ldots, v\}\} = \bigcap_{i=1}^{v} H_i(x_i)^-. \tag{3.7}$$

The faces of X will be denoted by

$$X_\lambda := X \cap \left\{ \bigcap_{i \in \lambda} H_i(x_i) \right\}. \tag{3.8}$$

The polytope $A = P(a)$ is the usual polar dual of A^*. Because of (3.5), $P(x)$ is always bounded and thus a polytope. Using properties of

dual cones [18], or linear programming [26], one can give a precise characterization of the possibilities for $P(x)$:

$$P(x) = \emptyset \text{ if and only if } \exists\, \alpha \in \mathbb{R}^v_+ \text{ such that } \sum_{i=1}^v \alpha_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^v \alpha_i x_i < 0. \tag{3.9}$$

$$P(x) \text{ is a point if and only if } x \text{ belongs to the row space of } \mathbf{U}, \text{ where} \tag{3.10}$$

$$\mathbf{U} := \begin{bmatrix} u_{11} & \cdots & u_{v1} \\ \vdots & & \vdots \\ u_{1n} & \cdots & u_{vn} \end{bmatrix}.$$

The correspondence above between points in $\mathbb{R}^v$ and polytopes in $\mathscr{P}$ gives rise to a partition of $\mathbb{R}^v$. Two points x and y are identified if $P(x)$ is (strongly) combinatorially isomorphic to $P(y)$, written $P(x) \approx P(y)$). The equivalence classes satisfy

Proposition 2. *If $P(x) \neq \emptyset$, then $\{y \mid P(y) \approx P(x)\}$ is a convex cone with apex at the origin.*

This proposition follows from some useful relationships between normal fans and Minkowski sums (see [36]). If X is a polytope with $0 \in \operatorname{int} X$, then its normal fan $\mathscr{N}(X)$ equals the collection of polyhedral cones which are the positive hulls over faces of the polar dual X^*. The Minkowski sum of two sets X and Y is

$$X + Y = \{p + q \mid p \in X \text{ and } q \in Y\}.$$

If X and Y are polytopes, then

$$\mathscr{N}(X + Y) = \mathscr{N}(X) \cap \mathscr{N}(Y), \tag{3.11}$$

where $\mathscr{N}(X) \cap \mathscr{N}(Y)$ contains the intersections of all cones in $\mathscr{N}(X)$ and $\mathscr{N}(Y)$. The key property we need is

$$\forall\, \alpha, \beta \geqslant 0, \quad \alpha P(x) + \beta P(y) \subseteq P(\alpha x + \beta y), \tag{3.12}$$

with equality if $\mathscr{N}(P(x)) \prec \mathscr{N}(P(y))$ ($\mathscr{N}(P(x))$ is refined by $\mathscr{N}(P(y))$), or $\mathscr{N}(P(y)) \prec \mathscr{N}(P(x))$.

Clearly if $P(x) \approx P(y)$, then their normal cones are identical so equality holds in (3.12). Proposition 2 follows immediately. In case $P(x)$ is simple, there is an open neighborhood about x for which all the corresponding polytopes are isomorphic to $P(x)$. Even when $P(x)$ is not simple, a small perturbation of $H_i(x_i)$ gives rise to a refinement of $\mathscr{N}(P(x))$ and equality in (3.12).

We now list some basic facts from the theory of mixed volumes (see [2], [7], [8], [27] or [31]). Mixed volumes are defined as coefficients in the (homogeneous) polynomial expansion of the volume of a Minkowski sum $\alpha_1 X^1 + \ldots + \alpha_n X^n$, $\alpha_i > 0$, of convex bodies in $\mathbb{R}^n$;

$$V(\alpha_1 X^1 + \ldots + \alpha_n X^n) = \sum_{i_1, \ldots, i_n \in \{1, \ldots, n\}} V(X^{i_1}, \ldots, X^{i_n})\, \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n}. \tag{3.13}$$

The mixed volume $V(X^1, \ldots, X^n)$ is chosen to be invariant under permutations of its arguments. Moreover, it is monotone, linear, and translation invariant in each argument. Also

$$V(X^1, \ldots, X^n) \geqslant 0, \tag{3.14}$$

and is non-zero if and only if there exists a set of line segments $L^1, \ldots, L^n$, $L^i \in X^i$, such that $\dim(L^1 + \ldots + L^n) = n$.

Our main tool below will be the following asymmetrical expansion for $V(X^1, \ldots, X^n)$. Let $X = X^2 + \ldots + X^n$ and let $v_1, \ldots, v_m$ be the outer normals to the facets of X. If L_i^k is the hyperplane supporting X^k with outer normal v_i, we define $X_i^k := X^k \cap L_i^k$ and $h(X^k, v_i)$ as the distance from the origin to L_i^k. Then

$$V(X^1, \ldots, X^n) = \frac{1}{n} \sum_{i=1}^{m} h(X^1, v_i)\, V_i(X^2, \ldots, X^n), \tag{3.15}$$

where $V_i(X^2, \ldots, X^n)$ is the $(n-1)$-dimensional mixed volume of the parallel faces $X_i^2, \ldots, X_i^n$ ([31], 6B4).

