Mh. Math. 108, 255—276 (1989)

Monatshefte für
Mathematik

Complex Analytic Curves and Maximal Surfaces

By

K. Abe, Storrs, CT, and **M. A. Magid**,[1] Wellesley, MA

(Received 28 September 1987; revised 27 June 1989)

Abstract. Maximal immersions of a surface M^2 into n-dimensional Lorentz space which are isometric to a fixed holomorphic mapping of M^2 into complex Lorentz space are determined. The main tool is an adaption of Calabi's Rigidity Theorem. Such an adaption is necessary because of the existence of degenerate hyperplanes in complex Lorentz space.

Every minimal surface in Euclidean space is locally isometric to a complex analytic curve. Essentially, the minimal surface is the real part of this analytic curve.

The Calabi Rigidity Theorem [C1] implies that each class of isometric minimal immersions contains exactly one complex analytic curve. In [C2] Calabi considered a fixed holomorphic immersion $\lambda: M^2 \to \mathbb{C}^m$ and described the space of all non-congruent minimal immersions $\beta: M^2 \to \mathbb{R}^n$ which are isometric to λ. This space is parametrized by $n \times m$ complex matrices with $m \leqslant n \leqslant 2m$ which satisfy certain conditions. A full exposition of these results appears in [L].

In this paper we consider maximal immersions of a surface M^2 in Lorentz space $\mathbb{R}_1^n$ which are isometric to holomorphic immersions of M^2 in complex Lorentz space $\mathbb{C}_1^m$. Given a fixed holomorphic curve $\lambda: M^2 \to \mathbb{C}_1^m$ we describe all the maximal immersions $\beta: M^2 \to \mathbb{R}_1^n$ which are isometric to λ. The main tool is our Fundamental Lemma, which is the appropriate adaptation of Calabi's Rigidity Theorem. The existence of degenerate hyperplanes in $\mathbb{C}_1^m$ makes it clear that such an adaptation is necessary. We show in Theorem 2 and Proposition 11 that if the image of λ is contained in no hyperplane of $\mathbb{C}_1^m$ the result is virtually the same as in the positive definite case, while if λ is contained in a degenerate hyperplane (but no non-degenerate

[1] Partially supported by a grant from Wellesley College.

hyperplane) we show that $m-1 \leqslant n \leqslant 2m$, and the space of $n \times m$ complex matrices needed to describe all β up to congruence must be supplemented by the choice of an arbitrary holomorphic function on M^2.

1. Preliminaries

The metric g on Lorentz space $\mathbb{R}_1^n$, is given by $g(v,v) = -v_1^2 + v_2^2 + \ldots + v_n^2$ for all $v = (v_1, \ldots, v_n) \in \mathbb{R}_1^n$. The indefinite Kähler metric on $\mathbb{C}_1^n$ is given by

$$\langle (z_1, \ldots, z_n), (z_1, \ldots, z_n) \rangle = -z_1 \bar{z}_1 + z_2 \bar{z}_2 + \ldots + z_n \bar{z}_n$$

for any $(z_1, \ldots, z_n)$ in $\mathbb{C}_1^n$. We also set

$$((z_1, \ldots, z_n), (z_1, \ldots, z_n)) = -z_1^2 + z_2^2 + \ldots + z_n^2.$$

Throughout this paper a Riemannian metric h is fixed on the surface M^2. All immersions of M^2 into $\mathbb{R}_1^n$ for any n are assumed to be isometric immersions, inducing the metric h. Our local coordinates (u_1, u_2) on M^2 will always be isothermal with respect to h, i.e., if $h = h_{ij}\, du_i\, du_j$ then $h_{11} = h_{22}$ and $h_{12} = 0$. The local complex parameter $w = u_1 + i u_2$ make M^2 into a Riemann surface with the conformal structure determined by h. Thus we can speak of holomorphic functions on M^2, or of a holomorphic immersion $\lambda: M^2 \to \mathbb{C}_1^m$.

Let $\beta: M^2 \to \mathbb{R}_1^n$ be an immersion of M^2 into Lorentz space $\mathbb{R}_1^n$. Following [H-O] we write

$$\beta(u_1, u_2) = (\beta_1(u_1, u_2), \ldots, \beta_n(u_1, u_2))$$

and set

$$\frac{d\beta}{dw} = \frac{1}{2}\left[\frac{\partial \beta_1}{\partial u_1} - i\frac{\partial \beta_1}{\partial u_2}, \ldots, \frac{\partial \beta_n}{\partial u_1} - i\frac{\partial \beta_n}{\partial u_2}\right].$$

$\dfrac{d\beta}{dw}$ takes its values in $\mathbb{C}_1^n$. Since (u_1, u_2) is isothermal,

$$\left(\frac{d\beta}{dw}, \frac{d\beta}{dw}\right) = \tfrac{1}{4}(h_{11} - h_{22} - 2 i h_{12}) = 0$$

and since h is Riemannian

$$\left\langle \frac{d\beta}{dw}, \frac{d\beta}{dw} \right\rangle = \tfrac{1}{4}(h_{11} + h_{22}) > 0.$$

An immersion $\beta: M^2 \to \mathbb{R}_1^n$ is called maximal if the mean curvature vector $\eta \equiv 0$. In terms of the isothermal coordinates u_1, u_2 on M^2,

$$2\eta = \Delta\beta = (\Delta\beta_1, \ldots, \Delta\beta_n).$$

Thus β is maximal iff the β_k are harmonic functions, that is, $\Delta\beta_k = 0$. Equivalently β is maximal iff $\frac{\partial\beta_k}{\partial u_1} - i\frac{\partial\beta_k}{\partial u_2} =: 2\left(\frac{d\beta}{dw}\right)_k$ is analytic for $k = 1, \ldots, n$.

Given a maximal $\beta: M^2 \to \mathbb{R}_1^n$ we can locally define a holomorphic function f_k so that $\mathrm{Re}(f_k) = \beta_k$. The holomorphic curve defined locally by

$$\gamma(w) = \gamma(u_1 + iu_2) = \frac{1}{\sqrt{2}}(f_1(w), \ldots, f_n(w)) \in \mathbb{C}_1^n$$

can be considered as a maximal immersion $\gamma: M^2 \to \mathbb{R}_2^{2n} = \mathbb{R}_1^n \times \mathbb{R}_1^n$ isometric to β by writing

$$\gamma(u_1, u_2) = \frac{1}{\sqrt{2}}(\mathrm{Re} f_1, \ldots, \mathrm{Re} f_n, \mathrm{Im} f_1, \ldots, \mathrm{Im} f_n).$$

The Cauchy—Riemann equations give

$$\frac{\partial\gamma}{\partial u_1} = \frac{1}{\sqrt{2}}\left[\frac{\partial\beta_1}{\partial u_1}, \ldots, \frac{\partial\beta_n}{\partial u_1}, -\frac{\partial\beta_1}{\partial u_2}, \ldots, -\frac{\partial\beta_n}{\partial u_2}\right]$$

and

$$\frac{\partial\gamma}{\partial u_2} = \frac{1}{\sqrt{2}}\left[\frac{\partial\beta_1}{\partial u_2}, \ldots, \frac{\partial\beta_n}{\partial u_2}, \frac{\partial\beta_1}{\partial u_1}, \ldots, \frac{\partial\beta_n}{\partial u_1}\right].$$

This shows that $\Delta\gamma = 0$ and that h is the metric induced by $\gamma: M^2 \to \mathbb{R}_2^{2n}$. If M^2 is simply connected then γ is well defined globally on M^2.

