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d_f-Cohomology of Lagrangian Foliations

By

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Abstract. The cohomology of various sheaves associated with a Lagrangian foliation is discussed. Obstructions for the foliation either to behave like in the case of a cotangent bundle or to have an affine transversal distribution are found in this cohomology. Furthermore, one proves a formula for all the symplectic structures of a cotangent bundle such that the fibers constitute a Lagrangian foliation.

In this paper we discuss cohomology of various sheaves naturally associated with Lagrangian foliations, and which have resolutions based on the operator d_f of exterior differentiation along the leaves. The applications consist in obtaining cohomological obstructions, namely: the obstruction for the manifold to behave like a cotangent bundle (Proposition 3.1), the obstructions for the foliation to have an affine transversal distribution (Proposition 4.1) [M3], and the obstructions to the existence of a symplectic form that makes a given foliation into a Lagrangian foliation (Proposition 5.3). Among other results, we obtain a geometric characterization of cotangent bundles (Proposition 3.3), and a formula for all the symplectic forms of T^*M for which the fibers define a Lagrangian foliation (Proposition 5.5).

1. Examples and Basic Structure

We send to [AM] for general information on symplectic geometry and Lagrangian foliations, and we shall usually denote by (V^{2n}, ω) a symplectic manifold, and by $\mathscr{L}^n$ a Lagrangian foliation on V. (Everything is C^∞ here.) Following are some examples:

[1] This paper has been written while the author was a visiting professor at the Universities of Montpellier and Strasbourg (France). He wants to acknowledge here his gratitude to these host institutions, and, particularly, to Pierre Molino, Claude Godbillon and Daniel Bernard.

a) $V = \mathbb{R}^{2n} = \{(x^i, y^i)_{i=1}^n\}$, $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ (*the canonical form*), $\mathscr{L}$ is defined by $x^i = \text{ct.}$, (*vertical foliation*) or $y^i = \text{ct.}$ (*horizontal foliation*). By a quotientization, similarly defined foliations are obtained on the torus T^{2n}.

b) $V = T^*M^n$ (an arbitrary cotangent bundle) with (x^i) = local coordinates on M, (y^i) = associated natural covector coordinates, with the canonical form ω of example a), and the *vertical foliation* $\mathscr{L} = \mathscr{V}$ is defined by the fibres ($x^i = \text{ct.}$). Generally, we do not have a horizontal Lagrangian foliation as in the case of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, but, for instance, we have one in the case of T^*T^n.

c) $V_\lambda = [(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})]/K_\lambda$, with the coordinates (x^i), (y^i) in the two copies of $\mathbb{R}^n$, and where K_λ is the group generated by the transformation $x^i \to \lambda x^i$, $y^i \to (1/\lambda) y^i$ $(0 < \lambda < 1)$; the symplectic form is induced by the canonical form ω, and we have the vertical and horizontal foliations like in example a). By performing the transformation $(x, y) \to (x/\| x \|, \ln \| x \|, y/\| y \|, \ln (\| x \| \cdot \| y \|))$, we see that V_λ is diffeomorphic to $S^{n-1} \times S^1 \times S^{n-1} \times \mathbb{R}^1$, and that the leaves of the foliations considered are $S^{n-1} \times \mathbb{R}^1$.

d) $V = T^{2n}$ like in example a), $\mathscr{L}$ is defined by $y^i - \alpha^i x^i = \text{ct.}$ ($\alpha^i \in \mathbb{R}$; *foliations of slopes* α^i). If all α^i are rational, the leaves are diffeomorphic to T^n, and if all α^i are irrational the leaves are dense in T^{2n}.

e) [CFG] $V^{2p+2q+2} = M(p, q) \overset{\text{def}}{=} \Gamma(p, q) \setminus [H(1, p) \times H(1, q)]$ where $H(1, p)$ is the *generalized Heisenberg group*

$$H(1,p) = \left\{ \alpha_1 = \begin{pmatrix} \mathrm{Id}_p & X_1 & Z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \mid X_1,\ Z_1 \in \mathbb{R}^p,\ y_1 \in \mathbb{R} \right\},$$

$H(1, q) = \{\alpha_2\}$ defined similarly, and $\Gamma(p, q)$ is made up by similar pairs of matrices with integral entries; on this V, one takes

$$\omega = F \overset{\text{def}}{=} {}^t dX_1 \wedge (dZ_1 - X_1 dy_1) + {}^t dX_2 \wedge (dZ_2 - X_2 dy_2) + dy_1 \wedge dy_2,$$

and $\mathscr{L} = \mathscr{L}_\alpha$ defined by $X_1 = \text{ct.}$, $X_2 = \text{ct.}$, $y_1 - \alpha y_2 = \text{ct.}$ $(\alpha \in \mathbb{R})$. $\mathscr{T} \overset{\text{def}}{=} M(1, 0)$ is Thurston's example [Th].

f) [CFG] $V^{4p+2} = I(p) \overset{\text{def}}{=} \Gamma_c(1,p) \backslash H_c(1,p)$, where H and Γ are like in example e) above but have complex and Gauss integer entries, respectively;

$$\omega = Q \overset{\text{def}}{=} {}^t\mathrm{Re}\,(dX) \wedge \mathrm{Re}\,(dZ - X\,dy) - {}^t\mathrm{Im}\,(dX) \wedge \mathrm{Im}\,(dZ - X\,dy) + \\ + \mathrm{Re}\,(dy) \wedge \mathrm{Im}\,(dy);$$

and $\mathscr{L}$ is given by $X = \text{ct.}$, $\mathrm{Re}\,y - \alpha\,\mathrm{Im}\,y = \text{ct.}$ ($\alpha \in \mathbb{R}$). $I(p)$ is called the *Iwasawa manifold.*

g) [FGG] $V = E^4 =$ principal circle bundle over E^3, where $E^3 =$ principal circle bundle over T^2; $\omega = (a\alpha + b\beta) \wedge \gamma + (e\alpha + f\beta) \wedge \eta$, where $a, b, e, f \in \mathbb{R}$, $af - be \neq 0$, α, β are parallel 1-forms on T^2 which generate $H^1(T^2, \mathbb{Z})$, γ is a connection form on $E^3 \to T^2$, η is a connection form on $E^4 \to E^3$ such that

$$d\gamma = n\alpha \wedge \beta,\ d\eta = p\alpha \wedge \gamma + q\beta \wedge \gamma \quad (n, p, q \in \mathbb{Z},\ fp - eq = 0);$$

$\mathscr{L}$ is defined by $e\alpha + f\beta = 0$, $\gamma = 0$. If the bundles $E^4 \to E^3 \to T^2$ are both nontrivial, and E^4 is blown up at $k \geqslant 1$ points one gets a compact symplectic manifold with no Lagrangian foliation [Gt].

Now, let us come back to the general case of $(V^{2n}, \omega, \mathscr{L}^n)$. Then the basic local structure is described by a theorem which tells us that V has a *canonical atlas* $(U_\alpha; x_\alpha^i, y_\alpha^j)$ ($\alpha \in$ some set A) such that $\mathscr{L}$ is $x^i = \text{ct.}$, and ω takes the canonical form (see, for instance, [L]). The coordinate transformations of this atlas are locally of the form

$$x_\beta^i = x_\beta^i(x_\alpha^j), \quad y_\beta^i = \sum_{k=1}^n \frac{\partial x_\alpha^k}{\partial x_\beta^i} y_\alpha^k + \varphi_{\alpha\beta}^i(x_\alpha^j) \tag{1.1}$$

$$\left(\sum_{i=1}^n \frac{\partial x_\beta^i}{\partial x_\alpha^j} \frac{\partial \varphi_{\alpha\beta}^i}{\partial x_\alpha^k} = \sum_{i=1}^n \frac{\partial x_\beta^i}{\partial x_\alpha^k} \frac{\partial \varphi_{\alpha\beta}^i}{\partial x_\alpha^j}\right).$$

Such transformations form a pseudogroup Ω_n in $\mathbb{R}^{2n}$ (the *Lagrangian pseudogroup*), hence $(V, \omega, \mathscr{L})$ can be seen as an Ω_n-structure. It will be technically convenient to enlarge such a structure to an $\hat{\Omega}_n$-structure, where $\hat{\Omega}_n$ (called the *weak Lagrangian pseudogroup*) is given by (1.1)

with no condition imposed to $\varphi^i_{\alpha\beta}$. The enlarged atlas is called *semicanonical*, and it is characterized by the expression

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i + \tfrac{1}{2} \sum_{i,j=1} F_{ij}(x^h)\, dx^i \wedge dx^j \quad (F_{ij} = -F_{ji}) . \tag{1.2}$$

For a later utilization, we also introduce the subpseudogroup $\Omega^0_n \subset \Omega_n$, defined by (1.1) with $\varphi^i_{\alpha\beta} = 0$.

