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Conormal Bundles with Vanishing Maslov Form

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Abstract. If M is a Riemannian manifold, and L is a Lagrangian submanifold of T^*M, the Maslov class of L has a canonical representative 1-form which we call the *Maslov form* of L. We prove that if $L = \nu^*N =$ the conormal bundle of a submanifold N of M, its Maslov form vanishes iff N is a minimal submanifold. Particularly, if M is locally flat ν^*N is a minimal Lagrangian submanifold of T^*M iff N is a minimal submanifold of M. This strengthens a result of Harvey and Lawson [H L].

Let (M^m, g) be a Riemannian manifold, and N^n $(n \leq m)$ be a submanifold of M with the g-normal bundle νN. Then it is well known that $L(N) = L \stackrel{\text{def}}{=} \nu^*N$ is a Lagrangian submanifold of the cotangent manifold T^*M endowed with its canonical symplectic structure. The metric g leads to a canonically defined Riemannian metric γ, and to a canonically defined metric but not torsionless connection ∇ on T^*M such that the Maslov class $\mu_1(L) \in H^1(L, \mathbb{R})$ of an arbitrary Lagrangian submanifold L of T^*M is represented by the 1-form $\mathcal{M}_1(L) = (1/2\pi)\,\mathrm{tr}(\beta)$, where β is the second fundamental form of L with respect to the connection ∇. These facts will be partially recalled in the first section, and we refer to [V] for a full exposition. Moreover, we say that $\mathcal{M}_1(L)$ is the (first) *Maslov form* of L.

Hereafter, we look only at the case $L = \nu^*N$, and our aim is to compute the ∇-second fundamental form β of L. As a consequence we get that L is a conormal bundle with a vanishing Maslov form iff N is a minimal submanifold of (M, g). Particularly, if (M, g) is a locally flat manifold ∇ is torsionless [V], β is the usual second fundamental form of L in (T^*M, γ), and L is a minimal Lagrangian submanifold of (T^*M, γ) iff N is a minimal submanifold of (M, g). This latter result is more precise than the one given by Harvey and Lawson in the case $M = \mathbb{R}^n$ namely, that ν^*N is a minimal Lagrangian submanifold of a certain special type iff N is an *austere* submanifold of (M, g) ([H L], p. 102).

1. Cotangent Bundles and Submanifolds

We begin by recalling some known facts about cotangent bundles T^*M. T^*M is endowed with the *Liouville* 1-*form*

$$\lambda_p(v) = p(\pi_* v) \quad (\pi\colon T^*M \to M,\ p \in T^*M,\ v \in T_p T^*M) \tag{1.1}$$

and with the *canonical symplectic form*

$$\Omega = -\mathrm{d}\lambda. \tag{1.2}$$

On the other hand one has

$$TT^*M = \mathscr{H} \oplus \mathscr{V}, \tag{1.3}$$

where $\mathscr{V}$ is the *vertical* (i. e., tangent to the fibers) distribution of T^*M, and $\mathscr{H}$ is the *horizontal* distribution of T^*M associated with the Levi—Civita connection D of g. Accordingly g has a natural lift $g_{\mathscr{H}}$ to $\mathscr{H}$, and since the tangent spaces of the fibers of T^*M may be identified with the fibers themselves g also defines a metric $g_{\mathscr{V}}$ on $\mathscr{V}$. Hence T^*M has the *canonical Riemannian metric*

$$\gamma = g_{\mathscr{H}} + g_{\mathscr{V}} \qquad (\mathscr{H} \perp \mathscr{V}). \tag{1.4}$$

Furthermore, Ω and γ are interrelated by a compatible almost complex structure J determined by

$$J|_{\mathscr{H}} = -(\Omega\text{-duality}) \circ (\gamma\text{-duality}) \qquad (J^2 = -Id.). \tag{1.5}$$

Finally, T^*M also has the *canonical connection* ∇ determined by asking it to preserve the distributions $\mathscr{H}$ and $\mathscr{V}$, and the tensors γ and J, and to have a vertical torsion. ∇ is defined by the natural action of the Levi—Civita connection D on $\mathscr{H}$ and $\mathscr{V}$.

For computational purposes we shall use along the submanifold N of M local coordinates (x^a, x^u) such that $x^a = 0$ defines N. Here and in the sequel our conventions for indices are

$$a, b, \ldots = 1, \ldots, m-n;\ u, v, \ldots = m-n+1, \ldots, m; \tag{1.6}$$
$$i, j, \ldots = 1, \ldots, m$$

($m = \dim M$, $n = \dim N$). Accordingly, on open neighborhoods of points of N, TM has bases of the form

$$TN = \operatorname{span}\left\{X_u \overset{\text{def}}{=} \frac{\partial}{\partial x^u}\right\}, \quad vN = \operatorname{span}\left\{X_a \overset{\text{def}}{=} \frac{\partial}{\partial x^a} - t_a^u \frac{\partial}{\partial x^u}\right\} \quad (vN \perp TN) \tag{1.7}$$

(the Einstein sum convention is used overall), and the dual cobases

$$\mathrm{d}x^a, \quad \theta^u = \mathrm{d}x^u + t^u_a \, \mathrm{d}x^a \tag{1.8}$$

(t^u_a are some well defined local functions).

