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The Foliation of Semigroups by Congruence Classes

By

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Abstract. Let S be an open subsemigroup of a Lie group G with $\mathbf{1} \in \bar{S}$. We shall show that for every congruence $\varkappa$ on S with closed congruence classes there exists an open neighborhood U of $\mathbf{1}$ in G and a foliation of $S \cap U$ whose leaves locally coincide with both the congruence classes of $\varkappa$ and the cosets of a normal analytic subgroup of G which is uniquely determined by $\varkappa$.

Let $\varkappa$ be a congruence on a group G. Then $\varkappa$ is determined by the congruence class $N = \varkappa(\mathbf{1})$ of the identity, since the congruence class of every other element $g \in G$ is given by $\varkappa(g) = gN$. If, in addition, G is a Lie group and N is closed then the congruence $\varkappa$ induces a foliation of the manifold underlying the Lie group G. In fact, write $\mathbf{L}(G)$ for the Lie algebra associated with G, and I for the ideal of $\mathbf{L}(G)$ which belongs to the closed normal subgroup N. Then for any vector space complement E of I in $\mathbf{L}(G)$ we can find open cell neighborhoods A of 0 in E and B of 0 in I such that for each $g \in G$ the map $\Phi_g: A \times B \to G$, $(X, Y) \mapsto (\exp X)(\exp Y)g$, is a diffeomorphism mapping from $A \times B$ onto an open neighborhood V of g in G, and such that each coset $(\exp X)Ng = N(\exp X)g$ meets V exactly in $\Phi_g(\{0\} \times B)$. In particular, it is implicit in these classical results that the congruence classes $\varkappa(g) = Ng$ are analytic submanifolds of G.

Let us now consider congruences $\varkappa$ on an open subsemigroup S of a Lie group G such that the closure $\bar{S}$ of S in G contains the identity 1. (Such semigroups arise naturally in the context of Control Theory and the Lie theory of semigroups (cf. [3]) as the interior of semigroups generated by one-parameter subsemigroups of G.) Not unexpectedly, for semigroups in place of groups, the situation is much more com-

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plex. There are examples of congruences on open subsemigroups of such a familiar Lie group G as the additive group $\mathbb{R}^2$ whose congruence classes have a highly intricate structure in every neighborhood of the identity of G (cf. Example 28 below). It is also by no means true that $\varkappa$ is closed if and only if its congruence classes are closed (cf. Examples 28 and [4]).

It is, therefore, somewhat surprising that on an open subsemigroup S of a Lie group G with $\mathbf{1}\in\bar{S}$ a congruence $\varkappa$ with closed congruence classes gives rise to a foliation of S at all points near the identity. Indeed it is the aim of this paper to show this fact and to prove, more precisely, that $\varkappa$ determines an ideal I of $\mathbf{L}(G)$ such that for every vector space complement E of I in $\mathbf{L}(G)$ and any point $u\in U\cap S$, where U is a suitable open neighborhood of the identity in G, there are open cells A_u and B_u around 0 in E and I, respectively, so that the map $\Phi_u\colon A_u\times B_u\to V=(\exp A_u)(\exp B_u)$ is a diffeomorphism and satisfies

$$\varkappa(\Phi_u(X,0))\cap U\cap S=\Phi_u(\{X\}\times B_u) \text{ for all } X\in A_u\,.$$

Thus on the open set $S\cap U$, that is, near the identity, the open manifold S is foliated by the congruence classes modulo $\varkappa$. In particular, our result implies that the subsets $\varkappa(u)\cap S\cap U$ are analytic manifolds of $S\cap U$. This is one of those instances where purely topological assumptions (such as the closedness of the congruences classes) have strong analytical consequences.

Historically, the first author to prove theorems of this type was Klaus Keimel who showed that on any closed convex (not necessarily pointed) cone in a finite dimensional real vector space, every closed congruence relation agrees, on a suitable open neighborhood of 0, with the congruence induced by some linear subspace. (See [9], Theorem 2.6; cf. also the papers [6], [7] and [8].) We gratefully acknowledge Klaus Keimel's participation in a seminar in which the subject matter of this paper was discussed and developed.

Throughout the paper $\varkappa$ will denote a congruence on an open subsemigroup S of a Lie group G such that $\mathbf{1}\in\bar{S}$. We do not assume that $\varkappa$ is closed (as a subset of $S\times S$) but make the weaker assumption that all congruence classes $\varkappa(s)$ are closed in S. It is known from the general theory of quotient spaces that the quotient of a locally com-

pact topological space by a closed equivalence need not be Hausdorff in general; it will be a Hausdorff k-space if it is, in addition, σ-compact. (We refer the reader to the paper by LAWSON and MADISON [10] for more information about quotients of semigroups on locally compact spaces and k-spaces.)

If not stated otherwise, we follow the notation and terminology of [3]. In particular, we write $\mathbf{L}(G)$ for the Lie algebra of G, and $X * Y$ for the Campbell-Hausdorff product $X + Y + \frac{1}{2} \cdot [X, Y] + \ldots$ in $\mathbf{L}(G)$.

The Foliation Lemma

In this section, our main goal is to show that sufficiently close to the identity, the congruence classes modulo $\varkappa$ agree locally with the cosets of a normal analytic subgroup N of G. To this end it is necessary to compare congruence classes at different points; we do this by transporting them to the identity via right translations and considering their tangent structure at the origin.

We start by recalling the notion of a subtangent vector (cf. [3], sections I.5 and IV.1). Let M be a subset of a topological vector space L and let $x \in \bar{M}$. Then a vector $y \in L$ is called a *subtangent vector of M at x* if we can find a sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(m_n)_{n\in\mathbb{N}}$ of points $m_n \in M$, such that

$$\lim r_n = \infty \text{ and } \lim r_n \cdot (m_n - x) = y \,. \tag{1}$$

The set of all subtangent vectors at x is denoted with $L_x(M)$.

1. *Definition.* For $s \in S$ write $K_s = \varkappa(s)s^{-1} = \varrho_s^{-1}(\varkappa(s))$. The set $L_0(\exp^{-1} K_s)$ of subtangent vectors of K_s at the origin will be called $\mathbf{L}[s]$. We define

$$I \overset{\text{def}}{=} \bigcap_{s\in S} \mathbf{L}[s] \,. \tag{2}$$

Clearly, $\mathbf{L}[s]$ is a closed subset of $\mathbf{L}(G)$ and closed under nonnegative scalar multiplications. Hence I is a closed subset satisfying $\mathbb{R}^+ \cdot I = I$. Most of our efforts will focus on this set.