Corresponding to (1.2), the Hamiltonian field of a function f is

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} + \sum_{j=1}^{n} F_{ij} \frac{\partial f}{\partial y^j} \right) \frac{\partial}{\partial y^i}, \tag{1.3}$$

and this formula gives two known fundamental facts:

a) $df \to -X_f$ for local foliate[2] functions f is an isomorphism $T\mathcal{L} \approx \nu^* \mathcal{L}$, where $\nu\mathcal{L} = TV/T\mathcal{L}$, thereby $T\mathcal{L}$ and $TV \approx$ $\approx T\mathcal{L} \oplus \nu\mathcal{L}$ become foliate vector bundles [V2] (but not in a canonical way, in the case of TV).

b) [W] The leaves of $\mathcal{L}$ have an induced locally affine structure given by the torsionless flat linear connection $\overset{\circ}{\nabla}$ which makes the fields X_f of a) into parallel fields.

Remark. If (V, ω) is only locally conformal symplectic (e.g. [V3]) the first term of the righthand side of the second equation (1.1) has to be multiplied by a constant cocycle $\{k_{\alpha\beta}\}$. Hence b) above is valid in this case too, while a) is replaced by $T\mathcal{L} \approx \nu^* \mathcal{L} \otimes \lambda$ where λ is the line bundle defined by $\{k_{\alpha\beta}\}$.

For any dimension $m = p + q$, equations (1.1) are generalized by

$$x^a_\beta = x^a_\beta(x^b_\alpha), \; y^u_\beta = \sum_{v=1}^{p} A^u_{\alpha\beta,v}(x^b)\, y^v_\alpha + B^u_{\alpha\beta}(x^b_\alpha) \tag{1.4}$$

$(a, b = 1, \ldots, q;\ u, v = 1, \ldots, p)$, and (1.4) defines the *affine pseudogroup of codimension* q, $\mathfrak{A}^q_m$. The latter has a subpseudogroup $\mathfrak{A}^{0q}_m$ defined by $B^u_{\alpha\beta} = 0$ in (1.4). An $\mathfrak{A}^q_m$-structure on a manifold M^m is a *leafwise locally affine* (l.l.a. in the sequel) foliation $\mathcal{F}$, with the connection $\nabla^0_{\partial/\partial y^u}(\partial/\partial y^v) = 0$ along the leaves. (1.4) also shows that $T\mathcal{F}$ is a foliate vector bundle, and, since so is also $\nu\mathcal{F} = TM/T\mathcal{F}$, we can get

[2] On a foliated manifold, we use the qualification *foliate* for everything projectable on the space of leaves.

foliate structures on TM itself. It suffices to fix a transversal distribution

$$\mathscr{F}' = \text{span}\{X_a\}, \quad X_a = \frac{\partial}{\partial x^a} - \sum_{a=1}^{p} t_a^u \frac{\partial}{\partial y^u}. \tag{1.5}$$

Then the local bases $(X_a, \partial/\partial y^u)$ of TM have foliate transition functions. Notice that the corresponding dual cobases are

$$dx^a, \ \theta^u = dy^u + \sum_{a=1}^{q} t_a^u dx^a. \tag{1.6}$$

Remark. Formulas (1.5) and (1.6) will also be used for Lagrangian foliations $\mathscr{F} = \mathscr{L}$, with $\mathscr{F}' = \mathscr{L}'$, and indices $i, j, \ldots$ instead of $a, u, \ldots$.

If $\mathscr{F}$ is a l.l.a. foliation, the restrictions of $\nu^* \mathscr{F}$ to the leaves of $\mathscr{F}$ are the leaves of a Lagrangian foliation $\mathscr{L}\mathscr{F}$ of T^*M with its canonical symplectic structure.

On the other hand, more examples of l.l.a. foliation are offered by the fact that any *symplectically complete isotropic foliation* $\mathscr{F}$ of a symplectic manifold (V, ω) [L] is l.l.a. Indeed, any Hamiltonian vector field X_f, where f is a foliate function with respect to the ω-orthogonal foliation $\mathscr{F}^\perp$ of $\mathscr{F}$, can be seen as a parallel vector field.

2. Sheaves and Cohomology

Let $\mathscr{F}$ be an arbitrary foliation on M^m, given by $x^a = \text{ct.}$, with leaf coordinates y^u where, again, $a, b, \ldots = 1, \ldots, q$; $u, v, \ldots = 1, \ldots, p$; $p + q = m$. Then, formulas (1.5) and (1.6) remain valid, and $TM = \mathscr{F}' \oplus \mathscr{F}$. Accordingly, we get *forms of the type* (h, k) which vanish unless h arguments are in $\mathscr{F}'$ and k arguments are in $\mathscr{F}$, and

$$d = d' + d'' + \partial, \tag{2.1}$$

where the terms are operators of type $(1,0)$, $(0,1)$, $(2,-1)$, respectively. Particularly, $d_f \overset{\text{def}}{=} d''$ is precisely the exterior differential along the leaves of $\mathscr{F}$, and it satisfies a local Poincaré Lemma [V1], [V2].

On $(M, \mathscr{F})$, the basic sheaf is the sheaf Φ of germs of foliate functions. In the sequel, if $E \to M$ is a vector bundle, we denote by $\underline{E}$ the sheaf of germs of sections of E, and by $\underline{\underline{E}}$ the sheaf of germs of foliate sections, which exists iff E is a foliate vector bundle. Then Φ has the fine resolution

$$0 \to \Phi \subseteq \underline{\Lambda}^{00}\mathcal{F} \xrightarrow{d_f} \underline{\Lambda}^{01}\mathcal{F} \xrightarrow{d_f} \dots \tag{2.2}$$

where $\Lambda^{hk}\mathcal{F}$ is the bundle of forms of type (h, k) on M [V2]. Furthermore, we also have the rather obvious

Proposition 2.1. *Let $\mathcal{S}$ be a locally free Φ-module sheaf of finite rank (i.e., $\mathcal{S} = \underline{E}$ for some E). Then*

$$0 \to \mathcal{S} \subseteq \mathcal{S} \otimes_\Phi \underline{\Lambda}^{00}\mathcal{F} \xrightarrow{d_f} \mathcal{S} \otimes_\Phi \underline{\Lambda}^{01}\mathcal{F} \xrightarrow{d_f} \dots \tag{2.3}$$

is a fine resolution of $\mathcal{S}$.

Proof. By definition, in (2.3) d_f acts on the second factor, which is correct since $d_f\Phi = 0$. The Poincaré lemma for this d_f follows easily if we express the local sections of the relevant sheaves

$$\mathcal{S}^{0k}(\mathcal{F}) = \mathcal{S} \otimes_\Phi \underline{\Lambda}^{0k}\mathcal{F} \tag{2.4}$$

in terms of Φ-bases. Finally, the sheaves $\mathcal{S}^{0k}(\mathcal{F})$ are fine since, on them, we may define multiplication by $f \in C^\infty(M)$ as acting on the second factor. Q.e.d.

Corollary 2.2.

$$H^k(M, \mathcal{S}) = \frac{\ker(d_f : \Gamma\mathcal{S}^{0k} \to \Gamma\mathcal{S}^{0,k+1})}{\operatorname{im}(d_f : \Gamma\mathcal{S}^{0,k-1} \to \Gamma\mathcal{S}^{0k})},$$

where Γ denotes global sections.

Examples of sheaves $\mathcal{S}$ of this type are $\underline{\Lambda}^{k0}\mathcal{F}$, $\underline{\nu}\mathcal{F}$, etc. Particularly, $H^1(M, \underline{\nu}\mathcal{F})$ contains the main parts of the infinitesimal deformations of $\mathcal{F}$ [Ht].

Now, if $\mathcal{F}$ is l.l.a., we have another interesting sheaf of the same type namely, the sheaf $\mathcal{A}(\mathcal{F})$ of germs of *affine functions*

$$f = \sum_{u=1}^{p} \alpha_u(x^a)\, y^u + \beta(x^a)\,, \tag{2.5}$$

which has the local Φ-bases $(y^u, 1)$. In the same class, we also have the sheaves $\mathcal{A}^h(\mathcal{F})$ of germs in $\underline{\Lambda}^{h0}\mathcal{F}$ with local coefficients of the form (2.5). The sheaves in $\mathcal{A}^h$ are again representable by (2.5) where, now, α_u, β are foliate h-forms. (α_u) defines in fact a $T^*\mathcal{F}$-valued foliate h-form, which proves the existence of the exact sequences

$$0 \to \underline{\Lambda}^{h0}\mathscr{F} \subseteq \mathscr{A}^h(\mathscr{F}) \xrightarrow{\pi} \underline{\underline{T^*\mathscr{F}}} \otimes_{\mathbb{R}} \underline{\Lambda}^{h0}\mathscr{F} \to 0 \tag{2.6}$$

for $h = 0, 1, 2, \ldots$, with $\pi(f) = (\alpha_u)$.