If $\alpha: M^2 \to \mathbb{R}_1^k$ and $\beta: M^2 \to \mathbb{R}_1^j$ are maximal isometric immersions and γ and δ are the holomorphic curves associated with them as above then

$$\left\langle\frac{d\gamma}{dw}, \frac{d\gamma}{dw}\right\rangle = \left\langle\frac{d\delta}{dw}, \frac{d\delta}{dw}\right\rangle.$$

The analogous statement in the Kähler case is enough to prove the following theorem.

Theorem 1. [L]. *In each class of isometric, non-congruent minimal surfaces in Euclidean space there exists exactly one holomorphic curve.*

The proof of this theorem depends on a result of Calabi (see p. 144 of [L]).

Proposition 1. *Let* $\varphi: D^0 \to \mathbb{C}^m$ *and* $\psi: D^0 \to \mathbb{C}^{m+n}$ *be holomorphic mappings of the unit disk such that*

$$\sum_{j=1}^{m} |\varphi_j|^2 = \sum_{k=1}^{m+n} |\psi_k|^2$$

and consider $\mathbb{C}^m \subset \mathbb{C}^{m+n}$ *as the span of the first m coordinates. Then there is a unitary transformation* $U: \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ *such that* $\psi = U \circ \varphi$.

In the indefinite case this proposition is not true. For example, $\varphi(w) = (0, 0, w)$ and $\psi(w) = (w^2, w^2, w)$ have $\|\varphi\|^2 = \langle\varphi, \varphi\rangle = = \langle\psi, \psi\rangle = \|\psi\|^2$, but φ is not rigidly equivalent to ψ. In a certain sense this is the only difficulty.

2. An Algebraic Lemma

Definition. A unitary transformation $U \in U(1, m-1)$ is a linear mapping U: $\mathbb{C}_1^m \to \mathbb{C}_1^m$ such that $\langle Uv, Uw\rangle = \langle v, w\rangle$. Equivalently $U \in U(1, m-1)$ iff U has a matrix representation, such that

$$U \begin{bmatrix} -1 & \\ & I_{m-1} \end{bmatrix}^{t} \bar{U} = \begin{bmatrix} -1 & \\ & I_{m-1} \end{bmatrix}.$$

The hypotheses of our Fundamental Lemma are stated in terms of the causal character of hyperplanes, which we now define and characterize without proof.

Definition. A subspace P of $\mathbb{C}_1^k$ is *degenerate* iff $\exists L \neq 0 \in P$ such that $\langle L, p\rangle = 0$, $\forall p \in P$.

Proposition 2. a) *A hyperplane P passing through the origin in* $\mathbb{C}_1^k$ *is degenerate iff P satisfies the equation* $z_1 = \sum_{j=2}^{k} a_j z_j$ *with* $|a_2|^2 + \ldots + |a_k|^2 = 1$.

b) *P is a degenerate hyperplane passing through the origin in* $\mathbb{C}_1^k$ *iff there is a unitary transformation of P onto the hyperplane defined by* $z_1 = z_2$.

Definition. A subspace P of C_1^k is *spacelike* iff $\langle p,p\rangle > 0$, $\forall p \neq 0$ in P.

Proposition 3. a) *A hyperplane P passing through the origin in $\mathbb{C}_1^k$ is spacelike iff P satisfies the equation $z_1 = \sum_{j=2}^{k} a_j z_j$ with $|a_2|^2 + \ldots + |a_k|^2 < 1$.*

b) *P is a spacelike hyperplane passing through the origin in $\mathbb{C}_1^k$ iff there is a unitary transformation of P onto the hyperplane defined by $z_1 = 0$.*

Fundamental Lemma. *Suppose $\varphi: D^0 \to \mathbb{C}_1^m$ and $\psi: D^0 \to \mathbb{C}_1^{m+n}$ are holomorphic mappings of the unit disk such that $\|\varphi\|^2 = \|\psi\|^2$. Consider $\mathbb{C}_1^m \subset \mathbb{C}_1^{m+n}$ as the span of the first m coordinates. If the image of ψ is not contained in a degenerate hyperplane of $\mathbb{C}_1^{m+n}$, then there is a unitary transformation $U \in U(1, m+n-1)$ such that $U \circ \varphi = \psi$. If the image of φ is not contained in a degenerate hyperplane or a spacelike hyperplane then the same conclusion holds.*

Proof: We reduce the problem to Proposition 1, first considering the case that the image of ψ is not contained in a degenerate hyperplane of $\mathbb{C}_1^{m+n}$. Consider the two maps $\tilde{\varphi}: D^0 \to \mathbb{C}^m$ and $\tilde{\psi}: D^0 \to \mathbb{C}^{m+n}$ given by $\tilde{\varphi} = (\psi_1, \varphi_2, \ldots, \varphi_m)$ and $\tilde{\psi} = (\varphi_1, \psi_2, \ldots, \psi_{m+n})$. Since

$$-|\varphi_1|^2 + |\varphi_2|^2 + \ldots + |\varphi_m|^2 = -|\psi_1|^2 + |\psi_2|^2 + \ldots + |\psi_{m+n}|^2,$$

we have

$$|\psi_1|^2 + |\varphi_2|^2 + \ldots + |\varphi_m|^2 = |\varphi_1|^2 + |\psi_2|^2 + \ldots + |\psi_{m+n}|^2.$$

By Proposition 1 there is a unitary matrix $\tilde{U}: \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ such that $\tilde{U} \circ \tilde{\varphi} = \tilde{\psi}$. Setting $\tilde{V} = \tilde{U}^{-1}$ we have a unitary matrix $\tilde{V}$ such that $\tilde{V} \circ \tilde{\psi} = \tilde{\varphi}$. Writing $\tilde{V} = [V_{ij}]$ gives

$$\begin{aligned}
\psi_1 &= v_{11}\varphi_1 + v_{12}\psi_2 + \ldots + v_{1\,m+n}\psi_{m+n} \\
\varphi_j &= v_{j1}\varphi_1 + v_{j2}\psi_2 + \ldots + v_{j\,m+n}\psi_{m+n} \qquad j = 2, \ldots, m \qquad (*) \\
0 &= v_{k1}\varphi_1 + v_{k2}\psi_2 + \ldots + v_{k\,m+n}\psi_{m+n} \qquad k = m+1, \ldots, m+n.
\end{aligned}$$

If $v_{11} = 0$ then, contrary to hypothesis, $\psi_1 = v_{12}\psi_2 + \ldots + v_{1\,m+n}\psi_{m+n}$ with $|v_{12}|^2 + \ldots + |v_{1\,m+n}|^2 = 1$. Thus $\varphi_1 = v_{11}^{-1}(\psi_1 - v_{12}\psi_2 - \ldots - v_{1\,m+n}\psi_{m+n})$. Substitution for φ_1 in the remaining equations of (*) yields

$$\varphi_j = \frac{1}{v_{11}}\left(v_{j1}\psi_1 + (v_{j2}v_{11} - v_{j1}v_{12})\psi_2 + \ldots + (v_{jm+n}v_{11} - v_{j1}v_{1m+n})\psi_{m+n}\right)$$

for $2 \leqslant j \leqslant m$ and

$$0 = \frac{1}{v_{11}}\left(v_{k1}\psi_1 + (v_{k2}v_{11} - v_{k1}v_{12})\psi_2 + \ldots + (v_{km+n}v_{11} - v_{k1}v_{1m+n})\psi_{m+n}\right).$$