First we introduce on G (and thus, a fortiori, on S) the transitive relation $\prec$ given by $x \prec y$ if and only if $y \in xS$. This relation is a preorder if and only if it is reflexive; and this is the case if and only

if S contains the identity. We are, however, interested in the case that $\mathbf{1} \notin S$ (because if $\mathbf{1} \in S$ then S contains the identity component of G and the issue becomes trivial from our point of view).

2. Lemma. (i) *If $s \prec t$ then $K_s \subseteq K_t$ and thus $\mathbf{L}[s] \subseteq \mathbf{L}[t]$.*

(ii) *The set $(S, \prec)$ is filtered. That is, for each pair $t, t' \in S$, there is an $s \in S$ with $s \prec t$ and $s \prec t'$.*

Proof. (i) If $s \prec t$, then there is an $s' \in S$ with $t = s\,s'$. Since $\varkappa$ is a congruence, $\varkappa(s)\,s' \subseteq \varkappa(s\,s')$ and thus $\varkappa(s) \subseteq \varkappa(s\,s')\,s'^{-1}$. Therefore $K_s = \varkappa(s)\,s^{-1} \subseteq \varkappa(s\,s')\,s'^{-1}\,s^{-1} = K_{ss'} = K_t$ and the assertion follows.

(ii) If $t, t' \in S$, then $t\,S^{-1}$ and $t'\,S^{-1}$ are open neighborhoods of $\mathbf{1}$ in G. Since $\mathbf{1} \in \bar{S}$ we therefore find an element $s \in t\,S^{-1} \cap t'\,S^{-1} \cap S$. Then $t \in s\,S$ and thus $s \prec t$. Similarly, $s \prec t'$. □

Remark. The above Lemma implies the following assertions (proofs are left to the reader): (i) The families $\{K_s : s \in S\}$ and $\{\mathbf{L}[s] : s \in S\}$ are filter bases. (ii) Every sequence $(s_n)_{n\in\mathbb{N}}$ in S converging to $\mathbf{1}$ is $\prec$-cofinal in S. Consequently, $\{K_{s_n} : n \in \mathbb{N}\}$ is cofinal in $\{K_s : n \in \mathbb{N}\}$, and $\{\mathbf{L}[s_n] : n \in \mathbb{N}\}$ is cofinal in $\{\mathbf{L}[s] : n \in \mathbb{N}\}$. (iii) Every sequence $(s_n)_{n\in\mathbb{N}}$ with $\lim s_n = \mathbf{1}$ has a $\prec$-decreasing subsequence.

Our next lemma is purely algebraic.

3. Lemma. (i) *Suppose that $s \prec t$ in S. If $g_1, g_2, \ldots, g_n \in K_s$ satisfy $s \prec g_k g_{k-1} \ldots g_1\, t$ for any $k \in \{1, 2, \ldots, n\}$, then $g_n g_{n-1} \ldots g_1 \in K_t$.*

(ii) *If $g \in K_s$, and $g\,s \prec t$, then $g^{-1} \in K_t$.*

Proof. (i) By induction on k, we show that $g_k g_{k-1} \ldots g_1 \in K_t$, or, equivalently, that $g_k \ldots g_1\, t \sim_\varkappa t$ for $k = 1, 2, \ldots, n$. If $k = 1$, then the assertion follows from Lemma 2 (i). Suppose that $g_k g_{k-1} \ldots g_1\, t \sim_\varkappa t$ for some $k < n$. By hypothesis, there is an element $s_k \in S$ such that $s\,s_k = g_k g_{k-1} \ldots g_1\, t$. Then $s\,s_k \sim_\varkappa t$. Also $g_{k+1} \in K_s$, that is, $g_{k+1}\,s \sim_\varkappa s$. Hence $g_{k+1}\,s\,s_k \sim_\varkappa s\,s_k$, since $\varkappa$ is a congruence. Transitivity now implies $g_{k+1}\,s\,s_k \sim_\varkappa t$ and thus $g_{k+1} g_k \ldots g_1\, t \sim_\varkappa t$ which we had to show.

(ii) Since $g\,s \prec t$ there is an $s' \in S$ with $g\,s\,s' = t$, that is $s^{-1} g^{-1}\, t = s'$. Also, $g \in K_s$ means that $g\,s \sim_\varkappa s$. Now $g^{-1}\, t = s\,s' \sim_\varkappa g\,s\,s' = g\,s\,s^{-1} g^{-1}\, t = t$, so $g^{-1} \in K_t$. □

4. Lemma. (i) *Let A be a subset of a Banach space with $0 \in \bar{A}$. If X is a subtangent vector of A at 0 then there exist sequences $(k_n)_{n\in\mathbb{N}}$ of natural numbers and $(X_n)_{n\in\mathbb{N}}$ of elements in A such that*

$$X = \lim n k_n \cdot X_n \,. \tag{3}$$

(Note that (3) *implies* $\lim k_n \cdot X_n = 0$.)

(ii) *If L is a Dynkin algebra and A, A' are subsets of L containing 0 in their closures, and if $X \in L_0(A)$, $X' \in L_0(A')$, then there exist sequences $(k_n)_{n\in\mathbb{N}}$, $(k'_n)_{n\in\mathbb{N}}$ of natural numbers and $(X_n)_{n\in\mathbb{N}}$, $(X'_n)_{n\in\mathbb{N}}$ of elements in A and A', respectively, such that*

$$X + X' = \lim n (k_n \cdot X_n * k'_n \cdot X'_n). \tag{4}$$

Proof. (i) Suppose that $Y \in L_0(A)$; say, $Y = \lim r_n \cdot Y_n$, with $Y_n \in A$ and $\lim r_n = \infty$. Now we decompose r_n as $r_n = k_n + d_n$, where k_n is the largest natural number not exceeding r_n and $0 \leqslant d_n < 1$. Then $\lim d_n \cdot Y_n = 0$ and thus $Y = \lim r_n \cdot Y_n = \lim k_n \cdot Y_n$.

Let $n \in \mathbb{N}$. Since every positive multiple of a subtangent vector is also a subtangent vector at the point in question, we have $1/n \cdot X \in \in L_0(A)$, and thus the above observation shows that there is a natural number k_n and a point $X_n \in A$ such that $\| 1/n \cdot X - k_n \cdot X_n \| < 1/n^2$. Then $\| X - n k_n X_n \| < 1/n$, whence the claim.

(ii) We know from Trotter's formulas that whenever $X = \lim n \cdot Z_n$ and $X' = \lim n \cdot Z'_n$, then $X + X' = \lim n (Z_n * Z'_n)$. By Part (i) of the Lemma we know $X = \lim n k_n \cdot X_n$ and $X' = \lim n k'_n \cdot X'_n$ with natural numbers k_n, k'_n and $X_n \in A$, $X'_n \in A'$. Now we take $Z_n = k_n \cdot X_n$, $Z'_n = k'_n \cdot X'_n$ and immediately obtain the assertion.