If $\mathscr{F}_a$ are l.l.a. foliations on M_a $(a = 1, 2)$, and $\varphi: (M_1, \mathscr{F}_1) \to$ $\to (M_2, \mathscr{F}_2)$ sends leaves into leaves by locally affine mappings, we say that φ is an *affine morphism*. In obvious notation, φ has local equations of the form

$$x_2^a = x_2^a(x_1^b), \quad y_2^u = \sum_{v=1}^{p_1} E_v^u(x_1^b)\, y_1^v + G^u(x_1^b)\,, \tag{2.7}$$

which proves the existence of induced morphisms

$$\varphi^{\#}: \mathscr{A}^h(\mathscr{F}_2) \to \mathscr{A}^h(\mathscr{F}_1), \ \varphi^*: H^*(M_2, \mathscr{A}^h(\mathscr{F}_2)) \to H^*(M_1, \mathscr{A}^h(\mathscr{F}_1)). \tag{2.8}$$

Therefore, $H^*(M, \mathscr{A}^h(\mathscr{F}))$ are global invariants of l.l.a. foliations. Moreover, two affine morphisms $\varphi_{0,1}: M_1 \to M_2$ will be *affine homotopic* if they are homotopic by some $H: M_1 \times [0, 1] \to M_2$ which is affine for the foliations $\mathrm{pr}_{M_1}^{-1}(\mathscr{F}_1)$ and $\mathscr{F}_2$. This means that the local equations of H are of the form

$$x_2^a = x_2^a(x_1^b), \quad y_2^u = \sum_{v=1}^{p_1} \alpha_v^u(x_1^b)\, y_1^v + \beta^u(x_1^b)\, t + \gamma(x_1^b)\,. \tag{2.9}$$

If this happens, it follows in the standard way (see [K] for general foliations) that $\varphi_0^* = \varphi_1^*$. Like in [K], it is also possible to obtain a Mayer-Vietoris sequence.

The next important sheaf to be considered for an l.l.a. foliation $\mathscr{F}$ is the sheaf Θ of germs of infinitesimal automorphisms of $\mathscr{F}$, because of its role in the deformation theory of $\mathfrak{U}_m^q$-structures (see [GS], for instance). The field

$$X = \sum_{a=1}^{q} \xi^a \frac{\partial}{\partial x^a} + \sum_{u=1}^{p} \eta^u \frac{\partial}{\partial y^u} \tag{2.10}$$

is a section of Θ if $\exp(t X)$ are affine morphisms i.e., if $Y \in T\mathscr{F}$ then $[X, Y] \in T\mathscr{F}$, and if, moreover, Y is parallel so is $[X, Y]$. This is equivalent to the local formulas [M3]

$$\xi^a = \xi^a(x^b), \quad \eta^u = \sum_{v=1}^{p} \lambda_v^u(x^b)\, y^v + \mu^u(x^b), \tag{2.11}$$

which shows that Θ is a Φ-module with the local bases $\left(y^v\frac{\partial}{\partial y^u}, \frac{\partial}{\partial y^u}, \frac{\partial}{\partial x^a}\right)$. Hence $H^k(M, \Theta)$ are given like in Corollary 2.2. We also notice the exact sequence of Φ-modules

$$0 \to \Theta_0 \subseteq \Theta \xrightarrow{\text{pr}} \underline{\nu}\,\mathscr{F} \to 0\,, \tag{2.12}$$

where Θ_0 is the subsheaf which leaves invariant every leaf of $\mathscr{F}$.

If $\mathscr{F} = \mathscr{L}$ is a Lagrangian foliation on (V^{2n}, ω), the sheaves $\mathscr{A}^h(\mathscr{L})$ $(h = 0, 1, 2, \ldots)$ are important since a foliation preserving symplectomorphism is affine, hence the spaces $H^k(V, \mathscr{A}^h(\mathscr{L}))$ *are global invariants of a Lagrangian foliation.* For $h = 0, f \in \mathscr{A}(\mathscr{L})$ iff the Hamiltonian field X_f is foliate. We also make the remark that π of (2.6) may now be composed with $T^*\mathscr{L} \approx \nu\mathscr{L}$, which leads to the exact sequences

$$0 \to \underline{\Lambda}^{h0}\mathscr{L} \subseteq \mathscr{A}^h(\mathscr{L}) \xrightarrow{\sigma} \underline{\nu}\,\mathscr{L} \otimes_{\mathbb{R}} \underline{\Lambda}^{h0}\mathscr{L} \to 0\,. \tag{2.13}$$

For $f \in \mathscr{A}(\mathscr{L})$ (i.e., $h = 0$), $\sigma(f)$ is the transversal part of the Hamiltonian field X_f (use (1.3)).

Furthermore, the deformations of an $\mathfrak{L}_n$-structure (Lagrangian foliation) are related to the sheaf Ξ of germs of fields

$$X = \sum_{i=1}^{n} \xi^i(x^j)\frac{\partial}{\partial x^i} + \sum_{j=1}^{n} \eta^i(x^j, y^k)\frac{\partial}{\partial y^i} \tag{2.14}$$

such that $L_X\omega = 0$, i.e.,

$$\frac{\partial \eta^h}{\partial y^i} = -\frac{\partial \xi^i}{\partial x^h}, \quad \frac{\partial \eta^i}{\partial x^j} = \frac{\partial \eta^j}{\partial x^i}, \tag{2.15}$$

while the deformations of an $\hat{\mathfrak{L}}_n$-structure are related to $\hat{\Xi}$ consisting of germs (2.14) such that $L_X\omega$ is of type (2,0), which is equivalent to the first condition (2.15) alone. Ξ and $\hat{\Xi}$ are not Φ-modules. But it is easy to see that (2.14) is equivalent with germ $i(X)\omega \in d(\mathscr{A}(\mathscr{L}))$, i.e., $\Xi \approx d\mathscr{A}$, and we have the exact sequence

$$0 \to \mathbb{R} \subseteq \mathscr{A} \xrightarrow{d} (d\mathscr{A} \approx \Xi) \to 0\,. \tag{2.16}$$

Other interesting exact sequences are obtained by using the subsheaves $\mathscr{X} \subset \Xi$, $\hat{\mathscr{X}} \subset \hat{\Xi}$ of germs (2.14) which preserve every leaf of $\mathscr{L}$, i.e., have $\xi^i = 0$. Then, symplectic duality yields $\hat{\mathscr{X}} \approx \underline{\Lambda}^{10}\mathscr{L}$, $\mathscr{X} \approx$

$\approx \mathscr{Z}^{10}(\mathscr{L})$ = the sheaf of germs of closed foliate 1-forms, and the exact sequences mentioned are

$$0 \to \underline{\underline{\Lambda}}^{10}\mathscr{L} \to \hat{\Xi} \to \underline{\underline{\nu}}\,\mathscr{L} \to 0\,,$$
$$0 \to \mathscr{Z}^{10}(\mathscr{L}) \to \Xi \to \underline{\underline{\nu}}\,\mathscr{L} \to 0\,. \tag{2.17}$$

In (2.17), $\underline{\underline{\Lambda}}^{10}$ and $\underline{\underline{\nu}}$ are Φ-modules, and the cohomology of $\mathscr{Z}^{10}$ is also understood. Indeed, let $\mathscr{Z}^{h0}$ = the sheaf of germs of closed foliate h-forms with respect to any foliation $\mathscr{F}$ of a manifold M^m, and consider the sheaves

$$\mathscr{K}_h^u = \{\lambda \in \underline{\underline{\Lambda}}^{h+u} M \mid \lambda = \lambda_{hu} + \ldots + \lambda_{h+u,0}\} \quad (u = 0, 1, 2, \ldots)\,, \tag{2.18}$$

where the indices denote the type of the terms. Then (see [D] for complex manifolds; see also [V2] and [DD] for foliations)

$$0 \to \mathscr{Z}^{h0} \subseteq \mathscr{K}_h^0 \xrightarrow{d} \mathscr{K}_h^1 \xrightarrow{d} \ldots \tag{2.19}$$

is a fine resolution of $\mathscr{Z}^{h0}$. Indeed, by (2.1), $d^2 = 0$ becomes [V1]

$$d''^2 = 0,\ \partial^2 = 0,\ d'^2 + d''\partial + \partial d'' = 0,\ d'd'' + d''d' = 0,$$
$$d'\partial + \partial d' = 0\,. \tag{2.20}$$

This allows us to write down the various components of $d\lambda = 0$ for $\lambda \in \mathscr{K}_h^u$, and to find inductively, and locally, the components of a corresponding

$$\mu = \mu_{h,u-1} + \ldots + \mu_{h+u-1,0} \in \mathscr{K}_h^{u-1}$$

such that $d\mu = \lambda$, by means of a repeated application of the d''-Poincaré lemma. Therefore, (2.19) is an exact sequence of fine sheaves (except for $\mathscr{Z}^{h0}$). Q.e.d.