With these bases the connection D is expressible as

$$D X_u = \omega^v_u X_v + \alpha^a_u X_a, \quad D X_a = \alpha^v_a X_v + \varkappa^b_a X_b, \tag{1.9}$$

where, since $X_a \perp X_u$ one has

$$\alpha^v_a g_{uv} + \alpha^b_u g_{ab} = 0. \tag{1.10}$$

The coefficients of (1.9) are local 1-forms, and we write

$$\alpha^a_u = A^a_{uv} \theta^v + A^a_{ub} \, \mathrm{d}x^b, \tag{1.11}$$

and

$$\varkappa^b_a = K^b_{au} \theta^u + K^b_{ac} \, \mathrm{d}x^c. \tag{1.12}$$

Along N (i. e., for $x^a = 0$) (1.9) are the Gauss—Weingarten equations, and (α^a_u) defines the second fundamental form of N in M.

Now, let us come back to $T^* M$. On M, and in a neighbourhood of a point of N, a 1-form has the general expression

$$\xi = \xi_a \, \mathrm{d}x^a + \xi_u \theta^u$$

hence (x^a, x^u, ξ_a, ξ_u) may be used as local coordinates of T^*M along $L \overset{\text{def}}{=} \nu^* N$, and then L will have the local equations

$$x^a = 0, \quad \xi_u = 0. \tag{1.13}$$

Furthermore, in (1.3) $\mathscr{V}$ is clearly defined by $\mathrm{d}x^i = 0$, and $\mathscr{H}$ is defined by the equations of parallelism

$$\begin{aligned} \zeta_a &\overset{\text{def}}{=} \mathrm{d}\xi_a - \varkappa^b_a \xi_b - \alpha^u_a \xi_u = 0, \\ \zeta_u &\overset{\text{def}}{=} \mathrm{d}\xi_u - \alpha^b_u \xi_b - \omega^v_u \xi_v = 0. \end{aligned} \tag{1.14}$$

Hence $(\mathrm{d}x^i, \zeta_i)$ are local tangent cobases of $T^* M$ along L, and their dual bases can be easily computed. It will follow that

$$\mathscr{V}|_L = \operatorname{span}\left\{\frac{\partial}{\partial \xi_a}, \frac{\partial}{\partial \xi_u}\right\}, \quad \mathscr{H}|_L = \operatorname{span}\{Z_a, Z_u\} \tag{1.15}$$

where, modulo (1.13), we have

$$Z_a = X_a + K^c_{ba}\xi_c\frac{\partial}{\partial\xi_b} + A^c_{va}\xi_c\frac{\partial}{\partial\xi_v},$$
$$Z_u = \frac{\partial}{\partial x^u} + K^c_{bu}\xi_c\frac{\partial}{\partial\xi_b} + A^c_{vu}\xi_c\frac{\partial}{\partial\xi_v}. \tag{1.16}$$

The other geometric elements which we defined on T^*M are now computable as follows along L:

$$\lambda = \xi_a\,\mathrm{d}x^a + \xi_u\theta^u, \quad \Omega = -\mathrm{d}\lambda; \tag{1.17}$$

$$\gamma(Z_i, Z_j) = g_{ij}, \quad \gamma\left(\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\xi_j}\right) = g^{ij}, \quad \gamma\left(Z_i, \frac{\partial}{\partial\xi_j}\right) = 0, \tag{1.18}$$

where g_{ij} are the local components of g on M with respect to the basis X_i of (1.7);

$$JY^i = \frac{\partial}{\partial\xi_i}, \quad J\frac{\partial}{\partial\xi_i} = -Y^i, \quad Y^i = g^{ij}Z_j; \tag{1.19}$$

and finally

$$\nabla Y^u = -\omega^u_v Y^v - \alpha^u_a Y^a, \qquad \nabla Y^a = -\alpha^a_v Y^v - \varkappa^a_b Y^b,$$
$$\nabla\frac{\partial}{\partial\xi_u} = -\omega^u_v\frac{\partial}{\partial\xi_v} - \alpha^u_a\frac{\partial}{\partial\xi_a}, \quad \nabla\frac{\partial}{\partial\xi_a} = -\alpha^a_v\frac{\partial}{\partial\xi_v} - \varkappa^a_b\frac{\partial}{\partial\xi_b}. \tag{1.20}$$

Just check that these formulas agree with the geometric definitions given earlier in this section (see also [V]).