5. Lemma. *For each element $s \in S$ there exists an open neighborhood B of 0 in $\mathbf{L}(G)$ such that*

$$((I \cap B) + (I \cap B)) \cap B \subseteq \mathbf{L}[s] \cap \exp^{-1} K_s \,. \tag{5}$$

Proof. We fix a Campbell-Hausdorff neighborhood B_0 of $\mathbf{L}(G)$ and choose an element $s' \in S$ with $s' \prec s$; such an element s' exists by Lemma 3 (ii). This means that $s = s's''$ for some $s'' \in S$, hence $s' S s^{-1} = s' S s''^{-1} s'^{-1}$ is an open neighborhood of $\mathbf{1}$. Thus we can find an open convex symmetric neighborhood B of 0 in $\mathbf{L}(G)$ such that $B * B * B \subseteq B_0$ and that

$$B * B * B \subseteq \exp^{-1}(s' S s^{-1}) . \tag{*}$$

Furthermore, since K_s is closed in S and S is open in G, we may choose B so small that also

$$\overline{B \cap \exp^{-1} K_s} \subseteq \exp^{-1} K_s . \tag{**}$$

Now we let $X, X', X + X' \in B$ with $X, X' \in I$. Then, by the definition of I we know $X, X' \in \mathbf{L}[s']$. In view of Lemma 4 this means that there are sequences of natural numbers k_n, k'_n and elements $X_n, X'_n \in \mathbf{L}(G)$ such that

$$\begin{aligned} X &= \lim n k_n \cdot X_n, \quad \exp X_n \in K_s, \\ X' &= \lim n k'_n \cdot X'_n, \quad \exp X'_n \in K_{s'}, \end{aligned} \tag{6}$$

and

$$X + X' = \lim n \cdot (k_n \cdot X_n * k'_n \cdot X'_n) . \tag{7}$$

Because of (6), the sequences $k_n \cdot X_n$ and $k'_n \cdot X'_n$ converge to 0, and in conjunction with (7) we can now pick a natural number N so large that

$$\{k_n \cdot X_n,\ k'_n \cdot X'_n,\ n \cdot (k_n \cdot X_n * k'_n \cdot X'_n)\} \subset B \text{ for all } n \geqslant N . \tag{8}$$

Let us fix $n \geqslant N$. Since B is convex, we conclude

$$\{j \cdot X_n,\ j' \cdot X'_n,\ k \cdot (k_n \cdot X_n * k'_n \cdot X'_n)\} \subset B \text{ for all } j \leqslant k_n,\ j' \leqslant k'_n,\ k \leqslant n. \tag{9}$$

Now we consider the product

$$n \cdot (k_n \cdot X_n * k'_n \cdot X'_n) = \underbrace{X_n * \ldots * X_n}_{k_n \text{ times}} * \underbrace{X'_n * \ldots * X'_n}_{k'_n \text{ times}} * \ldots * X'_n,$$

which is a product of $m = n(k_n + k'_n)$ factors $F_m * F_{m-1} * \ldots * F_2 * F_1$ with $F_p \in \{X_n, X'_n\}$, $p = 1, \ldots, m$. The partial products $F_p * \ldots * F_1$ have the form

$$F_p * \ldots * F_1 = \begin{cases} j' \cdot X'_n * k(k_n \cdot X_n * k'_n \cdot X'_n), & \text{with } j' \leqslant k'_n,\ k \leqslant n, \text{ or} \\ j \cdot X_n * k'_n \cdot X'_n * k(k_n \cdot X_n * k'_n \cdot X'_n) & \text{with } j \leqslant k_n,\ k \leqslant n. \end{cases}$$

By (9), we have $F_p * \ldots * F_1 \in B * B$ in the first case, and $F_p * \ldots * F_1 \in B * B * B$ in the second. Thus

$$\exp F_p \cdots \exp F_1 = \exp(F_p * \ldots * F_1) \in \exp(B * B * B) \subseteq s' S s^{-1} \tag{10}$$

for all $p = 1, 2, \ldots, m$ by (*). But we also have

$$\exp F_p \in \{\exp X_n, \exp X'_n\} \subseteq K_{s'}, \tag{11}$$

by (6). Since $p \in s' S s^{-1}$ says that $s' \prec ps$, conditions (10) and (11) secure the hypotheses of Lemma 3(i); the Lemma applies and yields

$$\exp(k_n \cdot X_n * k'_n \cdot X'_n) = F_{k_n + k'_n} * \ldots * F_1 \in K_s,$$

and

$$\exp(n \cdot (k_n \cdot X_n * k'_n) = F_m * \ldots * F_1 \in K_s.$$

Thus we conclude

$$X + X' = \lim n \cdot (k_n \cdot X_n * k'_n \cdot X'_n) \in L_0(\exp^{-1} K_s) \cap \exp^{-1} K_s, \quad (12)$$

since $B \cap \exp^{-1} K_s$ is closed in B by (**). This completes our proof. □

6. Lemma. *For each $s \in S$ there is a neighborhood B of 0 in $\mathbf{L}(G)$ such that $B \cap I \subseteq \mathbf{L}[s] \cap \exp^{-1} K_s$.*

Proof. We choose $s' \prec s$ and B as in the proof of the preceding Lemma 5, but with an additional requirement. Recall that $s = s' s''$, whence $s S^{-1} s'^{-1} = s' s'' S^{-1} s'^{-1}$ is an open neighborhood of $\mathbf{1}$. We therefore may choose B in fact so small that also

$$B \subseteq \exp^{-1}(s S^{-1} s'^{-1}) . \quad (13)$$

Now let $X \in B \cap I$. Then $X \in \mathbf{L}[s']$ and thus, as in the proof of the preceding Lemma, $X = \lim n k_n \cdot X_n$ with integers k_n and $\exp X_n \in K_s$. If we choose N so large that $n \geqslant N$ implies $n k_n \cdot X_n \in B$, then, by the convexity of B again, $X_n \in B$. Since B is symmetric,

$$-X_n \in B .$$

By (13), $g = \exp X_n$ satisfies $g^{-1} = \exp(-X_n) \in s S^{-1} s'^{-1}$, that is, we have $g s \in s' S$. Consequently,

$$(\exp X_n) s \prec s' \text{ for all } n \geqslant N .$$

By Lemma 3 (ii), this implies

$$\exp(-X_n) \in K_s, \text{ for all } n \geqslant N . \quad (14)$$

Thus $-X = \lim n k_n \cdot (-X_n) \in \mathbf{L}[s]$ by definition of $\mathbf{L}[s]$. Likewise, since $B \cap \exp^{-1} K_s$ is closed in B we also have $-X \in \exp^{-1} K_s$. □

Now we are ready for the crucial conclusion.