Particularly, $H^*(M, \mathscr{Z}^{10}(\mathscr{F})) = H^*(\Gamma\mathscr{K}_1 = \bigoplus_{u=0}^{m-1} \Gamma\mathscr{K}_1^u, d)$ where Γ denotes spaces of global sections, hence $\Gamma\mathscr{K}_1$ is the complex of forms which vanish on the leaves of $\mathscr{F}$. We notice also that $\Gamma\mathscr{K}_1$ has the subcomplex $\bigoplus_{u}^{m-1} \Gamma\underline{\underline{\Lambda}}^{u+1,0}\mathscr{F}$ which defines the *basic cohomology* of $\mathscr{F}$ [R], and the corresponding quotient complex is

$$\mathscr{C} = \bigoplus_{u=0}^{m-1} (\Gamma\mathscr{K}_1^u / \Gamma\underline{\underline{\Lambda}}^{u+1,0}\mathscr{F}, d)\,. \tag{2.21}$$

Unfortunately, the cohomolgies defined in this section are rarely calculable in an explicit form. However, we can give some examples based on results of [Ht] and [K].

a) $V = T^*M \xrightarrow{\pi} M$, $\mathscr{L}$ = the foliation by fibers (Example b, Section 1). For any sheaf $\mathscr{S}$ on T^*M, $H^*(T^*M, \mathscr{S})$ is the limit of a spectral sequence such that [Gd]

$$E_2^{hk} = H^h(M, \mathscr{H}^k(\text{“}F\text{”}, \mathscr{S})) , \tag{2.22}$$

where “F” means “fibre-like”, and $\mathscr{H}^k(\text{“}F\text{”}, \mathscr{S})$ is the sheaf defined by the presheaf $U \subset M \to H^k(\pi^{-1}(U), \mathscr{S})$. If $\mathscr{S}$ is a locally free Φ-module of rank s, then Corollary 2.2 and the contractibility of the fibers give $\mathscr{H}^k(\text{“}F\text{”}, \mathscr{S}) = 0$ for $k \geqslant 1$. On the other hand, we have $H^0(\pi^{-1}(U)) = \Gamma(\mathscr{S}|_{\pi^{-1}(U)}) = [C^\infty(U)]^s$, hence $\mathscr{H}^0(\text{“}F\text{”}, \mathscr{S})$ is a fine sheaf on M. Accordingly, we obtain $E_2^{hk} = 0$ for $k \geqslant 1$, and for $k = 0$, $h \geqslant 1$, which implies

$$H^k(T^*M, \mathscr{S}) = \begin{cases} \Gamma\mathscr{S} & \text{for } k = 0 \\ 0 & \text{for } k \geqslant 1 . \end{cases} \tag{2.23}$$

Remark. The same result holds for any foliation by the fibers of a vector bundle.

A nice consequence of (2.23) is

Proposition 2.3. *If $H^1(M, \mathbb{R}) = H^2(M, \mathbb{R}) = 0$, then the Lagrangian foliation $\mathscr{V}$ of T^*M is infinitesimally rigid as an $\mathfrak{L}_n$-structure.*

Proof. Using the hypotheses, the exact sequence (2.16), and (2.23), we get $H^1(T^*M, \Xi) = H^1(T^*M, \mathscr{A}) = 0$. Q.e.d.

b) $\mathscr{L}$ = the foliation $x^i = \text{ct.}$ on V_λ of Example c, Section 1. We saw that $V_\lambda = X \times F$, $X = S^{n-1} \times S^1$, $F = S^{n-1} \times \mathbb{R}^1$, leaves of $\mathscr{L} = \{x\} \times F$ ($x \in X$). In this case, $H^*(V_\lambda, \Phi)$ was calculated in [K], by using a similar spectral sequence argument as above, and this procedure [K] is also good for $\underline{\Lambda}^{h0}\mathscr{L}$, $\mathscr{A}^h(\mathscr{L})$. The results are

$$\begin{aligned} H^k(V_\lambda, \underline{\Lambda}_{h0}\mathscr{L}) &= H^k(S^{n-1}, \mathbb{R}) \otimes_{\mathbb{R}} \Gamma\underline{\Lambda}^h(S^{n-1} \times S^1) , \\ H^k(V_\lambda, \mathscr{A}^h\mathscr{L}) &= H^k(S^{n-1}, \mathbb{R}) \otimes_{\mathbb{R}} \Gamma Q , \end{aligned} \tag{2.24}$$

where $Q = [\underline{T}(S^{n-1} \times S^1) \oplus 1] \otimes_{\mathbb{R}} \underline{\Lambda^h T^*}(S^{n-1} \times S^1)$.

c) $\mathscr{L} = \mathscr{L}_\alpha$ = the foliation of slope $\alpha \in \mathbb{R}$ on T^2 (Example d, Section 1). We want $H^*(T^2, \mathscr{A}_\alpha = \mathscr{A}(\mathscr{L}_\alpha))$, and we shall use the exact cohomology sequence of (2.13) with $h = 0$, which is

$$\begin{aligned} 0 \to H^0(T^2, \Phi_\alpha) \xrightarrow{\iota_0^*} H^0(T^2, \mathscr{A}_\alpha) \xrightarrow{\sigma_0^*} H^0(T^2, \underline{\nu}\, \mathscr{L}_\alpha) \xrightarrow{\partial} \\ \xrightarrow{\partial} H^1(T^2, \Phi_\alpha) \xrightarrow{\iota_1^*} H^1(T^2, \mathscr{A}_\alpha) \xrightarrow{\sigma_1^*} H^1(T^2, \underline{\nu}\, \mathscr{L}_\alpha) \to 0, \end{aligned} \tag{2.25}$$

where $\Phi_\alpha = \Phi(\mathscr{L}_\alpha)$, ι^* are induced by $\Phi_\alpha \subset \mathscr{L}_\alpha$, and σ^* are induced by σ of (2.13).

If α is rational, $\mathscr{L}_\alpha$ is foliate affinely isomorphic to $\mathscr{L}_0$, and $H^*(T^2, \mathscr{A}_\alpha) = H^*(T^2, \mathscr{A}_0)$. First, we get

$$H^0(T^2, \mathscr{A}_0) = H^0(T^2, \Phi_0) = H^0(T^2, \underline{\nu}\, \mathscr{L}_0) \approx C^\infty(S^1)\,, \tag{2.26}$$

where the first equality follows because, if $\varphi \in \Gamma \mathscr{A}_0$, the lift $\tilde{\varphi}$ is a periodic function on $\mathbb{R}^2$, and the other equality signs were established in [Ht] and [K]. Then, using again the periodicity argument, the Corollary (2.2) for $\mathscr{S} = \Phi_0$, and the definition of ∂ lead straightforwardly to the fact that ∂ is an isomorphism. Therefore, by the exactness of (2.25), σ_1^* is also an isomorphism, and we have

$$H^1(T^2, \mathscr{A}_0) = H^1(T^2, \underline{\nu}\, \mathscr{L}_0) \approx C^\infty(S^1)\,, \tag{2.27}$$

where the last isomorphism was proven in [Ht]. (One also has $H^1(T^2, \Phi_0) \approx C^\infty(S^1)$.)

If α is irrational, the leaves of $\mathscr{L}_\alpha$ are dense in T^2, hence $H^0(T^2, \Phi_\alpha) = H^0(T^2, \underline{\nu}\, \mathscr{L}_\alpha) \approx \mathbb{R}$. Now, ∂ is seen to be injective, hence ι_0^* is an isormophism, and we get

$$H^0(T^2, \mathscr{A}_\alpha) \approx \mathbb{R}\,. \tag{2.28}$$

Furthermore, if α is not a *Liouville number* one has $H^1(T^2, \Phi_\alpha) = H^1(T^2, \underline{\nu}\, \mathscr{L}_\alpha) \approx \mathbb{R}$ [Ht], and ∂ is again an isormophism, hence

$$H^1(T^2, \mathscr{A}_\alpha) \approx \mathbb{R}\,. \tag{2.29}$$

d) $V = \mathscr{T} = M(1,0)$ of Example e, Section 1, which is equivalent to $\mathscr{T} = \mathbb{R}^4/G$, where G is the group

$$(x^1, x^2, x^3, x^4) \to (x^1 + s, x^2 + n, x^3 + p, x^4 + px^1 + q) \quad (s,n,p,q \in \mathbb{Z}) \tag{2.30}$$

with

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge (dx^4 - x^3 dx^1), \tag{2.31}$$

and $\mathscr{L} = \mathscr{L}_\alpha$ is given by $x^1 - \alpha x^2 = \text{ct.}$, $x^3 = \text{ct.}$ ($\alpha \in \mathbb{R}$). $\mathscr{T}$ may be seen as an S^1-bundle over $T^2 \times S^1$ with the coordinates (x^1, x^2) on T^2, and x^3 on S^1, and $\mathscr{L}_\alpha$ is the pull-back to $\mathscr{T}$ of the foliation $\tilde{\mathscr{L}}_\alpha$ on $T^2 \times S^1$ whose leaves are given by the leaves of the foliation of slope $1/\alpha$ on T^2.