In the special case when $X^i = P(x^i)$, $i = 1, \ldots, n$, is a family of strongly combinatorially isomorphic polytopes, then

$$V(X^1, \ldots, X^n) = \frac{1}{n} \sum_{i=1}^{v} x_i^1\, V_i(X^2, \ldots, X^n), \tag{3.16}$$

ALEXANDROV ([1], § 1) has used induction and (3.16) to expand $V(X^1, \ldots, X^n)$ as a homogeneous polynomial in $\{x_i^k\}$.

Recall that $A = P(a)$ is the usual polar dual of A^* since $a_i = 1/|p_i|$. Since A^* is simplicial, A is simple and so small movements of the hyperplanes H_i bounding A will not change its combinatorial type. We wish to determine how the volume of A changes under this disturbance.

Let $x \in \mathbb{R}^v$ be near a, and let $t := x - a$. Given $0 \leqslant \alpha \leqslant 1$, (3.12) implies

$$P(a + \alpha t) = P((1-\alpha)a + \alpha x) = (1-\alpha)A + \alpha X. \tag{3.17}$$

Combining terms with equal coefficients in (3.13), we obtain

$$V(P(a+\alpha t)) = \sum_{r=1}^{n} \binom{n}{r} V^r \alpha^r (1-\alpha)^{n-r}, \tag{3.18}$$

where

$$V^r := V(\underbrace{X, \ldots, X}_{r}, \underbrace{A, \ldots, A}_{n-r}). \tag{3.19}$$

The directional derivatives of $V(A)$ in the direction t can be obtained by letting $\alpha \to 0$ in (3.18). For example,

$$V'(a; t) = n(V^1 - V^0) \tag{3.20}$$

$$V''(a; t) = n(n-1)[V^2 - 2V^1 + V^0]. \tag{3.21}$$

Using (3.16) and elementary geometry, we may expand these derivatives:

$$V'(a; t) = \sum_{i=1}^{v} x_i V_i(A) - \sum_{i=1}^{v} a_i V_i(A) = \sum_{i=1}^{v} t_i V_i(A) \tag{3.22}$$

$$V''(a; t) = \sum_{i=1}^{v} t_i \sum_{j \neq i} t_{ij} V_{ij}(A), \tag{3.23}$$

where $V_{ij}(A)$ is the $(n-2)$-dimensional volume of A_{ij} and

$$t_{ij} := \frac{(t_j - t_i \cos\theta_{ij})}{\sin\theta_{ij}}. \tag{3.24}$$

If we let $a_i u_i$ represent the origin in H_i, then a_{ij} is the (signed) support distance from $a_i u_i$ to A_{ij} in H_i. Note that $V_{ij}(A) = 0$ if A_{ij} is not an $(n-2)$-dimensional face of A, i.e., $\operatorname{conv}\{p_i, p_j\}$ is not an edge of A^*.

We have already seen in (3.1) that $(V_i(A))$, $i = 1, \ldots, n$, lies in $\mathscr{D}^1$. Similarly, we obtain a stress $d \in \mathscr{D}^2$ by defining

$$d_{ij} := \begin{cases} \dfrac{V_{ij}(A)}{\sin\theta_{ij}} & i \neq j \\ -\displaystyle\sum_{j \neq i} \frac{\cos\theta_{ij}}{\sin\theta_{ij}} V_{ij}(A) & i = j. \end{cases} \tag{3.25}$$

Note that $d_{ij} = \partial V(A)/\partial a_i \, \partial a_j$. To prove $d \in \mathscr{D}^2$, note that for any $z \in \mathbb{R}^n$

$$\begin{aligned} \sum_{j=1}^{v} d_{ij} \langle u_j, z\rangle &= \sum_{j=1}^{v} \frac{(\langle u_j, z\rangle - \langle u_i, z\rangle \cos\theta_{ij})}{\sin\theta_{ij}} V_{ij}(A) = \\ &= (n-1)\, V_i(Z, \underbrace{A, \ldots, A}_{n-2}) = 0 \end{aligned} \tag{3.26}$$

since $Z = P(\langle u_i, z\rangle)$ is a point (or $P(a_i + \langle u_i, z\rangle)$ is a translate of A). Equation (3.26) implies the equilibrium condition

$$\sum_{j=1}^{v} d_{ij} u_j = 0$$

holds at each vertex p_i. The stress (d_{ij}) is identical to the canonical stress discovered by Maxwell and Cremona. Section 5 contains the original geometric formulation of this stress in terms of "reciprocal" diagrams.