7. Lemma. (A Key Lemma) *The set I is a vector subspace of $\mathbf{L}(G)$, and for each $s \in S$ there is an open neighborhood B of 0 in $\mathbf{L}(G)$ such that*

$$\exp(I \cap B) \subseteq K_s . \tag{15}$$

Proof. Using Lemma 5, for each $s \in S$ we find an open neighborhood B with

$$((I \cap B) + (I \cap B)) \cap B \subseteq \mathbf{L}[s] .$$

Since $\mathbb{R}^+ \cdot \mathbf{L}[s] = \mathbf{L}[s]$ we conclude

$$I + I \subseteq \mathbf{L}[s] \text{ for all } s \in S.$$

By the definition of I as intersection of all $\mathbf{L}[s]$ we thus have

$$I + I \subseteq I . \tag{16}$$

By Lemma 6,

$$-I \cap B \subseteq \mathbf{L}[s].$$

Once again by $\mathbb{R}^+ \cdot I = I$ it follows that

$$\mathbb{R} \cdot I \subseteq \mathbf{L}[s] \text{ for all } s \in S .$$

Thus

$$\mathbb{R} \cdot I \subseteq I . \tag{17}$$

In Lemma 5 it was also shown that

$$I \cap B \subseteq \exp^{-1} K_s . \tag{18}$$

The Key Lemma now follows from (16), (17), and (18). □

Remark. A more direct way of characterizing the elements of I is the following: An element $X \in L(G)$ is in I if and only if there are sequences k_n of natural numbers and X_n, Y_n of elements in $\mathbf{L}(G)$ such that $X = \lim k_n(X_n * -Y_n)$, $\lim X_n = \lim Y_n = 0$, and $\exp X_n \sim_* \exp Y_n$. (We leave the proof to the reader.) An immediate consequence of this characterisation is that $-I = I$ and thus that $\mathbb{R} \cdot I = I$.

We now formulate and prove a simple lemma on vector spaces.

8. Lemma. *Let L denote a finite dimensional vector space and $\mathcal{F}$ a filter basis of closed subsets C satisfying $\mathbb{R}^+ \cdot C = C$. If $D = \bigcap \mathcal{F}$ is a vector space, then there is a $C \in \mathcal{F}$ with $D = C$.*

Proof. Passing to the factor space L/D we may assume $D = \{0\}$, so that we have to prove that $\{0\} \in \mathcal{F}$. Suppose that this not the case. Let Σ be the boundary sphere of some compact ball neighborhood of 0

with respect to any norm on L. Now $\{C \cap \Sigma\colon C \in \mathscr{F}\}$ is a filter basis of compact sets due to the fact that $C \cap \Sigma \neq \emptyset$, as $\mathbb{R}^+ C = C \neq \{0\}$. Hence it has an element $s \in \Sigma$ in its intersection, and this is a contradiction to $D = \{0\}$. □

Applying this lemma to the filterbasis $\mathscr{F} = \{\mathbf{L}[s] : s \in S\}$ (see Lemma 2(iii)) we conclude that there is an $s \in S$ with $I = \mathbf{L}[s]$. If $s' \prec s$, then this implies $I = \mathbf{L}[s']$. Since the set $\{s' \in G : s' \prec s\} = s S^{-1}$ is an open neighborhood of $\mathbf{1}$ in G, we therefore have

9. Lemma. *There is an open neighborhood U of $\mathbf{1}$ in G such that*

$$\mathbf{L}[u] = I \text{ for all } u \in S \cap U. \tag{19}$$

10. Lemma. (i) *If $s, t \in S$, then $s K_t s^{-1} \subseteq K_{st}$.*

(ii) *I is an ideal.*

Proof. (i) If $s, t \in S$ then $s K_t = s \varkappa(t) t^{-1} \subseteq \varkappa(s t) t^{-1} = \varkappa(s t)(s t)^{-1} s = K_{st} s$, consequently, $s K_t s^{-1} \subseteq K_{st}$.

(ii) Let U be as in Lemma 9 and choose an element $s \in U \cap S$. Find an open neighborhood V of $\mathbf{1}$ such that

$$V V \subseteq U \cap s S^{-1}.$$

Now let $u, v \in V \cap S$. Then $u v \prec s$ since $u v \in S^{-1}$. Therefore $K_{uv} \subseteq K_s$ by Lemma 2(i). By (i) above, $u K_v u^{-1} \subseteq K_{uv}$. Hence

$$u K_v u^{-1} \subseteq K_s \text{ for all } u, v \in V \cap S. \tag{20}$$

Now let $X \in I$. Then for each $v \in V$ we have $X \in \mathbf{L}[v]$, that is, $X = \lim p_n \cdot X_n$ with a sequence of natural numbers p_n, and a sequence $X_n \in \mathbf{L}(G)$ converging to 0 with $\exp X_n \in K_v$. Hence for $u \in V$ and the automorphism $\mathrm{Ad}(u)$ of $\mathbf{L}(G)$ given by $\exp \mathrm{Ad}(u)(Y) = u(\exp Y) u^{-1}$, we have $\mathrm{Ad}(u)(X) = \lim p_n \cdot \mathrm{Ad}(u)(X_n)$ and $\exp \mathrm{Ad}(u)(X_n) = u(\exp X_n) u^{-1} \subseteq u K_v u^{-1} \subseteq K_s$. Hence $\mathrm{Ad}(u)(X) \in \mathbf{L}[s]$. Since we picked $s \in U \cap S$ we know $\mathbf{L}[s] = I$ from Lemma 9. Therefore, $\mathrm{Ad}(u) I \subseteq I$ for all $u \in U \cap S$. For reasons of dimension, this implies $\mathrm{Ad}(u) I = I$ for $u \in U \cap S$. Now the closed subgroup $H = \{g \in G : \mathrm{Ad}(g) I = I\}$ of G contains the open set $U \cap S$. Since $\mathbf{1} \in \bar{S}$, this set is nonempty and so H is open and closed. Hence H contains the identity component G_0 of G. Thus for any $Y \in \mathbf{L}(G)$ we

have $e^{\operatorname{ad} Y} I = \operatorname{Ad}(\exp Y) I = I$. Since $\operatorname{ad} Y$ is the differential of $e^{\operatorname{ad} Y}$ at 0 we conclude that I is an ideal. □

11. Lemma. *For each $u \in U$, with U as in* Lemma 9, *there is a neighborhood B of* 0 *in* $\mathbf{L}(G)$ *such that* $\exp(I \cap B)$ *is a neighborhood of* **1** *in* K_u.