The transformation

$$u^1 = x^1 - \alpha x^2,\ u^2 = x^2,\ u^3 = x^3,\ u^4 = x^4 - x^1 x^3, \tag{2.32}$$

yields canonical coordinates, where $\mathscr{L}_\alpha$ is $u^1 = \text{ct.}$, $u^3 = \text{ct.}$, and (2.30) becomes

$$(u^1, u^2, u^3, u^4) \to (u^1 + s - \alpha n, u^2 + n, u^3 + p, u^4 - s u^3 + q - ps)\ . \tag{2.33}$$

As a consequence of (2.33) it is easy to see that

$$H^0(\mathscr{T}, \mathscr{A}(\mathscr{L}_\alpha)) = H^0(\mathscr{T}, \Phi(\mathscr{L}_\alpha));\ H^k(\mathscr{T}, \underline{\underline{\nu}}\, \mathscr{L}_\alpha) = 2\, H^k(\mathscr{T}, \Phi(\mathscr{L}_\alpha)) \tag{2.34}$$

If $\alpha = p/q$ is rational, $\mathscr{L}_\alpha$ is equivalent to $\mathscr{L}_0$ and we have

$$H^0(\mathscr{T}, \mathscr{A}(\mathscr{L}_{p/q})) \approx C^\infty(T^2)\ . \tag{2.35}$$

If α is irrational, the usual density argument along T^2, kills one variable, and

$$H^0(\mathscr{T}, \mathscr{A}(\mathscr{L}_\alpha)) \approx C^\infty(S^1)\ . \tag{2.36}$$

3. Linearity Obstructions, Liouville Manifolds

Let $\mathscr{F}$ be a l.l.a. foliation i.e., an $\mathfrak{A}_m^q$-structure on M, where $\mathfrak{A}_m^q$ is defined by (1.4). When is this structure reducible to an $\mathfrak{A}_m^{0\,q}$-structure?

In order to answer this question, let us define the *local Euler fields* (notation of (1.4))

$$E_\alpha = \sum_{u=1}^{p} y_\alpha^u \frac{\partial}{\partial y_\alpha^u} \tag{3.1}$$

over the neighbourhood U_α. Then, over $U_\alpha \cap U_\beta$, (1.4) implies

$$E_\beta - E_\alpha = \sum_{u=1}^{p} B_{\alpha\beta}^u \frac{\partial}{\partial y_\beta^u}\ , \tag{3.2}$$

and we see that the righthand side of (3.2) defines a 1-cocycle of $\underline{\underline{T}}\mathscr{F}$. This cocycle defines a class

$$\mathscr{E}(\mathscr{F}) \in H^1(M, \underline{\underline{T}}\mathscr{F}) , \tag{3.3}$$

which we call the *linearity obstruction* of $\mathscr{F}$ because of

Proposition 3.1. *An $\mathfrak{A}_m^q$-structure $\mathscr{F}$ is reducible to an $\mathfrak{A}_m^{0q}$-structure iff $\mathscr{E}(\mathscr{F}) = 0$.*

Proof. Necessity is obvious. Conversely, if $\mathscr{E}(\mathscr{F}) = 0$, there is a sufficiently fine adapted atlas where, at the level of germs, one has

$$\sum_{u=1}^{p} B_{\alpha\beta}^u \frac{\partial}{\partial y_\beta^u} = \sum_{u=1}^{p} \psi_\beta^u(x_\beta^a) \frac{\partial}{\partial y_\beta^u} - \sum_{u=1}^{p} \psi_\alpha^u(x_\alpha^a) \frac{\partial}{\partial y_\alpha^u} . \tag{3.4}$$

Then, the new coordinates

$$\tilde{x}_\alpha^a = x_\alpha^a , \; \tilde{y}_\alpha^u = y_\alpha^u - \psi_\alpha^u(x_\alpha^a) \tag{3.5}$$

yield an $\mathfrak{A}_m^{0q}$-atlas of $\mathscr{F}$. Q.e.d.

Remarks. 1) An $\mathfrak{A}_m^q$-structure may have many $\mathfrak{A}_m^{0q}$-reductions since the cochain ψ of (3.4) is defined only up to the addition of a 0-cocycle i.e., a global foliate cross-section of $T\mathscr{F}$.

2) For every $\mathfrak{A}_m^{0q}$-reduction the fields (3.1) glue up to a *global Euler field* $E \in T\mathscr{F}$, and two such fields differ by a foliate cross-section of $T\mathscr{F}$.

Proposition 3.2. *Let $\mathscr{F}$ be a l.l.a. foliation. Then M consists of the leaves of a vector bundle over some manifold N iff*: i) *the leaves of $\mathscr{F}$ are simply connected;* ii) *the flat connection $\mathring{\nabla}$ along the leaves of $\mathscr{F}$ (Section* 1) *is complete;* iii) $\mathscr{E}(\mathscr{F}) = 0$.

Proof. Clearly, i), ii), iii) are necessary conditions. Conversely, i), ii) imply that every leaf F of $\mathscr{F}$ is an affine space where global affine coordinates are given by exp: $T_{x_0}F \to F$ for $x_0 \in F$, and by iii) we may choose an $\mathfrak{A}_m^{0q}$-atlas of $\mathscr{F}$. Let $x_0 \in U_\alpha$ of this atlas. Then the geodesics of $\mathring{\nabla}|_F$ through x_0 are given by $y_\alpha^u = \xi_F^u t + y_\alpha^u(x_0)$ on $F \cap U_\alpha$, where ξ_F^u are the global affine coordinates of the point $t = 1$ of F. Notice that these geodesics equations hold on all the slices of $F \cap U_\alpha$ as we may see by going from one slice to another along a finite chain of neighbourhoods U_{α_k}. Hence the F-affine coordinates of $x \in F \cap U_\alpha$ are $\xi_F^u = y_\alpha^u - y_\alpha^u(x_0)$.

Now, let E be the global Euler vector field of the chosen $\mathfrak{A}_m^{0q}$-atlas of $\mathscr{F}$. Then (3.1), and the previous formula of ξ_F^u yield

$$E|_F = \sum_{u=1}^{p} (\xi_F^u + y_\alpha^u(x_0)) \frac{\partial}{\partial \xi_F^u}$$

globally, which shows that F has a unique point $z(F)$ where $E(z(F)) = 0$. If N is the locus of $z(F)$ ($F \in \mathscr{F}$), and if $z(F) \in U_\sigma$, (3.1) shows that $W_\sigma = N \cap U_\sigma$ has the local coordinates $y_\sigma^u = 0$. Hence N is an embedded q-dimensional submanifold of M with the local coordinates x_σ^a. Furthermore, put $\xi_\sigma^u = \xi_F^u$ for $F \subset \tilde{U}_\sigma =$ the $\mathscr{F}$-saturation of U_σ, where the exponential mapping used to define ξ_F is supposed to start at $z(F)$. Then $(x_\sigma^u, \xi_\sigma^u)$ are coordinates of M in $\tilde{U}_\sigma$ which show the vector bundle structure $\pi: M \to N$, with $\pi(x) = z(F)$ ($x \in F$), and with the transition cocycle $(A_{\alpha\beta,v}^u)$ of (1.4) of the $\mathfrak{A}_m^{0q}$-atlas considered in this proof. Q.e.d.

In the Lagrangian case, Proposition 3.2 becomes

Proposition 3.3. *Let (V^{2n}, ω) be a symplectic manifold and $\mathscr{L}$ a Lagrangian foliation on V which has simply connected leaves and a complete connection $\mathring{\nabla}$. Then $(V, \omega, \mathscr{L})$ is equivalent to a cotangent bundle T^*M^n, with a semicanonical symplectic form* (1.2) *and the foliation by fibers $\mathscr{V}$, iff $\mathscr{E}(\mathscr{L}) = 0$.*

Indeed, it suffices to apply Proposition 3.2 to the case of formula (1.1).

(See a different characterization of cotangent bundles in [Q].)