Let V be the matrix of this transformation, so that

$$V \circ \psi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ i.e.,}$$

$$V = \frac{1}{v_{11}} \begin{bmatrix} 1 & -v_{12} & \cdots & -v_{1m+n} \\ v_{21} & & & \\ \vdots & & v_{11}v_{ji} - v_{j1}v_{1i} \quad \cdots & \\ v_{m+n1} & & \vdots & \end{bmatrix} i^{\text{th}}. \quad (j^{\text{th}} \text{ column})$$

A tedious check shows that $V \in U(1, m+n-1)$ and $V^{-1} = U$ satisfies $U \circ \varphi = \psi$.

Next we assume that the image of φ is not contained in a degenerate or spacelike hyperplane. With $\tilde{\varphi}, \tilde{\psi}$ as above $\exists\, U \in U(m+n)$ such that $U \circ \tilde{\varphi} = \tilde{\psi}$. If we write $U = [u_{ij}]$, we have $u_{11}\psi_1 + u_{12}\varphi_2 + \ldots + u_{1m}\varphi_m = \varphi_1$. If $u_{11} = 0$, then $\varphi_1 = u_{12}\varphi_2 + \ldots + u_{1m}\varphi_m$ with $|u_{12}|^2 + \ldots + |u_{1m}|^2 \leqslant 1$, since $(0, u_{12}, \ldots, u_{1m+n})$ is a row in a unitary matrix. The proof now continues as above. QED

In light of the Fundamental Lemma we introduce some notation. Let: $\varphi: M^2 \to \mathbb{C}_1^k$ be holomorphic. We say that φ satisfies condition

(D) if the image of φ is contained in some degenerate hyperplane;

(S) if the image of φ is contained in some spacelike but no degenerate hyperplane;

(F) if the image of φ is contained in no hyperplane or is contained in a timelike hyperplane, but no spacelike or degenerate hyperplane.

Now fix a holomorphic isometric immersion $\lambda: M^2 \to \mathbb{C}_1^m$, where the conformal class of h is used to make M^2 into a Riemann surface.

We will describe all non-congruent isometric immersions $\beta\colon M \to \mathbb{R}_1^n$ with $\eta = 0$ for any n. To say that λ and β are isometric means that

$$\frac{1}{2}\left\langle \frac{d\lambda}{dw}, \frac{d\lambda}{dw} \right\rangle = \left\langle \frac{d\beta}{dw}, \frac{d\beta}{dw} \right\rangle.$$

In order to normalize our immersions we introduce the following terminology.

Definition. An immersion $\beta\colon M \to \mathbb{R}_1^n$ is *full* if the image of β is not contained in any proper subspace of $\mathbb{R}_1^n$. (We make the analogous definition of $\lambda\colon M \to \mathbb{C}_1^m$).

Definition. An immersion $\beta\colon M \to \mathbb{R}_1^n$ is *degenerately full* if the image of β is contained in a degenerate hyperplane of $\mathbb{R}_1^n$, but not in any non-degenerate hyperplane of $\mathbb{R}_1^n$. The same definition is used for $\lambda\colon M \to \mathbb{C}_1^m$.

This definition reflects one difference between the Lorentzian and Riemannian ambient spaces. The curve $h(t) = (t, t, t^2)$ is contained in a degenerate plane but there is no non-degenerate plane to which we can restrict our attention. Note that a map cannot be both full and degenerately full.

For later use we record, without proof, the following fact.

Lemma 1. *If $\lambda\colon M^2 \to \mathbb{C}_1^m$ is holomorphic with $\lambda(p_0) = 0$ and the image of λ is not contained in any complex subspace of $\mathbb{C}_1^m$, then the image of λ is not contained in any real subspace of $\mathbb{C}_1^m = \mathbb{R}_2^{2m}$.*

3. Standard Forms for Holomorphic Functions with Equal Norms

We will always require that our immersions λ and β are either full or degenerately full and that for some $p_0 \in M$, $\lambda(p_0) = \beta(p_0) = 0$. If λ or β were contained in a spacelike or timelike hyperplane we could simply consider the maps as full or degenerately full in a lower dimensional vector space. Thus, by assumption, λ' satisfies either (F) or (D); in fact λ' is either full or satisfies (D). A priori $\dfrac{d\beta}{dw}$ can satisfy (F), (D), or (S). A large part of what follows determines which combinations are possible, as well as the relationships between m and n. In order to do that we need to find standard forms for holomorphic functions with equal norms.

We fix some notation which will be in force for the remainder of this paper. Let $\varphi: M \to \mathbb{C}_1^n$ and $\psi: M \to \mathbb{C}_1^m$ be holomorphic immersions with $\|\varphi\|^2 = \|\psi\|^2$ and $\varphi(p_0) = \psi(p_0) = 0$. We will replace φ by $\sqrt{2}\dfrac{d\beta}{dw}$ and ψ by λ' later. Before we give the standard forms we have two corollaries to the Fundamental Lemma.

Corollary 1. *Let ψ and φ be as above, with $n \leqslant m$. If ψ is full then there is a holomorphic isometry $F: \mathbb{C}_1^m \to \mathbb{C}_1^m$ such that $\psi = F \circ \varphi$. (F is a unitary transformation plus a translation.)*

Proof: [L], p. 147.

Corollary 2. *Let φ and ψ be as above, with $n \leqslant m$.*

a) *If φ satisfies* (D) *then ψ satisfies* (D).

b) *If ψ satisfies* (D) *then φ satisfies* (D) *or* (S).

Proof: (a) By Proposition 2, there is an $A \in U(1, n-1)$ such that $(A\varphi)_1 = (A\varphi)_2$. If ψ does not satisfy (D) then we have, by the Fundamental Lemma, $U \in U(1, m-1)$ such that $U \circ A\varphi = \psi$ or $A\varphi = V\psi$, where $V = U^{-1}$. If $V = [v_{ij}]$ this implies

$$v_{11}\psi_1 + \ldots + v_{1m}\psi_m = v_{21}\psi_1 + \ldots + v_{2m}\psi_m \text{ or}$$

$$\psi_1 = \frac{v_{22} - v_{12}}{v_{11} - v_{21}}\psi_2 + \ldots + \frac{v_{2m} - v_{1m}}{v_{11} - v_{21}}\psi_m.$$

($v_{11} \neq v_{21}$ because the first column of V has length -1).

But $\left|\dfrac{v_{22} - v_{12}}{v_{11} - v_{21}}\right|^2 + \ldots + \left|\dfrac{v_{2m} - v_{1m}}{v_{11} - v_{21}}\right|^2 = 1$ so that ψ does satisfy (D).