Proof. Assume the contrary. Then we find an element $u \in U$ and a sequence $X_n \to 0$ such that $\exp X_n \in K_u$ but $X_n \notin I$. Now we let E denote a vector space complement for I in $\mathbf{L}(G)$, and we write $X_n = X'_n * X''_n$ with $X'_n \in E$, $X''_n \in I$. (This decomposition is possible and unique since for sufficiently small 0 neighborhoods $B' \subseteq E$ and $B'' \subseteq I$ the map $(X', X'') \mapsto X' * X'' : B' \times B'' \to B' * B''$ is a diffeomorphism onto a neighborhood of 0 in $\mathbf{L}(G)$.) Since $X_n \notin I$ we have $X'_n \neq 0$. Now $\exp X'_n = \exp(X_n * - X''_n) = (\exp X_n)(\exp - X''_n)$. Further $\exp(- X''_n) \in K_u$ for all sufficiently large n by Lemma 7 (15). Hence $\exp(- X''_n) u \sim_\varkappa u$ and thus

$$\exp X'_n = (\exp X_n)(\exp - X''_n) \sim_\varkappa (\exp X_n) u \sim_\varkappa u$$

since $\exp X_n \in K_u$. Therefore

$$\exp X'_n \in K_u \quad \text{and} \quad 0 \neq X'_n \in E\,. \tag{21}$$

If we fix any norm in $\mathbf{L}(G)$, then the sequence $\| X'_n \|^{-1} \cdot X'_n$ has a cluster point Y with $\| Y \| = 1$, hence $Y \neq 0$, and with $Y \in$ $\in L_0(\exp^{-1} K_u) = \mathbf{L}[u] = I$. But also $Y \in E$. Hence $0 \neq Y \in I \cap E =$ $= \{0\}$, a contradiction. This contradiction proves the claim. □

We now pause a little bit to summarize the results of this section in the following Lemma. (Note that $K_u = \varkappa(u) u^{-1}$.)

12. Lemma. (The Foliation Lemma) *Suppose that G is a finite dimensional Lie group and S an open subsemigroup of G with $\mathbf{1} \in \bar{S}$. Suppose that $\varkappa$ is a congruence with closed classes on S. Then there exists an ideal I of* $\mathbf{L}(G)$ *and an open neighborhood U of* **1** *in G such that for each $u \in U \cap S$ there is an open cell neighborhood B_u of* 0 *in I for which* $(\exp B_u) u$ *is a neighborhood of u in $\varkappa(u)$.*

Consequences of the Foliation Lemma

The final result of the preceding section says that in the given situation the manifold $U \cap S$ *is foliated by the congruence classes of* $\varkappa$ *and that this foliation is induced by the foliation of* G *determined by the cosets* Ng *in* G *where* N *is a normal analytic subgroup of* G, namely, the analytic subgroup N with $I = \mathbf{L}(N)$. We shall make this statement more precise in a moment.

But first we recall at this point the general fact that the connected components of the classes of an equivalence relation μ on a topological space T are themselves the classes of an equivalence relation μ_0. If μ has closed classes, then the same holds for μ_0, since the connected components of a topological space are closed. If T_1 is a subspace of T, then we denote with $\mu \mid T_1$ the induced equivalence relation $\mu \cap (T_1 \times T_1)$. In the following, given an open set U of G, *we shall abbreviate the equivalence relation* $(\varkappa \mid (U \cap S))_0$ *with* $\varkappa'$.

13. Lemma. *In the notation of the* Foliation Lemma, *for each* $u \in U \cap S$, *the class* $\varkappa'(u)$ *is a closed connected submanifold of* $\varkappa(u) \cap U$ *and is the connected maximal integral submanifold on* $U \cap S$ *for the distribution on* G *which assigns to* $g \in G$ *the vector subspace* $d\varrho_g(\mathbf{1})(I)$, *where* ϱ_g *denotes the right translation* $x \mapsto xg$.

Proof. If N is the analytic subgroup generated by $\exp I$, then N is a maximal integral submanifold through $\mathbf{1}$ for the given distribution. Accordingly, Nu is a maximal connected integral submanifold for this distribution through u. If $u \in U \cap S$ then, with the notation of Lemma 12, $(\exp B_u)u$ is an open connected submanifold of Nu *in its intrinsic manifold topology* and it also belongs to $\varkappa(u)$ by the Foliation Lemma. Hence for each $u \in U \cap S$, the closed subset $\varkappa(u) \cap U$ is an integral manifold on $U \cap S$ of the distribution. Then, by the definition of $\varkappa'$, the classes $\varkappa'(u)$ with $u \in U \cap S$ are the maximal connected integral submanifolds of the distribution $g \mapsto d\varrho_g(\mathbf{1})(I)$ on $U \cap S$. □

In other words, for all elements $u \in U \cap S$, the connected components of $\varkappa(u) \cap U$ are uniquely determined by the ideal I and the distribution determined by I on G. The decomposition of $\varkappa(u) \cap U$ itself into connected components depends also on the geometric shape of $U \cap S$. The fact is illustrated in Example 28.

Lemma 13 has a very useful consequence. If a congruence is defined on a subsemigroup of a group and if x, y as well as the translates xg and yg lie in the semigroup such that x is congruent with y then we can not in general conclude that xg and yg are congruent too, unless g is an element of the semigroup. In the present situation, however, we have the following fact:

14. Lemma. *Let everything be as in the* Foliation Lemma 12. *If* $x \sim_{\varkappa'} y$ *and* C *is a smooth path joining* x *and* y *in* $\varkappa'(x)$, *then for all* $g \in G$ *with* $gC \subseteq U \cap S$ *we have* $gx \sim_{\varkappa'} gy$, *in particular,* $gx \sim_{\varkappa} gy$.