An interesting equivalent condition for $\mathscr{E}(\mathscr{L}) = 0$ is obtained as follows. If $\mathscr{L}$ is a Lagrangian foliation of (V, ω), then ω yields a well defined class $\tilde{\omega}(\mathscr{L}) \in H^1(\mathscr{C})$, for $\mathscr{C}$ from (2.21), and we have

Proposition 3.4. *$\mathscr{E}(\mathscr{L}) = 0$ iff $\tilde{\omega}(\mathscr{L}) = 0$.*

Proof. $\tilde{\omega}(\mathscr{L}) = 0$ means that V has a 1-form ε which vanishes on $\mathscr{L}$, and $\omega - d\varepsilon$ is a foliate 2-form. Then, rank $d\varepsilon = 2n$, and, if $\varepsilon = \sum_{i=1}^{n} y_\alpha^i(x_\alpha^j, z_\alpha^j)\, dx_\alpha^i$ with respect to coordinates (x_α^i, z_α^i) such that $x_\alpha^i = \text{ct.}$ defines $\mathscr{L}$, we see that (x_α^i, y_α^i) is a semicanonical atlas of $\mathscr{L}$, and $\mathscr{E}(\mathscr{L}) = 0$. Conversely, let $\mathscr{E}(L) = 0$, and let E be a global Euler vector field of the local expression (3.1). Then $\varepsilon = i(E)\omega = -\sum_{i=1}^{n} y_\alpha^i dx_\alpha^i$ is a well defined 1-form such that $\omega - d\varepsilon$ is foliate. Q.e.d.

We suggest to call such ε a *generalized Liouville form*, and to say that a symplectic manifold which has such a form is a *Liouville*

manifold. A Liouville manifold is, therefore, a triple $(V, \omega, \mathscr{L})$ with $\mathscr{E}(\mathscr{L}) = 0$. It also has the exact symplectic form $d\varepsilon$, and ω is precisely $d\varepsilon$ iff $[\omega] \in H^1(V, \mathscr{Z}^{10})$ vanishes.

Corollary 3.5. *A Liouville manifold is non compact.*

Indeed, $(d\varepsilon)^n$ has to be a volume element.

Corollary 3.6. *Let $\mathscr{L}$ be the foliation of $T^* T^n$ (T^n = torus) by its constant cross-sections. Then $\mathscr{E}(\mathscr{L}) \neq 0$.*

Proof. Let (x_α^i, y_α^i) be coordinates on $T^* T^n$ with transition relations $x_\beta^i = x_\alpha^i + n_{\alpha\beta}$, $y_\beta^i = y_\alpha^i$ ($n_{\alpha\beta} \in \mathbb{Z}$). Then $\mathscr{L}$ is $y_\alpha^i = \text{ct.}$, $\omega = \sum_{i=1}^n dx_\alpha^i \wedge dy_\alpha^i$, and $\mathscr{E}(\mathscr{L}) = 0$ would mean

$$\sum_{i=1}^n dx^i \wedge dy^i - d\Big(\sum_{i=1}^n a^i dy^i\Big) = \sum_{i,j=1}^n B_{ij} dy^i \wedge dy^j ,$$

where a^i are global functions on $T^* T^n$. Accordingly, we would have $dx^i - da^i = \sum_{h=1}^n b_h dy^h$, and by pulling this back to T^n via $y^i = \text{ct.}$, dx^i would be homologous to zero on T^n, which is false. Q.e.d.

We give a few more properties of Liouville manifolds.

Proposition 3.7. *$(V, \omega, \mathscr{L})$ is a Liouville manifold iff the sequence* (2.13) *splits for $h = 0$, and then it splits for every h.*

Proof. On a Liouville manifold, we may assume (1.1) with $\varphi_{\alpha\beta}^i = 0$, and, with the notation (2.5), we see that $\tau(\sum_{i=1}^n \alpha_i [\partial/\partial x^i]) = \sum_{i=1}^n \alpha_i y^i$ (where α_i are germs of foliate h-forms, and $[\partial/\partial x^i] \in \nu\mathscr{L}$) is a left inverse of σ of (2.13). Conversely, let τ: $\underline{\nu}\,\mathscr{L} \to \mathscr{A}(\mathscr{L})$ be a splitting of (2.13), for $h = 0$. Then, in semicanonical coordinates, we must have some expressions $\tau[\partial/\partial x_\alpha^i] = y_\alpha^i + \psi_\alpha^i(x_\alpha^h)$, and (1.1) implies

$$\psi_\beta^i = \sum_{h=1}^n \frac{\partial x_\alpha^h}{\partial x_\beta^i} \psi_\alpha^h - \varphi_{\alpha\beta}^i .$$

Accordingly, $\tilde{x}_\alpha^i = x_\alpha^i$, $\tilde{y}_\alpha^i = y_\alpha^i + \psi_\alpha^i$ is an $\hat{\Omega}_n^0$-atlas. Q.e.d.

Proposition 3.8. *If $(V, \omega, \mathscr{L})$ is a Liouville manifold, the exact sequences* (2.17) *split, and we also have the exact sequences of sheaves*

$$0 \to \underline{\nu}\,\mathscr{L} \xrightarrow{\mu} \hat{\Xi} \xrightarrow{\lambda} \underline{\Lambda}^{10}\mathscr{L} \to 0\,,$$
$$0 \to \underline{\nu}\,\mathscr{L} \xrightarrow{\mu} \Xi \xrightarrow{\lambda} \mathscr{X}^{10}(\mathscr{L}) \to 0\,. \tag{3.6}$$

Proof. Again we use (1.1) with $\varphi^i_{\alpha\beta} = 0$. This allows us to identify a germ of V with a germ of the cotangent bundle T^*N of a local transversal submanifold N of $\mathscr{L}$. Then the projection $\tilde{X}$ of X of (2.14) onto $\nu\mathscr{L}$ defines a local flow φ_t on N, and the lift of φ_t to T^*N yields a global vector field

$$Y = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} - \sum_{i,h=1}^{n} \frac{\partial \xi^h}{\partial x^i} y^h \frac{\partial}{\partial y^i} \tag{3.7}$$

on V. (This is the *complete lift* of [YI].) The mapping $\tilde{X} \to Y$ defines μ needed in (3.6), and $X \to$ symplectic dual of $(X - Y)$ defines λ. Q.e.d.

Remark. The technique used to get (3.7) can also be used to make various other kinds of *lifts* like for cotangent bundles [YI].

4. Affine Transversal Distributions

This notion was defined in [M3] for the Lagrangian case, and a corresponding existence obstruction is to be found in [M2]. We shall give a straightforward definition of such an obstruction, for l.l.a. foliations.

Let $\mathscr{F}$ be defined by the atlas of (1.4), and let $\mathscr{F}'$ be a transversal distribution defined by (1.5), (1.6). $\mathscr{F}'$ is said to be *affine* if the process of lifting paths of transversal submanifolds of $\mathscr{F}$ to paths tangent to $\mathscr{F}'$ yields affine (i.e., parallelism preserving) mappings between the leaves of $\mathscr{F}$ [M3]. It is obvious that this happens iff t^u_a of (1.5) are sections of $\mathscr{A}(\mathscr{F})$. The fact that this is an invariant property also follows from the transition formulas for the t^u_a as imposed by (1.5) and (1.4):

$$t^u_{\beta a} = \sum_{b,v} \frac{\partial y^u_\beta}{\partial y^v_\alpha} t^u_{\alpha b} \frac{\partial x^b_\alpha}{\partial x^a_\beta} - \sum_b \frac{\partial y^u_\beta}{\partial x^b_\alpha} \frac{\partial x^b_\alpha}{\partial x^a_\beta}\,. \tag{4.1}$$

Formulas (4.1) and (1.4) show that $d_f(\partial t^u_a/\partial y^v)$ glue up to a global d_f-closed form which yields a class

$$\tau(\mathscr{F}) \in H^1(M, \underline{\mathscr{B}}(\mathscr{F})) \quad (\mathscr{B}(\mathscr{F}) \overset{\text{def}}{=} T\mathscr{F} \otimes T^*\mathscr{F} \otimes \nu^*\mathscr{F}) \,. \tag{4.2}$$

$\tau(\mathscr{F})$ does not depend on the choice of $\mathscr{F}'$ since if the latter is changed to $\tilde{\mathscr{F}}'$ with the coefficients $\tilde{t}^u_a$, then $p^u_a = \tilde{t}^u_a - t^u_a$ defines a global cross-section of $T\mathscr{F} \otimes \nu^*\mathscr{F}$. Clearly, if an affine transversal distribution exists, $\tau(\mathscr{F}) = 0$. Conversely, if $\tau = 0$, we have

$$d_f\left(\frac{\partial t^u_a}{\partial y^v}\right) = -\, d_f s^u_{av}, \quad (s^u_{av}) \in \Gamma\mathscr{B}(\mathscr{F}) \,. \tag{4.3}$$

Then, the local forms $s^u_{av}\theta^v$ are d_f-closed, and provide some

$$s(\mathscr{F}) \in H^1(M, \underline{\underline{T\mathscr{F} \otimes \nu^*\mathscr{F}}}) \,, \tag{4.4}$$

which does not depend on the choice of s^u_{av} at the previous stage. Finally, iff $s(\mathscr{F}) = 0$, we shall obtain $(p^u_a) \in \Gamma(T\mathscr{F} \otimes \nu^*\mathscr{F})$ such that $s^u_{av} = \partial p^u_a / \partial y^v$, and

$$\tilde{\theta}^u = dy^u + \sum_{a=1}^{q} (t^u_a + p^u_a)\, dx^a = 0$$

defines an affine transversal distribution $\tilde{\mathscr{F}}'$. Hence, we proved

Proposition 4.1. *A l.l.a. foliation $\mathscr{F}$ has an affine transversal distribution iff $\tau(\mathscr{F}) = 0$ and $s(\mathscr{F}) = 0$.*

We also prove a few geometric results.