(b) Suppose, contrary to fact, that φ satisfies (F) and ψ satisfies (D). By the Fundamental Lemma there are unitary maps U and A such that $U \circ \varphi = A \circ \psi$ with

$$u_{11}\varphi_1 + \ldots + u_{1n}\varphi_n = u_{21}\varphi_1 + \ldots + u_{2n}\varphi_n.$$

This implies that $\varphi_1 = \dfrac{u_{22} - u_{12}}{u_{11} - u_{21}}\varphi_2 + \ldots + \dfrac{u_{2n} - u_{1n}}{u_{11} - u_{21}}\varphi_n$ with $|u_{22} - u_{12}|^2 + \ldots + |u_{2n} - u_{1n}|^2 \leqslant |u_{11} - u_{21}|^2$, so that the image of φ is contained in a spacelike or degenerate hyperplane. QED

Definition. Projection onto the last $k-2$ coordinates in $\mathbb{C}_1^k$ is denoted by $\mathrm{pr}: \mathbb{C}_1^k \to \mathbb{C}^{k-2}$.

Definition. Projection onto the last $k-1$ coordinates in $\mathbb{C}_1^k$ is denoted by $\mathrm{qr}: \mathbb{C}_1^k \to \mathbb{C}^{k-1}$.

Proposition 4. *If ψ satisfies* (D) *and $m \leqslant n$ then there is a $B \in U(1, m-1)$ and $V \in U(1, n-1)$ such that $(B\psi)_1 = (B\psi)_2$ and*

$$\varphi = V \begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\psi \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some holomorphic function h. Conversely any $\varphi: M \to \mathbb{C}_1^n$ defined from ψ in this way satisfies $\|\varphi\|^2 = \|\psi\|^2$.

Proof: By Corollary 2, φ also satisfies (D). Thus we can find an $A \in U(1, n-1)$ and $B \in U(1, m-1)$ so that $(A\varphi)_1 = (A\varphi)_2$ and $(B\psi)_1 = (B\psi)_2$. After including $\mathbb{C}^{m-2}$ into $\mathbb{C}^{n-2}$ Calabi's result gives a $U \in U(n-2)$ such that

$$\mathrm{pr}\, A\varphi = U \begin{bmatrix} \mathrm{pr}\, B\psi \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now $A\varphi = \begin{bmatrix} h \\ h \\ \mathrm{pr}\, A\varphi \end{bmatrix}$ for some holomorphic function $h = (A\varphi)_1$. Thus

$$A^{-1}(A\varphi) = \varphi = A^{-1} \begin{bmatrix} h \\ h \\ \mathrm{pr}\, A\varphi \end{bmatrix} = A^{-1} \begin{bmatrix} U \begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\psi \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{bmatrix} = V \begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\psi \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some $V \in U(1, n-1)$.

If $(B\psi)_1 = (B\psi)_2$ then it is clear that for any $V \in U(1, n-1)$,

$$\left\| V \begin{bmatrix} h \\ h \\ \mathrm{pr}\,(B\psi) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|^2 = \|\psi\|^2 .$$

There are two more propositions which are similar to Proposition 4.

Proposition 5. *If* φ *and* ψ *satisfy* (D) *and* $m > n$ *then there are* $V, B \in U(1, m-1)$ *such that*

$$\begin{bmatrix} \varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V \begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\psi \end{bmatrix}$$

for a holomorphic function h. *If* $(B\psi)_1 = (B\psi)_2$ *and* $V \in U(1, m-1)$ *then any* $(\varphi, 0, \ldots, 0) = V(h, h, \mathrm{pr}\, B\psi)$ *satisfies* $\|\varphi\|^2 = \|\psi\|^2$.

Proof. As above we can find $A \in U(1, n-1)$ and $B \in U(1, m-1)$ so that $(A\varphi)_1 = (A\varphi)_2$ and $(B\psi)_1 = (B\psi)_2$. In addition there is a $U \in U(n-2)$ so that

$$U \begin{bmatrix} \mathrm{pr}\, A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathrm{pr}\, B\psi \quad \text{or} \quad \begin{bmatrix} \mathrm{pr}\, A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = U^{-1} \mathrm{pr}\,(B\psi).$$

Now

$$\begin{bmatrix} \varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A^{-1} A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A^{-1} & \\ & I_{m-n} \end{bmatrix} \begin{bmatrix} h \\ h \\ \mathrm{pr}\, A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} A^{-1} & \\ & I_{m-n} \end{bmatrix} \begin{bmatrix} h \\ h \\ U^{-1} \mathrm{pr}\, B\psi \end{bmatrix} = V \begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\psi \end{bmatrix}. \quad \text{QED}$$

Proposition 6. *If ψ satisfies* (D), *φ satisfies* (S) *and $m > n$ then there is a $V \in U(1, m-2)$ and a $B \in U(1, m-1)$ such that*

$$\begin{bmatrix} \varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V \begin{bmatrix} 0 \\ \operatorname{pr} B\psi \end{bmatrix}.$$

If $(B\psi)_1 = (B\psi)_2$ then any φ defined from ψ in this way satisfies $\|\varphi\|^2 = \|\psi\|^2$.

Proof: There is an $A \in U(1, n-1)$ and a $B \in U(1, m-1)$ such that $(A\varphi)_1 = 0$ and $(B\psi)_1 = (B\psi)_2$.

After including $\mathbb{C}^{n-1}$ into $\mathbb{C}^{m-2}$ there is a $U \in U(m-2)$ such that

$$U \begin{bmatrix} \operatorname{qr} A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \operatorname{pr} B\psi \quad \text{or} \quad \begin{bmatrix} \operatorname{qr} A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = U^{-1} \operatorname{pr} B\psi.$$

As before

$$\begin{bmatrix} A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \operatorname{qr} A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ U^{-1} \operatorname{pr} B\psi \end{bmatrix},$$

so

$$\begin{bmatrix} A^{-1} & \\ & I_{m-n-1} \end{bmatrix} \begin{bmatrix} A\varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A^{-1} & \\ & I_{m-n-1} \end{bmatrix} \begin{bmatrix} 0 \\ U^{-1} \operatorname{pr} B\psi \end{bmatrix}$$

or $$\begin{bmatrix} \varphi \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V \begin{bmatrix} 0 \\ \operatorname{pr} B\psi \end{bmatrix}$$ for some $V \in U(1, m-2)$. QED

4. Relations Between the Dimensions of the Receiving Spaces

Now we assume that we have both a holomorphic $\lambda: M \to \mathbb{C}_1^m$ and a maximal $\beta: M \to \mathbb{R}_1^n$ which are isometric, full or degenerately full and that $\lambda(p_0) = \beta(p_0) = 0$, for some $p_0 \in M$. This implies that

$2\left\|\frac{d\beta}{dw}\right\|^2 = \|\lambda'\|^2$ and that $\frac{d\beta}{dw}$ is holomorphic. We also note that with these assumptions λ is full iff λ' is full.

The next several propositions determine the possible relationships between m and n.

Proposition 7. *Assume that* $\lambda: M \to \mathbb{C}_1^m$ *is holomorphic and full,* $\beta: M \to \mathbb{R}_1^n$ *is maximal and full or degenerately full and* $\|\lambda'\|^2 = 2\left\|\frac{d\beta}{dw}\right\|^2$. *Then* $m \leqslant n \leqslant 2m+1$.

(Note: This proof follows [L] p. 148. There the conclusion is $m \leqslant n \leqslant 2m$. We will be able to obtain the same result later.)