Proof. In the following, we write λ_g and ϱ_g for the left and the right translations by elements $g \in G$. Let $c: [0,1] \to \varkappa'(x)$ with $c(0) = x$ and $c(1) = y$ be a smooth curve such that $\dot{c}(t) \in d\varrho_{c(t)}(\mathbf{1})(I)$. If we consider an arbitrary $g \in G$ and define $c_g(t) = g\,c(t)$, then

$$\frac{d}{ds} c_g(s)\,|_{s=t} = d\lambda_g(c(t))\,\dot{c}(t) \in d\lambda_g(c(t))\,d\varrho_{c(t)}(\mathbf{1})(I).$$

Since $d\varrho_g(\mathbf{1})(I) = d\lambda_g(\mathbf{1})(I)$, as I is an ideal, this last expression equals

$$d\lambda_g(c(t))\,d\lambda_{c(t)}(\mathbf{1})(I) = d\lambda_{g\,c(t)}(\mathbf{1})(I) = d\varrho_{g\,c(t)}(\mathbf{1})(I) = d\varrho_{c_g(t)}(\mathbf{1})(I).$$

Thus $c_g([0,1])$ is contained in the integral manifold through $c_g(0) = gx$ given by the distribution $g \mapsto d\varrho_g(\mathbf{1})(I)$. If g is such that $c_g([0,1]) \subseteq U \cap S$, then $c_g([0,1])$ is contained in the integral manifold through gx for this distribution *on the open submanifold* $U \cap S$, which is $\varkappa'(gx)$ by Lemma 13 above. Hence $c_g([0,1]) \subseteq \varkappa'(gx)$ and thus $gx \sim_{\varkappa'} gy$ as asserted. □

Since the component $\varkappa'(x)$ is a manifold of the kind described by Lemma 13, a curve such as C in the preceding lemma exists for any pair of $\varkappa'$-related elements x and y in $U \cap S$.

As a consequence of Lemma 14, the $\varkappa'$-saturations of open subsets of $U \cap S$ are open:

15. Lemma. *Let* V *be an open subset of* $U \cap S$, *and let* $\varkappa'(V)$ *denote the saturation* $\bigcup_{v \in V} \varkappa'(v)$ *of* V *with respect to* $\varkappa'$ *on* $U(S)$. *Then* $\varkappa'(V)$ *is open. In particular, the quotient map* $p: U \cap S \to (U \cap S)/\varkappa'$ *is an open mapping.*

Proof. We consider $s \in \varkappa'(V)$; we must show that there is a whole open neighborhood W of s in S with $W \subseteq \varkappa'(V)$. By assumption there is a $v \in V$ with $s \sim_{\varkappa'} v$. Since the component $\varkappa'(v)$ is an integral manifold through v for the distribution determined by I on $U \cap S$ there is a smooth curve $c: [0,1] \to \varkappa'(v)$ with $c(0) = v$ and $c(1) = s$ such that $\dot{c}(t) \in d\varrho_{c(t)}(1)(I)$ where $\varrho_g(x) = xg$ in G.

Since $c([0,1])$ is compact there is an open neighborhood V_0 of $\mathbf{1}$ in G such that $V_0 c([0,1]) \in U(S)$ and that $V_0 v \subseteq V$. By the preceding, every element in the open set $W = V_1 c([0,1])$ is $\varkappa'$-related to some element in $V_0 v \subseteq V$. However, W contains $s = c(1)$ and thus is the required open neighborhood of s. □

The preceding results have global consequences for the relation of the $\varkappa$-classes and the cosets $sN \cap S$ for the analytic subgroup N generated by $\exp I$.

16. Proposition. *With the notation of the* Foliation Lemma *write N for the analytic subgroup of G generated by* $\exp I$ *and N_L for the group N with its intrinsic Lie group topology. Denote with i the corresponding immersion $N_L \to G$ mapping N_L onto N. Then for any element $s \in S$ the set*

$$C_s = \{c \in N_L : i(c)\, s \in \varkappa(s)\} = i^{-1}(\varkappa(s)\, s^{-1})$$

is an open closed neighborhood of $\mathbf{1}$ *in the subspace*

$$D_s = \{d \in N_L : i(d)\, s \in S\} = i^{-1}(S s^{-1})$$

of N_L. In particular, the identity component $(D_s)_0$ of D_s is contained in C_s.

Proof. Since $\varkappa(s)\, s^{-1}$ is closed in $S s^{-1}$ and i is continuous we see that C_s is closed in D_s. Next we show that C_s is open in D_s. Suppose that $c \in C_s$. Then $i(c)\, s \in \varkappa(s)$. Let U be an open set in G as it was constructed in the Foliation Lemma. Now $i(c)\, s\, S^{-1}$ is an open neighborhood of $\mathbf{1}$ in G, hence $U \cap S \cap i(c)\, s\, S^{-1} \neq \emptyset$. Thus we can write $i(c)\, s = u s'$ with suitable elements $u \in U$, $s' \in S$. By the Foliation Lemma, there is an open neighborhood B_u of 0 in I such that $(\exp B_u)\, u$ is a neighborhood of u in $\varkappa(u)$. It follows that $((\exp B_u)\, i(c))\, s = (\exp B_u)\, u s' \subseteq \varkappa(u)\, s' \subseteq \varkappa(u s') = \varkappa(s)$. This implies that the open neighborhood $(i^{-1}(\exp B_u))\, c$ of c in N_L is contained in C_s. Thus C_s is open as asserted. □

If N is the analytic subgroup generated by I, then the maximal integral manifolds for the distribution determined by I are the cosets Ng. Notice that one and the same leaf Ng of the foliation of G may — and will in general — intersect $U \cap S$ in countably many leaves (cf. Example 28). This pathology does not occur when N is closed; it thus cannot occur if the fundamental group $\pi_1(G)$ of G is finite. It is therefore important to observe that for the local study of congruence relations with closed classes on an open semigroup in a Lie group G the assumption that G be simply connected is no loss of generality:

17. Proposition. *Let G be a connected Lie group, S an open subsemigroup with $\mathbf{1} \in \bar{S}$ and $\varkappa$ a congruence relation with closed classes on S. Furthermore, let U be a neighborhood and I an ideal associated with $\varkappa$, as constructed in the* Foliation Lemma (Lemma 12). *Let $p: \tilde{G} \to G$ be the universal covering and $S^* = p^{-1}(S)$. Then the following conclusions hold:*

(i) *S^* is an open subsemigroup in the simply connected Lie group $\tilde{G}$ with $\mathbf{1} \in \bar{S}^*$ and $p \mid S^* : S^* \to S$ is a covering morphism.*

(ii) *If we set $\varkappa^* = \{(x, y) \in \tilde{G} \times \tilde{G} : p(x)\}$, then $\varkappa^*$ is a congruence relation on S^* with closed classes mapping onto the $\varkappa$-classes of S under $p \mid S^*$.*

(iii) *If we identify $\mathbf{L}(\tilde{G})$ with $\mathbf{L}(G)$ in such a fashion that $p \circ \exp_{\tilde{G}} = \exp_G$, then $U^* = p^{-1}(U)$ and I satisfy the conclusions of the* Foliation Lemma *for the congruence $\varkappa^*$ on the semigroup S^* in $\tilde{G}$.*

(iv) *If N is the analytic subgroup generated by $\exp_{\tilde{G}} I$ then N is closed, and the set of connected components of the submanifolds $uN \cap U^*$ as u ranges through $U^* \cap S^*$ agrees with the set of connected components of the submanifolds $\varkappa^*(u)$.*

Proof. (i) Clearly S^* is an open subsemigroup of $\tilde{G}$. Let W be an evenly covered cell neighborhood of $\mathbf{1}$ in G and W^* the component of $\mathbf{1}$ in $p^{-1}(W)$. Then $p \mid W^* : W^* \to W$ is a homeomorphism, and if $s_n \in W \cup S$ is a sequence converging to $\mathbf{1}$ in G, then $(p \mid W^*)^{-1}(s_n)$ is a sequence in $W^* \cup S^*$ converging to $\mathbf{1}$ in $\tilde{G}$. Hence $\mathbf{1} \in \bar{S}^*$. It is clear that $p \mid S^* : S^* \to S$ is a covering morphism.