The process above works as well for any x in some small open neighborhood $N(a)$ of a. We thereby obtain a commutative diagram

$$\begin{array}{ccc} N(a) & \xrightarrow{f} & \mathscr{D}^2 \\ & {\scriptstyle g}\searrow & \downarrow \cdot a \\ & & \mathscr{D}^1 \end{array} \tag{3.27}$$

The maps f and g are similar to those in (3.25) and (3.26), e.g.

$$f_{ij}(x) = \frac{V_{ij}(X)}{\sin \theta_{ij}}, \quad i \neq j, \tag{3.28}$$

$$g_i(x) = (n-1)\, V_i(A, \underbrace{X, \ldots, X}_{n-2}). \tag{3.29}$$

The original question about the infinitesimal rigidity of A^* is now reduced to showing that g is onto. It turns out to be easier to use the linear approximations of f and g at a given by the Jacobian maps. The resulting diagram

$$\begin{array}{ccc} R^v & \xrightarrow{f'} & \mathscr{D}^2 \\ & {\scriptstyle g'}\searrow & \downarrow \cdot a \\ & & \mathscr{D}^1 \end{array} \tag{3.30}$$

commutes by the chain rule. The directional derivative $g'(t) = g'(a; t)$ can be determined as in (3.17)—(3.20):

$$\frac{g_i'(t)}{(n-1)} = \sum_{j \neq i} t_{ij} V_{ij}(A) = \mathbf{D}\, t, \tag{3.31}$$

where $\mathbf{D} = (d_{ij})$ is the stress matrix given by (3.25). Combining diagram (3.30) with Proposition 1, we obtain

Proposition 3. *The 1-skeleton of a simplicial d-polytope A^*, $n \geqslant 3$, is infinitesimally rigid if* $\operatorname{rank} \mathbf{D} = v - n$.

We should point out that although $\mathbf{D}$ can be defined when $n = 2$ (and in fact has rank $v - 2$), the functions f and g are constant, so $f' = g' = 0$. In this case, $\dim \mathscr{D}^2 = 1$ and $\cdot a$ is onto if and only if $v = 3$ when A^* is a triangle.

Fortunately, the quadratic form $V''(a; t)$ associated to $\mathbf{D}$ is well understood. The following proposition about $V''(a; t)$ plays a crucial role in the proof of the Alexandrov-Fenchel inequality in [31, 6C2-6C7].

Proposition 4. *If $P(x) \approx P(a)$ and $t = x - a$, then the following are equivalent:*

(i) $(V^1)^2 \geqslant V^0 V^2$, *with equality if and only if X is homothetic to A.*

(ii) *If $V^1 = V^0$, then $V''(a; t) \leqslant 0$ with equality if and only if t belongs to the row space of* $\mathbf{U}$.

(iii) $V''(a; t)$ *has* 1 *positive eigenvalue and* $(v - n - 1)$ *negative eigenvalues (and thus rank $v - n$).*

The inequality in (i) is a special case of the Alexandrov-Fenchel inequality known as Minkowski's quadratic inequality. When $X \approx A$, equality in (i) implies X is homothetic to A by the Brunn-Minkowski theorem (which states that $\sqrt[n]{V((1 - \alpha) A + \alpha X)}$ is a concave function of α). Thus (iii) holds, and so Proposition 3 gives

Corollary 5. *The 1-skeleton of a simplicial polytope in $\mathbb{R}^n$, $n \geqslant 3$, is infinitesimally rigid.*

The Alexandrov-Fenchel inequality is stronger than the statement that A is rigid. For if A is rigid, $\cdot a$ in (3.30) is onto by Proposition 1. Since f' is onto (see [26]), g' must also be onto and thus $\operatorname{rank} \mathbf{D} = v - n$. However, we don't get any information about the eigenvalues of $\mathbf{D}$ and cannot derive the inequality $(V^1)^2 \geqslant V^0 V^2$.

4. Alexandrov's Theorem

Let A^* be an arbitrary convex polytope in $\mathbb{R}^n$, $n \geqslant 3$, and let F be a framework obtained by triangulating the 2-faces of A^*. Then F is infinitesimally rigid if and only if F has no points lying in the interior of a 2-face of A^*. In 3-dimensions, this result is due to ALEXANDROV [2] and for $n > 3$ it was recently proved by WHITELEY [42]. We shall see that these theorems follow from a characterization of the case of equality in Minkowski's quadratic inequality [6].

Suppose the points of F, $\{p_1, \ldots, p_v\}$, satisfy (3.5) and vert $A^* \subset$ $\subset \{p_1, \ldots, p_v\}$. We define $X = P(x)$ by (3.7), so that $A = P(a)$, $a_i = 1/|p_i|$, is the polar dual of A^*. We shall work through Section 3, highlighting key spots, until we finally reach a point when the method breaks down.

We have already noted that if x is close enough to a, $\mathcal{N}(A) \prec \mathcal{N}(X)$ and equality holds in (3.12). Equation (3.16) however must be modified. If $\mathcal{N}(A) \prec \mathcal{N}(X)$, a special case

$$V(X, \underbrace{A, \ldots, A}_{n-1}) = \frac{1}{n} \sum_{i=1}^{v} x_i V_i(A) \tag{4.1}$$

follows from (3.15). One verifies (4.1) by noting that since $\mathcal{N}(A) \prec \mathcal{N}(X)$, each facet A_i has a corresponding facet X_i with support distance x_i.

The procedure for finding the directional derivative of $V(A)$ in (3.17)—(3.22) remains valid. The next formula (3.23) is the first place where we start to have real difficulties. This expansion may not hold, and in the situation described below, V'' is not even continuous in a neighborhood of a.