Proposition 4.2. *Fet $\mathscr{F}$ be a l.l.a. foliation on M. If $\mathscr{F}$ has an affine transversal distribution, $T\mathscr{F}$ has a foliate connection. The converse also holds if $\mathscr{E}(\mathscr{F}) = 0$. If $\mathscr{F} = \mathscr{L}$ is a Lagrangian foliation, the same holds for $\nu\mathscr{L}$.*

Proof. Any transversal distribution $\mathscr{F}'$ allows us to extend the flat connection $\mathring{\nabla}$ along the leaves to a connection on $T\mathscr{F}$ by

$$\mathring{\nabla}^{\mathscr{F}'}_X Y = \mathrm{pr}_{T\mathscr{F}}[X, Y] \quad (X \in \mathscr{F}',\ Y \in T\mathscr{F}) \,, \tag{4.5}$$

and, if we use the basis $(X_a, \partial/\partial y^u)$ of (1.5), we see that $\mathring{\nabla}^{\mathscr{F}'}$ is a foliate connection [M1], [V2].

Conversely, if $\mathscr{E}(\mathscr{F}) = 0$, a germ of the manifold M may be seen as a germ of a vector bundle with fibre coordinates y^u, and some fibre basis e_u, along a transversal submanifold N of $\mathscr{F}$. Then, the foliate connection of $T\mathscr{F}$ induces a connection on the previous vector

bundle germ with local equations $\nabla_{(\partial/\partial x^a)} e_u = \sum_v \Gamma^v_{ua}(x)\, e_v$, and the tangent vectors of the paths of the ∇-parallel vectors y define an $\mathscr{F}'$ of equations $dy^v + \sum_u \Gamma^v_{ua}(x)\, y^u\, dx^a = 0$ i.e., an affine transversal distribution.

The final assertion follows by symplectic duality. Q.e.d.

Remark. If $\mathscr{F} = \mathscr{L}$ with $\mathscr{E}(\mathscr{L}) = 0$, and if $\nu\mathscr{L}$ has a foliate connection whose torsion is tangent to $\mathscr{L}$, then the affine transversal distribution constructed in the previous proof will be a Lagrangian distribution.

In the case of any Lagrangian foliation $\mathscr{L}$ we may always use a Lagrangian transversal distribution $\mathscr{L}'$ which will be given by (1.5) with indices $i, j, \ldots$, and where $t^j_i = t^i_j$. Moreover, the manifold V has a *canonical connection* which preserves $\mathscr{L}$ and $\mathscr{L}'$ (e.g., [Hs]), and which is the sum of the connection (4.5) for $\mathscr{F}' = \mathscr{L}'$ with its dual connection (we have $\mathscr{L}' \approx T^*\mathscr{L}$). The local equations of this connection are

$$\nabla_{\partial/\partial y^i} \frac{\partial}{\partial y^j} = 0, \quad \nabla_{X_i} \frac{\partial}{\partial y^j} = \sum_{k=1}^n \frac{\partial t^k_i}{\partial y^j} \frac{\partial}{\partial y^k},$$

$$\nabla_{\partial/\partial y^i} X_j = 0, \quad \nabla_{X_i} X_j = -\sum_{k=1}^n \frac{\partial t^j_i}{\partial y^k} X_k\,.$$

The torsion and curvature of ∇ are given by

$$T\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, T\left(\frac{\partial}{\partial y^i}, X_j\right) = 0,\ T(X_i, X_j) = -\sum_{k=1}^n \tau_{ijk} \frac{\partial}{\partial y^k},$$

$$\Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0,\ \Omega\left(\frac{\partial}{\partial y^i}, X_j\right) \frac{\partial}{\partial y^k} = \sum_{h=1}^n \frac{\partial^2 t^h_j}{\partial y^k \partial y^i} \frac{\partial}{\partial y^h},$$

$$\Omega\left(\frac{\partial}{\partial y^i}, X_j\right) X_k = -\sum_{h=1}^n \frac{\partial^2 t^k_j}{\partial y^i \partial y^h} X_h,$$

$$\Omega(X_i, X_j) \frac{\partial}{\partial y^k} = -\sum_{h=1}^n \frac{\partial \tau_{ijk}}{\partial y^k} \frac{\partial}{\partial y^h},\ \Omega(X_i, X_j)\, X_k = \sum_{h=1}^n \frac{\partial \tau_{ijk}}{\partial y^h} X_h,$$

where

$$\tau_{ijh} = \frac{\partial t^h_i}{\partial x^j} - \frac{\partial t^h_j}{\partial x^i} + \sum_{l=1}^n t^l_i \frac{\partial t^h_j}{\partial y^l} - \sum_{l=1}^n t^l_j \frac{\partial t^h_i}{\partial y^l},$$

and $\tau_{ijh} = 0$ iff $\mathscr{L}'$ is integrable.

The geometric consequences of these formulas are contained in

Proposition 4.3. *Let $\mathscr{L}$ be a Lagrangian foliation of (V, ω) which has a Lagrangian affine transversal distribution $\mathscr{L}'$. Then:*

i) ∇ *is a foliate connection of TV,*

(ii) *if $\mathscr{L}'$ is integrable $(V, \mathscr{L}, \mathscr{L}')$ is locally equivalent to $\mathbb{R}^{2n}$ with its horizontal and vertical foliations,*

(iii) *conversely, if the equivalence of* ii) *holds good, $\mathscr{L}'$ is an affine transversal distribution of $\mathscr{L}$.*

ii) and iii) are true since the equivalence with $\mathbb{R}^{2n}$ holds iff ∇ has no torsion and no curvature (e.g., [Hs]).

Remark. The existence of a foliate connection on $T\mathscr{L}$ implies $\text{Chern}^h(V) \overset{\text{def}}{=} \text{Chern}^h(\mathscr{L} \otimes \mathbb{C}) = 0$ for $h > n$, and, therefore, $\text{Pont}^l(V) = 0$ for $l > [n/2]$ and $\text{Euler}(V) = 0$. All these are necessary conditions for $\mathscr{L}$ to admit a Lagrangian affine transversal distribution.

A particularly interesting case is that of a Riemannian-Lagrangian foliation $\mathscr{L}$, i.e., one endowed with a foliate metric h on $\nu\mathscr{L}$ (e.g., [V2]). If, morever, $\mathscr{E}(\mathscr{L}) = 0$, the construction of Proposition 4.2 yields an affine transversal distribution $\mathscr{L}'$ given by

$$\theta^i = dy^i - \sum_{j,k=1}^{n} \Gamma_{ij}^k y^k \, dx^j = 0, \tag{4.6}$$

where Γ_{ij}^k are the Christoffel symbols of h, and

$$g = \sum_{i,j=1}^{n} h_{ij}(x) \, dx^i \, dx^j + \sum_{i,j=1}^{n} h^{ij}(x) \, \theta^i \theta^j \tag{4.7}$$

is a *quasi-cotangent metric* of V, as defined in [M3].

Of course, conversely, the existence of a quasi-cotangent metric, i.e., a metric of type (4.7) with θ^i of (4.6), implies the existence of an affine (metric preserving, in fact) transversal distribution. We can easily give a number of examples (besides the cotangent bundle itself and the torus) of Lagrangian foliations with a quasi-cotangent metric and, therefore, with an affine transversal distribution. These are:

a) the manifolds V_λ of Example c, Section 1, with the metric

$$g = [\sum_{i=1}^{n} (dx^i)^2]/\sum_{i=1}^{n} (x^i)^2 + \sum_{i=1}^{n} (x^i)^2[\sum_{i=1}^{n} (dy^i)^2]; \tag{4.8}$$

b) the manifolds $M(p, q)$ of Example e, Section 1, with the metric

$$\begin{aligned} g = {}& {}^t dX_1 \otimes dX_1 + {}^t dX_2 \otimes dX_2 + [d(y_1 - \alpha y_2)]^2 + \\ & + {}^t(dZ_1 - X_1 dy_1) \otimes (dZ_1 - X_1 dy_1) + \\ & + {}^t(dZ_2 - X_2 dy_2) \otimes (dZ_2 - X_2 dy_2) + (dy_2)^2; \end{aligned} \tag{4.9}$$

c) the manifolds $I(p)$ of Example f, Section 1, with the metric

$$\begin{aligned} g = {}& {}^t(\mathrm{Re}\, dX) \otimes (\mathrm{Re}\, dX) + {}^t(\mathrm{Im}\, dX) \otimes (\mathrm{Im}\, dX) + \\ & + [d(\mathrm{Re}\, y - \alpha\, \mathrm{Im}\, y)]^2 + \\ & + {}^t\mathrm{Re}\,(dZ - X\, dy) \otimes \mathrm{Re}\,(dZ - X\, dy) + \\ & + {}^t\mathrm{Im}\,(dZ - X\, dy) \otimes \mathrm{Im}\,(dZ - X\, dy) + (\mathrm{Im}\, dy)^2. \end{aligned} \tag{4.10}$$

The manifolds of Examples b, c above are compact, hence they have a *structural sheaf of Lie algebras* (Theorem II of [M3].) In order to check that (4.9) and (4.10) are quasi-cotangent metrics, one has to go over to canonical coordinates of the respective manifolds. We leave this as an exercise.