(ii) is straightforward.

(iii) For $s \in S^*$ we write $K_s^* = \varkappa^*(s)\, s^{-1}$ and note

$$p(K_s) = \varkappa(p(s))\, p(s)^{-1} = K_{p(s)}\,.$$

Because of our identification of $\mathbf{L}(G)$ with the Lie algebra of $\tilde{G}$ the set $\mathbf{L}[p(s)] = L_0(\exp_G^{-1} K_{p(s)})$ of subtangent vectors of $K_{p(s)}$ at the origin agrees with the set $\mathbf{L}[s] = L_0(\exp_{\tilde{G}}^{-1} K_s)$ of subtangent vectors of K_s at the origin. Hence, since $S = p(S^*)$, we have $\bigcap_{s \in S^*} \mathbf{L}[s] = \bigcap_{t \in S} \mathbf{L}[t] = I$. This shows that I is the ideal constructed in the Foliation Lemma for $\tilde{G}$ and S^*. Moreover, if $u \in U^*$ then $p(u) \in U$ and thus $I = \mathbf{L}[p(u)] = \mathbf{L}[u]$ and if B_u is an open neighborhood of 0 in I such that $(\exp_G B_u)(u)$ is a neighborhood of $p(u)$ in $\varkappa(p(u))$, then $(\exp_{\tilde{G}} B_u)\, u$ is also a neighborhood of u in $\varkappa^*(u) = p^{-1}\varkappa((p(u))$ since p is a local isomorphism.

(iv) Since $\tilde{G}$ is simply connected, every normal analytic subgroup is closed. The remainder then follows from an application of the Foliation Lemma to $\tilde{G}$ and S^*. □

In view of the above proposition the closedness of N is not a very stringent condition.

18. Corollary. *Let $G, S, \varkappa$ and N be as in* Proposition 16. *Then for each $s \in S$ the set $sN \cap \varkappa(s)$ is open and closed in sN. In particular, the connected component of s in the submanifold $sN \cap S$ of sN is contained in $\varkappa(s)$.*

Proof. If N is closed then we have $N_L = N$, and Proposition 16 says that $Ns \cap \varkappa(s)$ is open and closed in sN.

19. Corollary. *If $sN \cap S$ is connected, then $sN \cap S \subseteq \varkappa(s)$.*

The Foliation Theorem

In the present section we strengthen our basic foliation result in Lemma 12 by adding more information about the local product structure. The results thus obtained will be summarized as the Foliation Theorem and the Local Factorization Theorem.

We retain the notation of the preceding section. Let us start with a lemma about the existence of local cross sections.

20. Lemma. *Let $u \in (U \cap S) S^{-1}$ and E a vector space complement of I in $\mathbf{L}(G)$. Then there is an open neighborhood A_u of 0 in E such that $C \stackrel{\text{def}}{=} (\exp A_u) u$ has the property that $c_1, c_2 \in C \cap U \cap S$ and $c_1 \sim_\varkappa c_2$ imply $c_1 = c_2$. In other words $C \cap U \cap S$ is a local cross section for the $\varkappa$-classes. (Note that $C \cap U \cap S$ may be empty if $u \notin U \cap S$!)*

Proof. Let $A_u \subseteq E$ and $B_u \subseteq I$ be neighborhoods of 0 such that the map $A_u \times B_u \to A_u * B_u$, $(X', X'') \mapsto X' * X''$, is a diffeomorphism onto a neighborhood of 0 in $\mathbf{L}(G)$, just as in the proof of Lemma 11. We claim that if A_u is chosen sufficiently small, then for two elements $X, Y \in A_u$ the relations $\{(\exp X) u, (\exp Y) u\} \subset U \cap S$ and $(\exp X) u \sim_\varkappa (\exp Y) u$ always imply $X = Y$.

Suppose that this claim is false. Then for any natural number n we find a neighborhood $A_u^{(n)}$ of 0 in A_u whose diameter is at most $1/n$ such that $A_u^{(n)}$ contains two different elements X_n and Y_n with $(\exp X_n) u$, $(\exp Y_n) u \in U \cap S$ and $(\exp X_n) u \sim_\varkappa (\exp Y_n) u$. Since $0 \neq P_n = Y_n - X_n \in E$, we may, after passing to a subsequence, assume that the sequence $Z_n \stackrel{\text{def}}{=} \| P_n \|^{-1} \cdot P_n$ converges to a unit vector Z in E:

$$Z = \lim Z_n \in E \,. \tag{22}$$

Since $u \in (U \cap S) S^{-1}$ there exist elements $v \in U \cap S$ and $s \in S$ with $u = v s^{-1}$. Then u is in the open set $v S^{-1}$, hence so is $(\exp X_n) u$ for all large enough n. We may and shall assume that this is the case for all $n \in \mathbb{N}$. Thus there are elements $s_n \in S$ with $(\exp X_n) u s_n = v$; note that

$$\lim (\exp X_n) u s_n = v = \lim u s_n = \lim (\exp Y_n) u s_n \text{ and } \lim s_n = s, \tag{23}$$

since $\lim \exp X_n = \lim \exp Y_n = \mathbf{1}$. Moreover, $(\exp X_n) u \sim_\varkappa (\exp Y_n) u$, hence $(\exp Y_n) u s_n \sim_\varkappa v$ for all $n \in \mathbb{N}$. Now the Foliation Lemma applies and shows that we can find elements $X_n'' \in B_u \subseteq I$ such that $(\exp Y_n) u s_n = (\exp X_n'') v = (\exp X_n'') (\exp X_n) u s_n$ for all sufficiently large n. Thus

$$\exp (Y_n * (-X_n)) = (\exp Y_n) (\exp X_n)^{-1} = \exp X_n''$$

and hence

$$Y_n * (-X_n) = X_n'' \in I \text{ for all large enough } n. \tag{24}$$

Now we recall from general Lie theory that for all sufficiently small X, P in $\mathbf{L}(G)$ we have