Suppose p_1 lies in the relative interior of an edge $\operatorname{conv}\{p_2, p_3\}$ of A^*. Let $x = (a_1 + \varepsilon, a_2, \ldots, a_v)$ with $|\varepsilon|$ small. Since $\mathcal{N}(A) \prec \mathcal{N}(X)$, $\mathcal{N}(X + A) = \mathcal{N}(X)$ and thus

$$V(X, X, \underbrace{A, \ldots, A}_{n-2}) = \frac{1}{n} \sum_{i=1}^{v} x_i V_i(X, \underbrace{A, \ldots, A}_{n-2}) \tag{4.2}$$

follows from (3.15) in the same manner as (4.1). However, the inner term for $i = 2$ is

$$V_2(X, \underbrace{A, \ldots, A}_{n-2}) = \frac{1}{n-1} \sum_{j \in \mathcal{F}(A_2)} x_{2j} V_{2j}(A), \tag{4.3}$$

where the indices in $\mathcal{F}(A_2)$ correspond to facets of A_2. The trouble is that A_{21} and A_{23} are identical facets of A_2. So, depending on the sign of ε, i.e., whether p_1 becomes a vertex or interior point of X^*, we must make a different choice of $j = 1$ or $j = 3$ in (4.3) (see Figures 1 and 2). A careful study of this phenomenon will be necessary to carry through the proof of the infinitesimal rigidity of F.

In any case, we must produce a commutative diagram similar to (3.30). The idea comes from the next proposition which relates volumes in A to sums over the graph of F.

Proposition 6. *The following expansions hold*:

$$V(A) = \frac{1}{n} \sum_{i=1}^{v} a_i V_i(A) \tag{4.4}$$

$$V(A_i) = \frac{1}{n-1} \sum_{j \in \mathscr{L}(i)} a_{ij} V_{ij}(A) \tag{4.5}$$

$$V(A_{ij}) = \frac{1}{n-2} \sum_{k \in \mathscr{L}(i,j)} a_{ijk} V_{ijk}(A) \tag{4.6}$$

where $\mathscr{L}(\lambda)$ *is the link of* λ *in* F, *and*

$$a_{ij} = \frac{(a_j - a_i \cos \theta_{ij})}{\sin \theta_{ij}} \tag{4.7}$$

$$a_{ijk} = \left[a_k - a_i \frac{\cos \theta_{ik} - \cos \theta_{ij} \cos \theta_{jk})}{(\sin \theta_{ij})^2} - \right. \tag{4.8}$$

$$\left. - a_j \frac{\cos \theta_{jk} - \cos \theta_{ij} \cos \theta_{ik})}{(\sin \theta_{ij})^2} \right] \sin \theta_{ij} / |u_{ijk}|$$

are support distances in H_i *and* $H_i \cap H_j$ *respectively. The last number* $|u_{ijk}|$ *is the volume of the parallelohedron spanned by* $\{u_i, u_j, u_k\}$.

Proof. Formula (4.4) holds since $V_i(A)$ is 0 if $\dim A_i < n-1$, or equivalently, $p_i \notin \operatorname{vert} A^*$. The proofs of (4.5) and (4.6) are similar, so we only give the latter.

Let A^*_λ be a 2-face of A^* with λ being a list of all points of F on A^*_λ. Suppose $\operatorname{conv}\{p_i, p_j\}$ lies on an edge of A^*_λ. Then whenever $\operatorname{conv}\{p_i, p_j, p_k\}$ and $\operatorname{conv}\{p_i, p_j, p'_k\}$ are non-degenerate triangles in A^*_λ,

$$A_{ijk} = A_{ijk'} \quad \text{and} \quad a_{ijk} = a_{ijk'}.$$

Thus we may choose any such k, in particular $k = \mathscr{L}(i, j) \cap \lambda$, to obtain the volume of the corresponding pyramid in A_{ij}.

If $\operatorname{conv}\{p_i, p_j\} \cap \operatorname{int} A^*_\lambda \neq \emptyset$, then $V_{ij}(A) = 0$ since $\dim A_{ij} = n-3$. In this case, $\mathscr{L}(i, j) = \{k, k'\} \subset \lambda$,

$$A_{ijk} = A_{ijk'} \quad \text{and} \quad a_{ijk} = -a_{ijk'}.$$

Thus the right side of (4.6) is also 0. ∎

Note that the sums in (4.4)–(4.6) do not depend on the underlying graph of F.

Theorem 7. *There is a commutative diagram*

$$\begin{array}{ccc} R^v & \xrightarrow{\bar{f}} & \mathscr{D}^2 \\ & \bar{g} \searrow & \downarrow \cdot a \\ & & \mathscr{D}^1 \end{array} \tag{4.9}$$

where

$$\bar{g}_i(t) := \sum_{j \in \mathscr{L}(i)} t_{ij} V_{ij}(A) \tag{4.10}$$

$$\bar{f}_{ij}(t) := \begin{cases} \dfrac{1}{n-2} \displaystyle\sum_{k \in \mathscr{L}(i,j)} t_{ijk} V_{ijk}(A) & \{i,j\} \in E \\ \dfrac{-1}{n-2} \displaystyle\sum_{j \in \mathscr{L}(i)} \dfrac{\cos\theta_{ij}}{\sin\theta_{ij}} \bar{f}_{ij}(t) & i = j \\ 0 & \{i,j\} \notin E \end{cases} \tag{4.11}$$

(E *is the set of bars of* F).