Proof: If λ is full then there is a unitary matrix $U \in U(1, N-1)$, $N = \max\{m, n\}$, such that

$$\frac{d\beta}{dw} = \frac{U}{\sqrt{2}}\lambda'. \tag{*}$$

If $m > n$ then

$$\begin{bmatrix} \left(\frac{d\beta}{dw}\right)_1 \\ \vdots \\ \left(\frac{d\beta}{dw}\right)_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{U}{\sqrt{2}} \begin{bmatrix} \lambda'_1 \\ \vdots \\ \lambda'_m \end{bmatrix}.$$

The $(n+1)^{\text{st}}$ entry gives a linear relationship among $\{\lambda'_1, \ldots, \lambda'_m\}$, which would imply that λ' is not full. Since

$$\beta(w) = 2\,\mathrm{Re}\int_{p_0}^{w} \frac{d\beta}{dz}\,dz,$$

it follows from (*) that

$$\beta(w) = \sqrt{2}\,\mathrm{Re}\int_{p_0}^{w} U\frac{d\lambda}{dz}\,dz = \sqrt{2}\,\mathrm{Re}\,U\lambda(w) = \frac{1}{\sqrt{2}}(U\lambda + \bar{U}\bar{\lambda})(w).$$

Denoting the first m columns of U by S we have

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \sqrt{2}\,[\operatorname{Re} S, \operatorname{Im} S]\begin{bmatrix} \operatorname{Re}\lambda \\ -\operatorname{Im}\lambda \end{bmatrix}.$$

If $n - 2m > 0$ we can find at least $n - 2m$ linearly independent real vectors $d \in \mathbb{R}^n$ with $d[\operatorname{Re} S, \operatorname{Im} S] = 0$, so that

$$d\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = 0.$$

Since β is either full or degenerately full, there can be at most one such $d \neq 0$, and d must be null. QED

Proposition 8. *Assume that* $\lambda: M^2 \to \mathbb{C}_1^m$ *is holomorphic and degenerately full,* $\beta: M \to \mathbb{R}_1^n$ *is maximal and full or degenerately full,* $\|\lambda'\|^2 = 2\left\|\frac{d\beta}{dw}\right\|^2$ *and* $m \leqslant n$*. Then* $n \leqslant 2m + 1$.

Proof: By the hypothesis λ' satisfies (D) and since $m \leqslant n$ by Corollary 2 we know $\frac{d\beta}{dw}$ also satisfies (D). Thus, by Proposition 4,

$$\sqrt{2}\frac{d\beta}{dw} = V\begin{bmatrix} h' \\ h' \\ \operatorname{pr} B\lambda' \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some holomorphic h and $V \in (1, n-1)$. Thus

$$\beta = \sqrt{2}\operatorname{Re} V\begin{bmatrix} h \\ h \\ \operatorname{pr} B\lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Letting S be the first m columns of V, we have

$$\beta = \sqrt{2}\,\mathrm{Re}\, S\begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\lambda \end{bmatrix}.$$

As in Proposition 7, $n \leqslant 2m + 1$. QED

Proposition 9. *Assume that* $\lambda: M^2 \to \mathbb{C}_1^m$ *is holomorphic and degenerately full,* $\beta: M^n \to \mathbb{R}_1^m$ *is maximal and full or degenerately full and* $\|\lambda'\|^2 = 2\left\|\frac{d\beta}{dw}\right\|^2$. *If* $\frac{d\beta}{dw}$ *satisfies* (D), *then* $m \leqslant n$.

Proof: We examine the proof of Proposition 5 more closely. If $m > n$ the proof says

$$\begin{bmatrix} \frac{d\beta}{dw} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1/\sqrt{2}\begin{bmatrix} \begin{matrix} a_{11}^{-1} & \cdots & a_{1n}^{-1} \\ \vdots & \ddots & \vdots \\ a_{n1}^{-1} & \cdots & a_{nn}^{-1} \end{matrix} & \\ & I_{m-n} \end{bmatrix}\begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \\ & \begin{matrix} u_{11}^{-1} & \cdots & u_{1\,m-2}^{-1} \\ \vdots & \ddots & \vdots \\ u_{m-21}^{-1} & \cdots & u_{m-2\,m-2}^{-1} \end{matrix} \end{bmatrix}\begin{bmatrix} h \\ h \\ \mathrm{pr}\, B\lambda' \end{bmatrix}.$$

But the $(n + 1)$st entry gives a linear relation among $\lambda'_1, \ldots, \lambda'_m$ in addition to $(B\lambda')_1 = (B\lambda')_2$, which contradicts the assumption that λ' is degenerately full. QED

Proposition 10. *Assume that* $\lambda: M^2 \to \mathbb{C}_1^m$ *is holomorphic and degenerately full,* $\beta: M^2 \to \mathbb{R}_1^n$ *is maximal, full or degenerately full and* $\|\lambda'\|^2 = 2\left\|\frac{d\beta}{dw}\right\|^2$. *If* $\frac{d\beta}{dw}$ *satisfies* (S) *and* $m > n$ *then* $m = n + 1$.

Before the proof we note that there are examples of these kinds of λ and β.

Example 1. Let $m = 3$, $n = 2$, $\lambda(z) = (z^2, z^2, z)$ and

$$\frac{d\beta}{dz} = \begin{bmatrix} \cosh(\sigma + i\tau) & \sinh(\sigma + i\tau) \\ \sinh(\sigma + i\tau) & \cosh(\sigma + i\tau) \end{bmatrix}\begin{bmatrix} 0 \\ z \end{bmatrix}$$

with $(\cosh\sigma)(\sinh\sigma)(\cos\tau)(\sin\tau) \neq 0$. Then $\beta(x, y) = (\sinh\sigma\cos\tau\,(x^2 - y^2) - 2\cosh\sigma\sin\tau\,(xy),\ \cosh\sigma\cos\tau\,(x^2 - y^2) - 2\sinh\sigma\sin\tau\,(xy))$, and $\frac{d\beta}{dz}$ is not full, while $\beta(x, y)$ is full. (See [H-O] p. 56.)

Proof: By Proposition 6 we have

$$\begin{bmatrix} \dfrac{d\beta}{dw} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V \begin{bmatrix} 0 \\ \dfrac{\operatorname{pr} B\lambda'}{\sqrt{2}} \end{bmatrix}$$

for $V \in U(1, m-2)$. Again

$$\begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sqrt{2}\operatorname{Re} V \begin{bmatrix} 0 \\ (B\lambda)_3 \\ \vdots \\ (B\lambda)_m \end{bmatrix}.$$

If $m - n - 1 > 0$, this yields $0 = v_{n+1,2}(B\lambda)_3 + \dots + v_{n+1,m-1}(B\lambda)_m$, which is a contradiction if $(v_{n+1,2}, \dots, v_{n+1,m-1}) \neq 0$. If $(v_{n+1,2}, \dots, v_{n+1,m-1})$ were the zero vector then the $(n+1)$st row of V would be $(v_{n+1,1}, \dots, v_{n+1,m-1})$ and would not have positive length. Thus $m = n + 1$. QED

Combining the results in Corollary 2, and Propositions 7 through 10 we have so far established the following.