2. The Maslov Form of Conormal Bundles

In the introduction we defined the Maslov form $\mathcal{M}_1(L) = (1/2\pi)\,\mathrm{tr}(\beta)$, where β is the ∇-second fundamental form of L, and now we shall compute this form for a conormal bundle $L = \nu^* N$. (Notation is the same as in the first section.)

It follows from (1.13) that

$$TL = \mathrm{span}\left\{\frac{\partial}{\partial x^u}, \frac{\partial}{\partial\xi_a}\right\}. \tag{2.1}$$

Hence TL has a vertical component

$$\mathcal{V}L = \mathrm{span}\left\{\frac{\partial}{\partial\xi_a}\right\}. \tag{2.2}$$

On the other hand, the connection induced by D in νN, and which is locally defined by the second part of equations (1.9), defines a *relative horizontal subbundle* of TL given by

$$\mathscr{H}L = \operatorname{span}\left\{Z'_u \overset{\text{def}}{=} \frac{\partial}{\partial x^u} + K^b_{au}\,\xi_b \frac{\partial}{\partial \xi_a}\right\} = \operatorname{span}\{Y'^u \overset{\text{def}}{=} g^{uv} Z'_v\}. \tag{2.3}$$

In this way we have obtained a local basis $\{Y'^u, \partial/\partial\xi_a\}$ of TL.

Furthermore (1.16) and (2.3) imply

$$Z'_u = Z_u - A^c_{vu}\,\xi_c \frac{\partial}{\partial \xi_v}, \quad Y'^u = Y^u - g^{uv} A^c_{tv}\,\xi_c \frac{\partial}{\partial \xi_t}, \tag{2.4}$$

whence we see that Y^a belong to $\nu L =$ the γ-normal bundle of L. They can be supplemented to a full basis of νL by adding local vector fields

$$\Xi^u = \frac{\partial}{\partial \xi_u} + \lambda_{uv}\, Y'^v, \tag{2.5}$$

where λ_{uv} are well defined by the condition $\Xi^u \perp Y'^v$ for all u, v.

We shall use the bases defined above in order to get the ∇-Gauss equations of L, it is to express $\nabla Y'^u$ and $\nabla(\partial/\partial\xi_a)$ along L. Starting with (2.4), and using (1.20) and (2.5) we get by a straightforward computation

$$\begin{aligned}\nabla Y'^u = (\text{term in } TL) - \{\mathrm{d}(g^{uv} A^c_{tv}\,\xi_c) + \omega^u_v\, g^{vw} A^c_{tw}\,\xi_c - \\ - \omega^w_t\, g^{uv} A^c_{wv}\,\xi_c\}\,\Xi^t - \alpha^u_a\, Y^a.\end{aligned} \tag{2.6}$$

We may also use the metric character of D on M i.e.,

$$\mathrm{d}g^{uv} + \omega^u_t\, g^{tv} + \omega^v_t\, g^{ut} = 0 \tag{2.7}$$

to put (2.6) under the form

$$\begin{aligned}\nabla Y'^u = (\text{term in } TL) - g^{uv}\{\mathrm{d}(A^c_{tv}\,\xi_c) - \\ - \omega^w_v\, A^c_{tw}\,\xi_c - \omega^w_t\, A^c_{wv}\,\xi_c\}\,\Xi^t - \alpha^u_a\, Y^a.\end{aligned} \tag{2.8}$$

Using again (1.20) and (2.5), we see that the second part of the ∇-Gauss equations of L is

$$\nabla \frac{\partial}{\partial \xi_a} = (\text{term in } TL) - \alpha^a_v\, \Xi^v. \tag{2.9}$$

Now, formulas (2.8) and (2.9) provide us with the Maslov form of L:

$$2\pi \mathcal{M}_1(L) = -(g^{uv} A^c_{uv})\, \mathrm{d}\xi_c - \delta(g^{uv} A^c_{uv} \xi_c), \tag{2.10}$$

where δ denotes the covariant differential on the submanifold N of M while ξ_c are seen as the covariant coordinates of an arbitrary normal vector of $N \subset M$.

Furthermore, from (2.10) we get

Theorem. *The conormal bundle L of the submanifold N of (M, g) has a vanishing Maslov form iff N is a minimal submanifold. Particularly, if g is a flat metric L is a minimal Lagrangian submanifold of $T^* M$ iff N is a minimal submanifold of (M, g).*

The last assertion follows from the fact that g flat implies ∇ is the Levi—Civita connection of γ on $T^* M$, which leads to $2\pi \mathcal{M}_1(L) = -\mathrm{mi}(H)\,\Omega$, where H is the mean curvature vector of L [V].

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