Proof. The proofs that $\bar{g}(\mathbb{R}^v) \subset \mathscr{D}^1$, $\bar{f}(\mathbb{R}^v) \subset \mathscr{D}^2$ and

$$\bar{g}_i(t) = \sum_{j=1}^{v} a_j \bar{f}_{ij}(t) \tag{4.12}$$

follow essentially from the fact that mixed volumes are symmetric under permutations of their arguments. We illustrate the method with the proof that $\bar{g}(t) \subset \mathscr{D}^1$, and indicate briefly how it applies to (4.12).

Given $z \in \mathbb{R}^n$, let $Z = P(z_i := \langle u_i, z \})$ be the corresponding point. Then

$$\begin{aligned} \sum_{i=1}^{v} z_i \bar{g}_i(t) &= \sum_{i=1}^{v} z_i \sum_{j \in \mathscr{L}(i)} \frac{(t_j - t_i \cos\theta_{ij})}{\sin\theta_{ij}} V_{ij}(A) = \\ &= \sum_{\{i,j\} \in E} [(z_i t_j + z_j t_i) - (z_i t_i + z_j t_j)\cos\theta_{ij}] \, V_{ij}(A)/\sin\theta_{ij}. \end{aligned} \tag{4.13}$$

Since each term in (4.13) is symmetrical in z and t, we may reverse their order to obtain

$$\sum_{i=1}^{v} z_i \bar{g}_i(t) = \sum_{i=1}^{v} t_i \sum_{j \in \mathscr{L}(i)} \frac{(z_j - z_i \cos \theta_{ij})}{\sin \theta_{ij}} V_{ij}(A) =$$

$$= (n-1) \sum_{i=1}^{v} t_i V_i(Z, \underbrace{A, \ldots, A}_{n-2}) = 0.$$

The substitution of $V_i(Z, \underbrace{A, \ldots, A}_{n-2})$ follows from (4.5) with the distances z_{ij} to the point Z replacing a_{ij}. This shows that $\bar{g}(\mathbb{R}^v) \subset \mathscr{D}^1$.

The proof of (4.12) proceeds by substituting (4.6) in (4.10), and then switching the order of a and t using the same symmetry property. It is when performing this switch that we require F to be a triangulation. ■

As in Section 3, we study the rank of $\bar{g}$ by means of the associated quadratic form

$$Q(t) := \sum_{i=1}^{v} t_i \sum_{j \in \mathscr{L}(i)} t_{ij} V_{ij}(A). \tag{4.14}$$

A nice property of $Q(t)$ is a direct consequence of the fact that $V_{ij}(A) = 0$ if $\operatorname{conv}\{p_i, p_j\}$ intersects the interior of a 2-face of A^*.

Invariance Property. *Both $Q(t)$ and $\bar{g}(t)$ do not depend on the subdivisions induced by F on the interior of 2-faces of A^*.*

However, $Q(t)$ is sensitive to the behavior of points on the edges of A^*, as will become apparent in the next theorem.

Theorem 8. *If $\mathscr{N}(A) \prec \mathscr{N}(X)$, $x = a + t$, and V^r is given by* (3.19), *then*

$$Q(t) \leqslant V''(a; t) = n(n-1)[V^2 - 2V^1 + V^0]. \tag{4.15}$$

Proof. The difference between $Q(t)$ and $V''(a; t)$ resides in the term V^2. By the invariance property, any point which lies in the interior of a 2-face of A^* can safely be ignored. If all other points of F remain on the boundary of X^*, we claim that

$$V(\underbrace{X, X}_{n-2}, A, \ldots, A) = \frac{1}{n} \sum_{i=1}^{v} x_i V_i(X, \underbrace{A, \ldots, A}_{n-2}) = \tag{4.16}$$

$$= \frac{1}{n(n-1)} \sum_{i=1}^{v} x_i \sum_{j \in \mathscr{L}(i)} x_{ij} V_{ij}(A).$$

Under these conditions, each x_{ij} remains an appropriate supporting distance in $H_i(x_i) \cap H_i(x_j)$, so (4.16) follows in the same manner as (4.5). After substitution of (4.16), both sides of (4.15) can be arranged to have the same expansion.

The critical situation that remains is pictured in Figures 1 and 2.

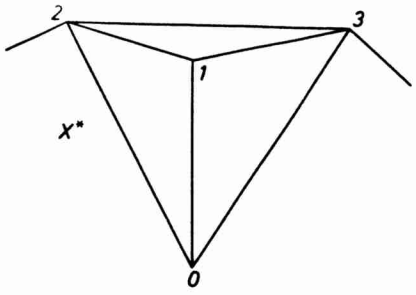

Fig. 1

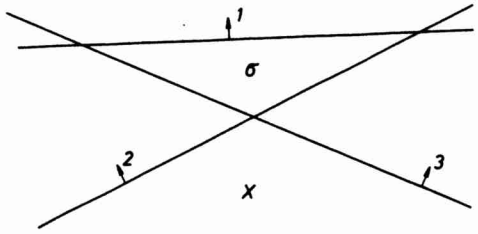

Fig. 2

We suppose $\{1, 2\}$ and $\{1, 3\}$ are bars of F, $p_1 \in \text{rel int conv}\{p_2, p_3\}$, and $\{p_1, p_2, p_3\}$ lies on an edge of A^*. The figures represent the intersection of X and X^* with $L = \text{lin}\{u_1, u_2, u_3\}$ in the case when u_1/x_1 lies in the interior of X^*.