If λ is full then $m \leqslant n \leqslant 2m + 1$, while if λ is degenerately full either $\dfrac{d\beta}{dw}$ satisfies (D) and $m \leqslant n \leqslant 2m + 1$ or $\dfrac{d\beta}{dw}$ satisfies (S) and $m = n + 1$.

We can actually show that in the first two cases that $n \leqslant 2m$. In the proof of Proposition 7 we noted that β is degenerately full iff there is a null vector $c \in \mathbb{R}^n_1$ such that

$$c[\operatorname{Re} S, \operatorname{Im} S] \begin{bmatrix} \operatorname{Re}\lambda \\ -\operatorname{Im}\lambda \end{bmatrix} = 0$$

and no vector $d \neq ac$ with this property. If the rank of $[\operatorname{Re} S, \operatorname{Im} S]$ is $n - 1$, then there is one vector d such that $d[\operatorname{Re} S, \operatorname{Im} S] = 0$. For the β defined using S to be degenerately full, this vector d must be null. The following lemma shows this cannot occur.

Lemma 2. *Let S be an $n \times m$ complex matrix formed from the first m columns of a matrix $V \in U(1, n-1)$. If the rank of $[\operatorname{Re} S, \operatorname{Im} S]$ is $n - 1$ and $d[\operatorname{Re} S, \operatorname{Im} S] = 0$ for some $d \neq 0$, then d is not null.*

Proof: For convenience, set $[\operatorname{Re} S, \operatorname{Im} S] = Q = [v_1 \ldots, v_m, w_1, \ldots, w_m]$. The hypothesis $dQ = 0$ is equivalent to $d \cdot v_j = 0 = d \cdot w_j$ for $1, \ldots, m$, where $\cdot$ denotes the standard dot product. This implies that d is perpendicular to the column space of Q in $\mathbb{R}^n$. Suppose that d is null in $\mathbb{R}^n_1$, i.e., $g(d, d) = 0$.

Given any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $\hat{x} = (-x_1, x_2, \ldots, x_n)$. Then $x \cdot y = g(\hat{x}, y)$, and we have $g(\hat{d}, v_j) = 0 = g(\hat{d}, w_j) = g(\hat{d}, \hat{d})$. If we denote the column space of Q by W then our hypothesis implies $\dim W = n - 1$. Since $d \cdot W = 0$, $\{d\} \oplus W = \mathbb{R}^n$. We also know that $\hat{d} \cdot d = 0$, which implies that $\hat{d} \in W \cap W^{\perp}$ where $W^{\perp}$ is the orthogonal complement of W in $\mathbb{R}^n_1$. This means W is a degenerate subspace. However $g(v_1, v_1) + g(w_1, w_1) = -1$, which holds because $V \in U(1, n-1)$, implies that v_1 or w_1 is timelike and a degenerate subspace has no non-zero timelike vectors. QED

We can conclude that $\operatorname{rank} Q = n$ and, in Propositions 7 and 8, $n \leqslant 2m$. We summarize these results in the following theorem.

Theorem 2. *Let* $\lambda: M^2 \to C^m_1$ *be holomorphic and* $\beta: M^2 \to \mathbb{R}^n_1$ *be maximal with* $\|\lambda'\|^2 = 2\left\|\dfrac{d\beta}{dw}\right\|^2$. *Assume in addition both are full or degenerately full while for some* $p_0 \in M^2$, $\lambda(p_0) = \beta(p_0) = 0$.

(a) *If* λ *is full then* $\beta = \sqrt{2} \operatorname{Re} S\lambda$, *where* S *is an* $n \times m$ *complex matrix satisfying*

(i) ${}^t\bar{S}\begin{bmatrix} -1 & \\ & I_{n-1}\end{bmatrix} S = \begin{bmatrix} -1 & \\ & I_{m-1}\end{bmatrix}$, *so that* S *is the first* m *columns of a unitary matrix in* $\mathbb{C}^n_1$,

(ii) *the* $n \times 2m$ *matrix* $[\operatorname{Re} S, \operatorname{Im} S]$ *has rank* n, *so that* β *is full*,

(iii) ${}^t\lambda' \, {}^tS\begin{bmatrix} -1 & \\ & I_{n-1}\end{bmatrix} S\lambda' = 0$, *so that* $\dfrac{d\beta}{dw}$ *lies on the quadric in* $\mathbb{C}P^{n-1}_1$,

(iv) $m \leqslant n \leqslant 2m$.

(b) *If* λ *and* $\dfrac{d\beta}{dw}$ *satisfy* (D) *then* $\beta = \sqrt{2} \operatorname{Re} S\begin{bmatrix} h \\ h \\ \operatorname{pr}(B\lambda)\end{bmatrix}$ *for some holomorphic* h *and* $B \in U(1, m-1)$ *with* $(B\lambda)_1 = (B\lambda)_2$, *where* S *satisfies* (i)—(iv).

(c) *If* λ *satisfies* (D) *and* $\dfrac{d\beta}{dw}$ *satisfies* (S) *then* $m = n + 1$, $\beta = \sqrt{2}\,\mathrm{Re}\,V\begin{bmatrix} 0 \\ \mathrm{pr}\,B\lambda \end{bmatrix}$ *where* $V \in U(1, m-2)$ *and* $B \in U(1, m-1)$ *and* (iii) *holds with* V *substituted for* S. *The rank of* $[\mathrm{Re}\,V, \mathrm{Im}\,V]$ *is always* n.

5. Determination of All Immersions Isometric to a Fixed $\lambda: M^2 \to \mathbb{C}_1^m$

Given $\lambda: M \to \mathbb{C}_1^m$ we determine all maximal $\beta: M \to \mathbb{R}_1^n$ isometric to λ. Unlike the case studied by Calabi, we must be concerned with whether β is full or degenerately full.

Lemma 3. *Let* S *be an* $n \times m$ *complex matrix formed from the first* m *columns of a unitary matrix in* $\mathbb{C}_1^n$. *If* $\lambda: M^2 \to \mathbb{C}_1^m$ *is holomorphic and full and rank* $[\mathrm{Re}\,S, \mathrm{Im}\,S] = n$ *then* $\beta = \sqrt{2}\,\mathrm{Re}\,(S\lambda)$ *is full.*

Proof: If β is not full then there is a $d \neq 0$ in $\mathbb{R}^n$ such that $d[\mathrm{Re}\,S, \mathrm{Im}\,S]\begin{bmatrix} \mathrm{Re}\,\lambda \\ -\mathrm{Im}\,\lambda \end{bmatrix} = 0$, giving $d[\mathrm{Re}\,S, \mathrm{Im}\,S]\cdot \mathrm{image}\,\lambda = 0$ in $\mathbb{R}^{2m}$. However, by Lemma 1, the image of λ spans $\mathbb{R}^{2m}$ so that $d[\mathrm{Re}\,S, \mathrm{Im}\,S] = 0$, a contradiction. QED

Now we can state our result in the simplest case, which is the exact analogy of the case, studied by Calabi.

Proposition 11. *If* $\lambda: M^2 \to \mathbb{C}_1^m$ *is holomorphic and full, and* S *is an* $n \times m$ *matrix satisfying* (i)—(iv) *in Theorem* 2 *then* $\beta = \sqrt{2}\,\mathrm{Re}\,(S\lambda)$ *is full, maximal in* $\mathbb{R}_1^n$ *and isometric to* λ.