5. Weak Lagrangian Structures

In this section, we consider the $\hat{\mathfrak{Q}}_n$-structures, where $\hat{\mathfrak{Q}}_n$ is the weak Lagrangian pseudogroup of Section 1.

Proposition 5.1. *An $\hat{\mathfrak{Q}}_n$-structure on a manifold V^{2n} is equivalent to a pair $(\mathscr{L}, \sigma)$ where $\mathscr{L}$ is an n-dimensional foliation, and σ is a non-degenerate d_f-closed form of type* (1,1) *on V.*

Proof. If the first line of (1.1) holds good, $\mathscr{L}$ will be defined by $x^i_\alpha = \mathrm{ct.}$, and, if $\mathscr{L}'$ is a transversal distribution given by (1.5), (1.6), the form σ is $\sum_{i=1}^{n} dx^i_\alpha \wedge \theta^i_\alpha$. Conversely, let us give $\mathscr{L}$ and

$$\sigma = \sum_{i,j=1}^{n} \sigma_{ij}\, dx^i \wedge \theta^j \quad (\mathrm{rank}\, \sigma = 2n,\ d_f \sigma = 0),$$

where the only thing known about the coordinates is that $x^i = \mathrm{ct.}$ defines $\mathscr{L}$. Then, locally

$$\sigma = -\, d_f(\sum_{i=1}^{n} \varphi_i dx^i) = \sum_{i=1}^{n} dx^i \wedge d_f \varphi_i = \sum_{i=1}^{n} dx^i \wedge d\varphi_i - \sum_{i=1}^{n} dx^i \wedge d' \varphi_i$$

(using (2.1)), and $\tilde{x}^i = x^i$, $\tilde{y}^i = \varphi_i$ may be used as new local coordinates. With respect to these coordinates, we have

$$\sigma = \sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}^i + \text{ form of type } (2,0) ,$$

and, since σ is a global form, $(\tilde{x}^i, \tilde{y}^i)$ will have transition functions in $\hat{\Omega}_n$. Q.e.d.

Remark. Since the type of forms depends on the choice of $\mathscr{L}'$, some readers might prefer to see σ as a foliate vector bundle isomorphism $T\mathscr{L} \approx \nu^* \mathscr{L}$, which has an invariant meaning straightforwardly.

Proposition 5.2. *Let $(\mathscr{L}, \sigma)$ define a weak Lagrangian structure on a compact manifold V. Then $H^k(V, \underline{\Lambda}^k \mathscr{L}) \neq 0$ for every $k = 0, 1, 2, \ldots$.*

Proof. σ^k defines a nonzero class in $H^k(V, \underline{\Lambda}^k \mathscr{L})$ since if σ^k would be d_f-exact, σ^n would be d-exact which is false. Q.e.d.

In the sequel, we give obstructions for an $\hat{\Omega}_n$-structure $(\mathscr{L}, \sigma)$ to be reducible to an Ω_n-structure.

Clearly, $\mathscr{L}$ is a Lagrangian foliation iff V has a symplectic form ω such that $\omega = \sigma - \lambda_{20}$ for some form λ of type $(2,0)$. (Here, and in the sequel, we denote the type of the forms by means of indices.) Since $d_f \sigma = 0$, by (2.19), $d\sigma$ defines a class

$$[d\sigma] \in H^1(V, \mathscr{Z}^{20}(\mathscr{L})) , \tag{5.1}$$

and we see that the symplectic form ω requested exists iff $[d\sigma] = 0$.

A more detailed answer to the same question is obtained as follows. By (2.1), $d\omega = 0$ means

$$d'\sigma = d''\lambda, \quad \partial\sigma = d'\lambda . \tag{5.2}$$

On the other hand, since by (2.20) we have $d''d'\sigma = -d'd''\sigma = 0$, $(\mathscr{L}, \sigma)$ defines a class

$$[d'\sigma] \in H^1(V, \underline{\Lambda}^2 \mathscr{L}) , \tag{5.3}$$

and, iff this class vanishes, a form $\varkappa_{20}$ exists such that $d'\sigma = d''\varkappa_{20}$. Put $\theta = \partial\sigma - d'\varkappa_{20}$. Then (2.20) leads to $d'\theta = d''\theta = 0$, and we have again a cohomology class

$$[\theta] \in H^3_{\text{basic}}(V, \mathscr{L}) . \tag{5.4}$$

Iff $[\theta] = 0$, foliate forms ν_{20} exist such that $\theta = d\nu_{20}$. Now, the definitions $\lambda_{20} = \varkappa_{20} + \nu_{20}$, $\omega = \sigma - \lambda_{20}$ yield the symplectic form requested.

Notice that if we take other choices of $\varkappa_{20}, \nu_{20}$ we get a form $\tilde{\omega}$ such that $\tilde{\omega} - \omega$ is a closed foliate 2-form. On the other hand, the existence of ω depends only on the d_f-cohomology class of σ since, if σ is replaced by $\sigma + d_f\beta_{10}$, the obstruction (5.1) will not be changed and ω will be replaced, say, by $\omega + d\beta_{10}$.

The results may be summarized as follows

Proposition 5.3. *Let $\mathscr{L}^n$ be a foliation of a manifold V^{2n}. Then V has a symplectic structure ω which makes $\mathscr{L}$ into a Lagrangian foliation iff:*

i) *V has a nondegenerate d_f-closed form σ of type* (1,1);

ii) *the cohomological obstructions, either* (5.1) *or* (5.3) *and* (5.4) *vanish. Actually, these symplectic structures are associated to $[\sigma] \in H^1(V, \underline{\Lambda}^1 \mathscr{L})$, and, if ω is one of them, all the others are*

$$\tilde{\omega} = \omega + F + d\beta_{10}, \tag{5.5}$$

where F is a closed foliate 2-form, and β_{10} is a form of type (1,0) *such that $\sigma + d_f\beta$ is nondegenerate.*

As applications we have

Proposition 5.4. *Let B^n be a differentiable manifold such that $H^3(B, \mathbb{R}) = 0$, and $\pi: V \to B$ be a vector bundle of rank n over B. Then, if the fibers of V are the leaves of a weak Lagrangian foliation $\mathscr{L}$ of V, $\mathscr{L}$ is, in fact, a Lagrangian foliation.*

Proof. Since $H^3_{\text{basic}}(V, \mathscr{L}) = H^3(B, \mathbb{R}) = 0$, the obstruction (5.4) vanishes. Then, by (2.23) and the Remark afterwards, we also have $H^1(V, \underline{\Lambda}^2 \mathscr{L}) = 0$, hence the obstruction (5.3) also vanishes. Q.e.d.

Proposition 5.5. *Let $T^* M^n$ be a cotangent bundle with the vertical foliation $\mathscr{V}$. The only symplectic forms on $T^* M$ which make $\mathscr{V}$ into a Lagrangian foliation are given by*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i + F + d\Big(\sum_{i=1}^{n} \alpha(x, y)\, dx^i\Big), \tag{5.6}$$

where x^i are coordinates on M, y^i are corresponding natural covector coordinates, F is a closed 2-form on M, and $\sum \alpha_i dx^i$ is a 1-form on $T^* M$, under the condition that the form ω is nondegenerate.

Proof. This follows from the last part of Proposition 5.3 since by (2.23), $H^1(T^*M, \underline{\Lambda}^1 \mathcal{V}) = 0$. Q.e.d.

Notes added in proof. 1) The manifold V_λ of Example c, Section 1 is just $T^*(S^{n-1} \times S^1)$ with the zero section removed. If, following a remark by G. Hector, we keep the zero section, the latter will be a single compact leaf for one of the Lagrangian foliations of the example. All the leaves of the foliations of this example, including the compact one mentioned above, are affinely noncomplete since they are covered by $\mathbb{R}^n \setminus \{0\}$.

2) The linearity obstruction defined by (3.3) is a foliated version of the radiance obstruction encountered in affine manifolds theory. (See: GOLDMAN, W., HIRSCH, M. W.: The radiance obstruction and parallel forms on affine manifolds. Trans. Amer. Math. Soc. **286**, 629—649 (1984).)

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