Since $H_1(x_1) \cap X = \emptyset$, no terms containing t_1 appear in the expansion of $V''(a; t)$ whereas $Q(t)$ contains such terms. Their difference can thus be written

$$Q(t) - V''(a; t) = \sum \delta_{i_1} V_{i_2 i_3}(A), \tag{4.17}$$

where for example

$$\begin{aligned} \delta_1 &= [t_1(t_{12} + t_{13}) + t_2 t_{21} + t_3 t_{31}] - [t_2 t_{23} + t_3 t_{32}] = \\ &= t_1(t_{12} + t_{23}) + t_2(t_{21} - t_{23}) + t_3(t_{31} - t_{32}). \end{aligned} \tag{4.18}$$

Here we have used the fact $A_{12} = A_{13} = A_{23}$ to factor this volume out in (4.14).

The terms in parenthesis are the (signed) lengths of sides in the triangle σ appearing in Figure 2. A straightforward calculation shows that (compare [18], (1.4) and (1.43))

$$\delta_1 = -2V(\sigma). \tag{4.19}$$

Thus $Q(t) - V''(a; t) < 0$ since it is a sum of negative terms. ■

We can now prove

Theorem 9. *A framework F obtained by triangulating the 2-faces of a convex n-polytope A^*, $n \geqslant 3$, is infinitesimally rigid if and only if F has no points in the interior of 2-faces of A^*.*

Proof. We have already seen in the introduction that when F has a point p_1 lying in the interior of a 2-face of A^* it is not infinitesimally rigid. Thus, we may assume all points of F lie on edges or vertices of A^*.

By means of a simple inductive argument (see [13]) it can be shown that the cone over F is infinitesimally rigid. This implies the dimension of $\mathscr{D}^2$ is again given by (2.7). Proposition 1 and Theorem 7 can then be used to show F is infinitesimally rigid if $Q(t)$ attains its maximum rank. Moreover, the equivalence of (ii) and (iii) in Proposition 4 holds with V'' replaced by $Q(t)$ since it relies only on he fact that $Q(a) > 0$ and $\ker Q \supset$ row space $\mathbf{U}$. Thus, we need only prove (ii). The proof below relies on a result of Bol ([6]).

Let $x = a + t$ with $\mathscr{N}(A) \prec \mathscr{N}(X)$, and suppose $V^1 = V^0$. Then Minkowski's inequality

$$(V^1)^2 \geqslant V^0 V^2 \tag{4.20}$$

implies $V^1 \geqslant V^2$ and

$$V''(a; t) = n(n-1)[V^2 - 2V^1 + V^0] = n(n-1)[V^2 - V^1] \leqslant 0.$$

Combining this with Theorem 8 gives

$$Q(t) \leqslant V''(a; t) \leqslant 0, \tag{4.21}$$

which proves the first part of (ii).

If $Q(t) = 0$, then $V''(a; t) = 0$ and equality holds in (4.20). In this case, [6] implies A is an $(n-2)$-tangential body of some translate $\bar{X}$ of X. This means that for every $p \in \mathrm{skel}_1 A$, the hyperplane orthogonal to p which supports A must also support $\bar{X}$. However, $\bar{X}$ differs from A only by motions of the planes H_i normal to points of F, and they all belong to $\mathrm{skel}_1 A$ by assumption. Therefore $\bar{X} = A$ and X is a translate of A, i.e., $t \in$ row space $\mathbf{U}$ as required. ■

Bol's theorem can be avoided in the proof of Theorem 9 by using an inductive argument similar to the proof in [31, 6C6].

Whenever F has a point in the interior of 2-faces of A^*, the matrix associated to $\bar{g}$ has a corresponding 0 row, and so rank $Q(t) < v - n$.

Corollary 10. *Let F be an arbitrary triangulation of the boundary of a polytope A^* in $\mathbb{R}^3$. Then F is infinitesimally rigid if and only if* rank $Q(t) = v - 3$.

This corollary does not hold in higher dimensions. For example, if we add one point p_1 in the interior of a 2-face σ of a simplicial 4-polytope, then rank $Q(t) < v - 4$. However, we can make this framework infinitesimally rigid by addings bars from p_1 to points in the two 3-faces adjacent to σ (see [42], Theorem 8.4).

5. Maxwell–Cremona Stress

A classical result of MAXWELL [29] and CREMONA [14] describes a method of obtaining a self-stress on planar graphs by using "reciprocal" diagrams. This process can easily be generalized to all dimensions (see [33] for 3-dimensions). A number of approaches have been given recently which all yield the same stress (see [12], [20] and [41]). We shall illustrate this method for a convex polytope in $\mathbb{R}^3$, and obtain the same stress as **D** in (3.25).