The following example shows that if λ satisfies (D) and S satisfies (i)—(iv) in Theorem 2 it is still possible for $\beta = \sqrt{2}\,\mathrm{Re}\,(S\lambda)$ to be neither full nor degenerately full.

Example 2. Let $m = 3$, $n = 5$.

Set

$$\lambda(w)=\begin{bmatrix}h(w)\\h(w)\\k(w)\end{bmatrix}\subset \operatorname{span}_{\mathbb{C}}\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}=\operatorname{Span}_{\mathbb{R}}\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}.$$

Let $S=\begin{bmatrix}\cosh\theta & 0 & 0\\ i\sinh\theta & 0 & 0\\ 0 & \sin\psi & 0\\ 0 & i\cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$. S satisfies (i) and (ii) and if

$k^2=h^2(-\cosh^2\theta-\sinh^2\theta+\sin^2\psi-\cos^2\psi)$ then S satisfies (iii). If

$$d=\left[\frac{\sin\psi}{\cosh\theta},-1,-1,\frac{\sinh\theta}{\cos\psi},0\right]$$

then $d\cdot[\operatorname{Re}S,\operatorname{Im}S]=[\sin\psi,-\sin\psi,0,-\sinh\theta,\sinh\theta,0]$ so that $d[\operatorname{Re}S,\operatorname{Im}S]\begin{bmatrix}\operatorname{Re}\lambda\\-\operatorname{Im}\lambda\end{bmatrix}=0$. Since d is non-null β is neither full nor degenerately full.

Lemma 4. *Let $\lambda\colon M^2\to\mathbb{C}_1^m$ be holomorphic and degenerately full. If $m=n+1$, $B\in U(1,m-1)$ so that $(B\lambda)_1=(B\lambda)_2$, $V\in U(1,n-1)$ and $\beta=\sqrt{2}\operatorname{Re}V\begin{bmatrix}0\\\operatorname{pr}(B\lambda)\end{bmatrix}$ then $\left(\frac{d\beta}{dw}\right)$ satisfies* (S).

Proof: $$V\begin{bmatrix}0\\(B\lambda)_3\\\vdots\\(B\lambda)_m\end{bmatrix}=\begin{bmatrix}v_{12}(B\lambda)_3+\dots+v_{1n}(B\lambda)_m\\\vdots\\v_{n2}(B\lambda)_3+\dots+v_{nn}(B\lambda)_m\end{bmatrix}.$$

Note that

$$-\bar v_{11}(v_{12}(B\lambda)_3+\dots+v_{1n}(B\lambda)_m)+\dots+\bar v_{n1}(v_{n2}(B\lambda)_3+\dots+v_{nn}(B\lambda)_m)=0,$$

i.e., there is a complex vector $\gamma=(\gamma_1,\dots,\gamma_n)$ such that

$$\gamma V \begin{bmatrix} 0 \\ (B\lambda)_3 \\ \vdots \\ (B\lambda)_m \end{bmatrix} = 0.$$

Equivalently $\sum_{j=1}^{n} \gamma_j \left(\frac{d\beta}{dw}\right)_j = 0$, so that $\left(\frac{d\beta}{dw}\right)_1 = \sum_{k=2}^{m} - \left(\frac{\gamma_k}{\gamma_1}\right)\left(\frac{d\beta}{dw}\right)_k$.
Since $-|\gamma_1|^2 + |\gamma_2|^2 + \ldots + |\gamma_n|^2 = -1$,

$$\frac{|\gamma_2|^2 + \ldots + |\gamma_n|^2}{|\gamma_1|^2} = \frac{-1 + |\gamma_1|^2}{|\gamma_1|^2} < 1.$$

It is also easy to see that $\frac{d\beta}{dw}$ is contained in no degenerate hyperplane.
QED

Lemma 5. *If* $V \in U(1, n-1)$ *then* rank $[\mathrm{Re}\, V, \mathrm{Im}\, V] = n$.

Proof: If rank $[\mathrm{Re}\, V, \mathrm{Im}\, V] < n$ then there is a $d \neq 0 \in \mathbb{R}^n$ such that

$$\sum_{m=1}^{n} d_m \mathrm{Re}\, v_{mj} = 0 = \sum_{m=1}^{n} d_m \mathrm{Im}\, v_{mj} \quad j = 1, \ldots, n.$$

But then $\sum_{m=1}^{n} (d_m + i\, d_m)\, v_{mj} = 0$, $j = 1, \ldots, n$, which would imply that rank $V < n$. QED

Lemma 6. *Let* $\lambda\colon M^2 \to \mathbb{C}_1^m$ *be holomorphic and degenerately full. Set* $n = m-1$, *choose* $V \in U(1, n-1)$ *and let* $\beta = \sqrt{2}\, \mathrm{Re}\, V \begin{bmatrix} 0 \\ \mathrm{pr}\,(B\lambda) \end{bmatrix}$ *for* $B \in U(1, m-1)$ *with* $(B\lambda)_1 = (B\lambda)_2$. *If* β *is not full then the first column of* V *is a vector of the form* $v + i\mu v$ *or* iv *with* $v \in \mathbb{R}_1^n$, $\mu \in \mathbb{R}$ *and* $0 > g(v, v) \geqslant -1$. *If the first column of* V *is of this form then* β *is neither full nor degenerately full.*

Proof: Assume first that β is not full. Then there is a non-zero vector $d \in \mathbb{R}_1^n$ such that

$$d[\mathrm{Re}\, V, \mathrm{Im}\, V] \begin{bmatrix} 0 \\ \mathrm{Re}\,(\mathrm{pr}\, B\lambda) \\ 0 \\ -\mathrm{Im}\,(\mathrm{pr}\, B\lambda] \end{bmatrix} = 0.$$

If $[\mathrm{Re}\, V, \mathrm{Im}\, V] = [v_1, \ldots, v_n, w_1, \ldots, w_n]$ then

$$d \cdot v_k = 0 = d \cdot w_k, \quad k = 2, \ldots, n, \tag{1}$$

because $\mathrm{pr}(B\lambda)$ is full in $\mathbb{C}^{n-1}$. Taking advantage of the fact that $V \in U(1, n-1)$, we write

$$d[\mathrm{Re}\, V, \mathrm{Im}\, V] \begin{bmatrix} -1 & & & \\ & I_{n-1} & & \\ & & -1 & \\ & & & I_{n-1} \end{bmatrix} \begin{bmatrix} {}^t\mathrm{Re}\, V \\ {}^t\mathrm{Im}\, V \end{bmatrix} =: C$$

in two ways. Using (1) C equals

$$[x, 0, \ldots, 0, y, 0, \ldots, 0] \begin{bmatrix} -1 & & & \\ & I_{n-1} & & \\ & & -1 & \\ & & & I_{n-1} \end{bmatrix} \begin{bmatrix} {}^t\mathrm{Re}\, V \\ {}^t\mathrm{Im}\, V \end{bmatrix}, \text{ where}$$

$$\begin{aligned} x &= d \cdot v_1 \\ y &= d \cdot w_1 . \end{aligned} \tag{2}$$

But C is also equal to $[-d_1, d_2, \ldots, d_n] = \hat{d}$. Thus we have

$$\begin{aligned} d_1 &= x \,\mathrm{Re}\, v_{11} + y \,\mathrm{Im}\, v_{11} \\ d_k &= -x \,\mathrm{Re}\, v_{k1} - y \,\mathrm{Im}\, v_{k1}, \quad k = 2, \ldots, n. \end{aligned} \tag{3}$$

Substituting (3) in (2) gives $x = Ax + By$ and $y = Bx + Cy$, where $A = -g(v_1, v_1)$, $B = -g(v_1, w_1)$ and $C = -g(w_1, w_1)$.