Let A and A^* be a dual pair of convex polyhedra in $\mathbb{R}^3$, and consider the frameworks obtained by joining their vertices to the origin. For each vertex p_i of A^*, there is a dual face A_i of A orthogonal to p_i, and for each edge $p_{ij} = \text{conv}\{p_i, p_j\}$, there is an edge A_{ij} with p_{ij} orthogonal to $A_{0ij} = \text{conv}\{0, A_{ij}\}$ (see Figure 3). Such frameworks form a "reciprocal" pair in $\mathbb{R}^3$.

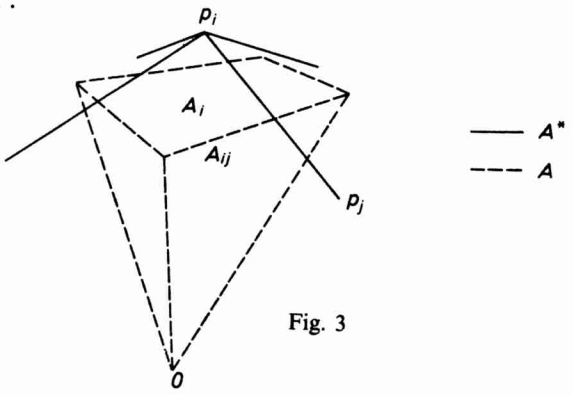

Fig. 3

If we apply the easy part of Minkowski's theorem to the solid $\operatorname{conv}\{0, A_i\}$ we obtain

$$0 = \sum_{j=1}^{v} \frac{p_j - p_i}{|p_j - p_i|} V(A_{0ij}) + \frac{p_i}{|p_i|} V(A_i). \tag{5.1}$$

The height of the triangle A_{0ij} is the inverse of the distance from the origin to p_{ij}, i.e.,

$$\frac{|p_j - p_i|}{|p_i|\,|p_j| \sin \theta_{ij}}. \tag{5.2}$$

Thus, the area of A_{0ij} is

$$V(A_{0ij}) = \frac{|p_j - p_i|\, V(A_{ij})}{2\,|p_i|\,|p_j| \sin \theta_{ij}}. \tag{5.3}$$

Substituting (5.3) in (5.1) gives

$$0 = \sum_{j=1}^{v} (p_j - p_i) \frac{V(A_{ij})}{|p_i|\,|p_j| \sin \theta_{ij}} + \frac{2p_i}{|p_i|} V(A_i). \tag{5.4}$$

Equation (5.4) is the equilibrium condition at p_i for the cone over A^*. Using (3.4), we can convert this stress $(V(A_{ij})/|p_i|\,|p_j| \sin \theta_{ij})$ into the one in (3.25).

Recall that $P(x)$ is a polytope with facets parallel to A. If $P(x) \approx A$, it is called a parallel redrawing of A. Any such $P(x)$ gives rise to a self-stress on the cone over A^* as above. The stresses obtained from parallel redrawings in fact span the space of stresses on the cone over A^*. In $\mathbb{R}^3$, an easy proof can be given using the projection method in [20].

KAHN ([21], Proposition 3.3) uses an idea in [10] to show that the matrix $\mathbf{D}$ has rank $v - 3$ when A^* is an arbitrary convex 3-polytope. Combining this result with Theorem 7 gives another proof of Alexandrov's theorem for triangulations using only the vertices of A^*.

The process above works as well for a reciprocal pair of non-convex polyhedra. However, mixed volumes are defined only for convex bodies so we cannot apply the methods of Section 3 in this case.

6. Remarks

The family V', V'', V''', ... in Section 3 gives a set of canonical stresses lying in the sequence $\mathscr{C}^1$, $\mathscr{C}^2$, $\mathscr{C}^3$, ..., of higher stress-spaces

which is defined in [25]. The original motivation of Kalai and Lee for studying these spaces was to give a "simple", i.e., geometric proof of MCMULLEN's g-theorem, which characterizes the face numbers of simplicial convex polytopes ([30]). The original proof of STANLEY used difficult techniques from algebraic geometry ([38]).

In order to carry out a proof of necessity in McMullen's g-theorem using the stress approach, one must prove that the analogous maps to (3.30) are onto in all dimensions $\leqslant \lfloor n/2 \rfloor$. This will be more difficult because the associated forms correspond to higher-order derivatives of $V(A)$, about which Brunn–Minkowski gives no obvious information. However, as in the first-order case, connections do arise with the equality case in certain Alexandrov–Fenchel inequalities.

Theorem 9 gives the signature of the stress matrix **D**. Moreover, a hyperplane is identified upon which the associated quadratic form is negative semi-definite. Such information has been useful to investigate questions about other kinds of rigidity (see [11]).

References

[1] ALEXANDROV, A. D.: On the theory on mixed volumes II. Mat. Sbornik **44**, 1205—1238 (1937).

[2] ALEXANDROV, A. D.: Konvexe Polyeder. Berlin: Akademie-Verlag. 1958.

[3] ASIMOW, L., ROTH, B.: The rigidity of graphs I. Trans. Amer. Math. Soc. **245**, 279—289 (1978).

[4] ASIMOW, L., ROTH, B.: The rigidity of graphs II. J. Math. Anal. App. **68**, 171—190 (1979).