We have $A + C = 1$, and by Lemma 5 we know $xy \neq 0$. Thus $\begin{vmatrix} A-1 & B \\ B & -A \end{vmatrix} = 0$ or $B^2 = A(A-1)$. This gives $A(1-A) \geqslant 0$ or $1 \geqslant A \geqslant 0$ and $1 \geqslant C \geqslant 0$. If $v_1 = 0$ or $w_1 = 0$ the first column of V is of the required form, so we may assume that $v_1 \neq 0$ and $w_1 \neq 0$. One of v_1 und w_1 is timelike. If $A = 0$, i.e., v_1 is null, then $B = 0$, but again we would have $v_1 = 0$. So we may assume that v_1 and w_1 are future timelike. Then $B^2 = A(A-1)$ implies, using the reverse Cauchy Schwarz inequality, that $w_1 = \mu v_1$ and $0 > g(v_1, v_1) \geqslant -1$.

Conversely, assume that the first column of V is of the form $v + i\mu v$ or iv with $0 > g(v_1, v_1) \geqslant -1$. In the first case write $[\mathrm{Re}\, V, \mathrm{Im}\, V] = [v, v_2, \ldots, v_n, \mu v, w_2, \ldots, w_n]$. Because V is unitary there are several equations relating the inner products of the columns of the matrix,

$$\begin{aligned} g(v, v_k) + \mu g(v, w_k) &= 0 \\ \mu g(v, v_k) &= g(v, w_k) \qquad\qquad k = 2, \ldots, n \\ g(v, v) + \mu^2 g(v, v) &= -1. \end{aligned}$$

The first two sets imply that $g(v, v_k) = 0 = g(v, w_k)$, i.e., $\hat{v} \cdot v_k = 0 = \hat{v} \cdot w_k$. Thus

$$\hat{v}[\operatorname{Re} V, \operatorname{Im} V] \begin{bmatrix} 0 \\ \operatorname{Re}\operatorname{pr}(B\lambda) \\ 0 \\ -\operatorname{Im}\operatorname{pr}(B\lambda) \end{bmatrix} = 0.$$

Since $g(\hat{v}, \hat{v}) \neq 0$, β is neither full nor degenerately full. The second case is similar. QED

Now we can state the theorem.

Theorem 3. *If* $\lambda: M \to \mathbb{C}_1^m$ *is holomorphic and degenerately full,* $n = m - 1$, $V \in U(1, n-1)$, *V satisfies* (iii) *of Theorem 2 and the first column of V is not a vector of the form* $v + i\mu v$ *or* iv *with* $v \in \mathbb{R}_1^n$ *and* $0 > g(v, v) \geqslant -1$ *then for* $B \in U(1, m-1)$ *with* $(B\lambda)_1 = (B\lambda)_2$

$$\beta = \sqrt{2} \operatorname{Re} V \begin{bmatrix} 0 \\ \operatorname{pr}(B\lambda) \end{bmatrix}$$

is maximal, full, isometric to λ *and* $\frac{d\beta}{dw}$ *satisfies* (*S*).

The last theorem describes those $\beta: M \to \mathbb{R}_1^n$ for which $\frac{d\beta}{dw}$ satisfies (D). In order to state the theorem we introduce some additional notation. If S is an $n \times m$ complex matrix and $[\operatorname{Re} S, \operatorname{Im} S] = [v_1, \ldots, v_m, w_1, \ldots, w_m]$ set

$$X = \operatorname{span}_{\mathbb{R}} \{v_1 + v_2, v_3, \ldots, v_m, w_1 + w_2, w_3, \ldots, w_m\} \subset \mathbb{R}_1^n.$$

Theorem 4. *Let* $\lambda: M \to \mathbb{C}_1^m$ *be holomorphic and degenerately full and S an* $n \times m$ *complex matrix satisfying* (i)—(iv). *If* $\dim X = n$ *or* $\dim X = n - 1$ *and X is degenerate then* $\beta = \sqrt{2} \operatorname{Re} S \begin{bmatrix} h \\ h \\ \operatorname{pr} B\lambda \end{bmatrix}$ *is maximal, isometric to* λ *and full or degenerately full. We note that* $\frac{d\beta}{dw}$ *satisfies* (D) *and* β *is degenerately full if* $\dim X = n - 1$.

Proof: We need only prove that β is full or degenerately full. Here β is full or degenerately full iff $d \in \mathbb{R}_1^n$ with

$$d_1\beta_1 + \ldots + d_n\beta_n = 0 \tag{4}$$

implies that $d = 0$ (so β is full) or that d is a multiple of a fixed null vector (so β is degenerately full). Because $B\,\mathrm{pr}\,\lambda$ is full in $\mathbb{C}^{m-2}$, (4) holds iff $d[\mathrm{Re}\,S, \mathrm{Im}\,S] \in \mathrm{span}_{\mathbb{R}}\{e_1 - e_2, e_{m+1} - e_{m+2}\}$ in $\mathbb{R}_2^{2m}$. Thus, (4) holds iff $d \cdot v_1 + v_2 = 0 = d \cdot v_k = d \cdot w_1 + w_2 = d \cdot w_k$, $k = 3, 4, \ldots, m$.

Here $\dim X = n - 2$ or $n - 1$ or n. Now $\dim X = n$ iff β is full, while if $X = n - 2$ there is a non-null d such that $d \cdot X = 0$ and β is not full nor degenerately full. Finally, if $\dim X = n - 1$, there is a null vector $d \in \mathbb{R}_1^n$ such that $d \cdot X = 0$ iff X is degenerate. To see this note that $d \cdot X = 0$ iff $\hat{d} \perp X$ in $\mathbb{R}_1^n$. If $\hat{d} \in X \cap X^\perp$ then $\hat{d} \perp X$ and $d \cdot X = 0$. If $d \cdot X = 0$ and d is null than $\hat{d} \in X \cap X^\perp$. QED

References

[C1] CALABI, E.: Isometric imbeddings of complex manifolds. Ann. Math. **58**, 1—23 (1953).

[C2] CALABI, E.: Quelques applications l'analyse complexe aux surfaces d'aire minima (together with Topics Complex Manifolds by H. ROSSI). Les Presses de l'Université de Montreal, 1968.

[H-O] HOFFMANN, D., OSSERMAN, R.: The Geometry of the Generalized Gauss Map. Memoirs of the A. M. S., No. 236, 1980.

[L] LAWSON, H. B. Jr.: Lectures on Minimal Submanifolds, Vol. I. Berkeley, CA: Publish or Perish, Inc.

K. ABE
Department of Mathematics
University of Connecticut
Storrs, CT 06268, U.S.A.

M. A. MAGID
Department of Mathematics
Wellesley College
Wellesley, MA 02181, U.S.A.