[5] BARNETTE, D.: A proof of the lower bound conjecture for convex polytopes. Pacific J. Math. **46**, 349—354 (1973).

[6] BOL, G.: Beweis einer Vermutung von H. Minkowski. Abh. Math. Sem. Univ. Hamburg **15**, 37—56 (1943).

[7] BONNESEN, T., FENCHEL, W.: Theorie der Konvexen Körper. Berlin: Springer. 1934.

[8] BUSEMANN, H.: Convex Surfaces. New York: Interscience Publ. 1958.

[9] CONNELLY, R.: The rigidity of certain cabled frameworks and the second order rigidity of arbitrary triangulated convex surfaces. Adv. Math. **37**, 272—299 (1980).

[10] CONNELLY, R.: Rigidity and energy. Invent. Math. **66**, 11—33 (1982).

[11] CONNELLY, R., WHITELEY, W.: Second-order rigidity and pre-stress stability. Preprint.

[12] CRAPO, H., WHITELEY, W.: Plane stresses and projected polyhedra. Univ. of Montreal. Preprint. 1977.

[13] CRAPO, H., WHITELEY, W.: Statics of frameworks and motions of panel structures: a projective geometry introduction. Struct. Top. **6**, 42—82 (1982).

[14] CREMONA, L.: Graphical Statics. Oxford: Univ. Press. 1890.

[15] DEHN, M.: Über die Starrheit Konvexer Polyeder. Math. Ann. **77**, 466—473 (1916).

[16] EWALD, G.: On the equality case in Alexandrov–Fenchel's inequality for convex bodies. Geom. Dedicata **28**, 218—220 (1988).

[17] FAVARD, J.: Sur les corps convexes. J. Math. Pures Appl. **12**, 219—282 (1933).

[18] FILLIMAN, P.: The volumes of duals and section of polytopes. Preprint.

[19] GLUCK. H.: Almost all simply connected closed surfaces are rigid. In Geometric Topology. Lect. Notes Math. **438**, pp. 225—239. Berlin—Heidelberg—New York: Springer. 1975.

[20] HOPCROFT, J., KAHN. P.: A paradigm for robust geometric algorithms. Algorithmica. To appear.

[21] KAHN, P.: Small perturbations of stressed graphs. Technical report. MSI, Cornell Univ. 1990.

[22] KALAI, G.: Rigidity and the lower bound theorem 1. Invent. Math. **88**, 125—151 (1987).

[23] KALAI, G.: The g-theorem for spheres. In preparation.

[24] KANN, E.: Infinitesimal rigidity of almost convex oriented polyhedra of arbitrary Euler characteristic. Preprint.

[25] LEE, C.: Some recent results on convex polytopes. Preprint.

[26] LEE, C.: PL-spheres and convex polytopes. In preparation.

[27] LEICHTWEISS, K.: Konvexe Mengen. Berlin—Heidelberg—New York: Springer. 1980.

[28] LYUSTERNIK, L. A.: Convex Figures and Polyhedra. Boston: D.C. Heath and Co. 1966.

[29] MAXWELL, J. C.: On reciprocal figures and diagrams of forces. Phil. Mag. (4), **27**, 205—261 (1864).

[30] MCMULLEN, P.: The maximum numbers of faces of convex polytope. Mathematika **17**, 179—184 (1970).

[31] MCMULLEN, P.: Metrical properties of convex sets. Notes.

[32] POGORELOV, A. V.: Extrinsic Geometry of Convex Surfaces. Providence: Amer. Math. Soc. 1973.

[33] RANKINE, W.: Principle of the equilibrium of polyhedral frames. Phil. Mag. (4) **27**, 92 (1862).

[34] ROTH, B.: Rigid and flexible frameworks. Amer. Math. Month. **88**, 6—21 (1981).

[35] SCHNEIDER, R.: On the Alexandrov—Fenchel Inequality. In: Discrete Geometry and Convexity (eds.: J. GOODMAN, et al.) Ann. New York Acad. Sci. **440**, 132—141 (1985).

[36] SIMLANSKY, Z.: Decomposability of polytopes and polyhedra. Geom. Dedicata **24**, 29—49 (1987).

[37] STANLEY, R.: Hilbert functions of graded algebras. Adv. Math. **28**, 57—83 (1978).

[38] STANLEY, R.: The number of faces of a simplicial convex polytope. Adv. Math. **35**, 236—238 (1980).

[39] TAY, T. S., WHITELEY, W.: Generating all isostatic frameworks. Struct. Top. **11**, 21—70 (1985).

[40] TUTTE, W.: How to draw a graph. Proc. London Math. Soc. (3) **13**, 743—768 (1963).

[41] WHITELEY, W.: Motions and stresses of projected polyhedra. Struct. Top. **7**, 13—38 (1982).

[42] WHITELEY, W.: Infinitesimally rigid polyhedra I, statics of frameworks. Trans. Amer. Math. Soc. **285**, 431—465 (1984).

[43] WHITELEY, W.: The combinatorics of bivariate splines. To appear in V. Klee Festschrift.

P. FILLIMAN
Mathematics Department
Western Washington University
Bellingham, WA 98225, U.S.A.