$$(X + P) * (- X) = f(- X)(P) + o(P) \text{ with}$$

$$f(- z) = (e^z - 1)/z = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

(cf., e.g., [1], Chap. II, § 6, n° 5, Proposition 5.) Thus

$$\| P_n \|^{-1} \cdot (Y_n * (- X_n)) = \| P_n \|^{-1} \cdot (X_n + P_n) * X_n =$$
$$= f(- X_n)(\| P_n \|^{-1} \cdot P_n) + O(P_n)$$

with $\lim O(P_n) = 0$, that is,

$$\| P_n \|^{-1} \cdot (Y_n * (- X_n)) = f(- X_n)(Z_n) + O(P_n) \tag{25}$$

for all sufficiently large n. Since $\lim X_n = 0$ we have $\lim f(- X_n) = \mathbf{1}$ and thus (22) and (25) imply

$$\lim \| P_n \|^{-1} \cdot (Y_n * (- X_n)) = Z \in E . \tag{26}$$

But (24) implies

$$Z = \lim \| P_n \|^{-1} \cdot (Y_n * (- X_n)) \in I .$$

Since $E \cap I = \{0\}$ we conclude $Z = 0$, a contradiction to the fact that Z is a unit vector. □

Remark. Suppose that U is an open neighborhood of $\mathbf{1}$ in G. If $u \in U \cap S$, then $u^{-1}(U \cap S)$ is an open neighborhood of $\mathbf{1}$ in G, hence contains an element $s \in S$. This means that $s = u^{-1} v$ for some $v \in$ $\in U \cap S$ and thus $u = v s^{-1} \in (U \cap S) S^{-1} = \{g \in G : (\exists\, u)\, u \in U \cap S$ and $g \prec u\}$. Also, $1 = u u^{-1} \in (U \cap S) S^{-1}$. It is not clear that $\overline{U \cap S} \subseteq$ $\subseteq (U \cap S) S^{-1}$. But in this direction we have at least the inclusion

$$(U \cap S) \cap \overline{V \cap S} \subseteq (U \cap S) S^{-1} ,$$

where V is any compact subset of U. (If V is a compact subset of U then there is an open neighborhood W of $\mathbf{1}$ in G such that $V W \subset U$. Then for any $v \in \overline{V \cap S} = V \cap \bar{S}$ and any $\omega \in W \cap S$ we have $v w \in S$ (since S is an ideal of the semigroup $\bar{S}$), hence $v \in (S \cap U)(W \cap S)^{-1} \subseteq (S \cap U) S^{-1}$.)

Thus the above Lemma 20 tells us something about cross sections through points in the boundary of $U' \cap S$, where U' is a suitable (precompact) neighborhood of $\mathbf{1}$ in G.

Now we have the following main result:

21. Theorem. (The Foliation Theorem) *Suppose that G is a finite dimensional Lie group and S an open subsemigroup of G with $\mathbf{1} \in \bar{S}$. Let $\varkappa$ be a congruence with closed classes on S. Then there exists an ideal I of $\mathbf{L}(G)$ and an open neighborhood U of $\mathbf{1}$ in G such that the following assertions hold: For each $u \in U \cap S$ and for each vector space complement E for I in $\mathbf{L}(G)$ there are open neighborhoods A_u of 0 in E and B_u of 0 in I such that*

(i) *the map*

$$\Phi_u\colon A_u \times B_u \to U \text{ given by } \Phi_u(X, Y) = (\exp X)(\exp Y)\, u \tag{27}$$

is a diffeomorphism mapping $A_u \times B_u$ onto an open neighborhood V of u in $U \cap S$, and

(ii) *for all $X \in A_u$ we have*

$$\Phi_u(\{X\} \times B_u) = \varkappa(\Phi_u(X, 0)) \cap V\,. \tag{28}$$

Under the hypotheses of the above Foliation Theorem, one also obtains information for points u which are not necessarily in $S \cap U$ but in the boundary of $S \cap U$. Indeed, we have the following information:

22. Corollary. *In the notation of* Theorem 21, *we may choose I, U, and E so that for the elements $u \in (U \cap S)\, S^{-1} \setminus U \cap S$ we have: For all $X \in A_u$ and $Y \in B_u$ with $\Phi_u(X, Y) \in S$, the diffeomorphism Φ_u maps the connected component of (X, Y) in $(\{X\} \times B_u) \cap \Phi_u^{-1}(S)$ onto the component of $\Phi_u(X, Y)$ in $\varkappa(\Phi_u(X, Y)) \cap V \cap \bar{S}$.*

Proof. The proof of Theorem 21 is organized in such a fashion that a proof of the associated Corollary 22 can be deduced readily. Thus we shall not assume *a priori* that $u \in U \cap S$ but mark carefully every occasion when this assumption is used.

We choose A_u and B_u so small that A_u satisfies the hypotheses of Lemma 20, that $(\exp(A_u * B_u))\, u \subseteq U$, and that the function Φ_u defined in (27) is a diffeomorphism onto its image V. Furthermore we require that $V \subseteq U \cap S$, and that B_u is an open cell contained in a Campbell-Hausdorff neighborhood of 0.

Let $X \in A_u$. We show that the set

$$M_X \overset{\text{def}}{=} \{Y \in B_u\colon \Phi_u(X, Y) \in \varkappa(\Phi_u(X, 0))\}$$

is both open and closed in the (open) submanifold $\{Y' \in B_u: \Phi_u(X, Y') \in S\}$ of the open cell B_u.

First we observe that $\Phi_u(\{X\} \times M_X) = \Phi_u(\{X\} \times B_u) \cap \varkappa(\Phi_u(X, 0))$. Since $\varkappa$-classes are closed, this set is closed in $\Phi_u(\{X\} \times B_u) \cap S = \Phi_u((\{X\} \times B_u) \cap \Phi_u^{-1}(S)$. Since Φ_u is a diffeomorphism, this shows that M_X is closed in $\{Y' \in B_u: \Phi_u(X, Y') \in S\}$.

To show that M_X is open in B_u, pick $Y \in M_X$. Then $\Phi_u(X, 0) \sim_\varkappa \Phi_u(X, Y)$, by the definition of M_X. Applying the Foliation Lemma 12, we find an open zero neighborhood B' in I such that $(\exp B')\, \Phi_u(X, Y)$ is a neighborhood of $\Phi_u(X, Y)$ in $\varkappa(\Phi_u(X, Y))$. Consider an element $Z \in B'$. Then, on one hand, $(\exp Z)\, \Phi_u(X, Y) \sim_\varkappa \sim_\varkappa \Phi_u(X, Y)$. Since $\Phi_u(X, Y) \sim_\varkappa \Phi_u(X, 0)$ this implies

$$(\exp Z)\, \Phi_u(X, Y) \sim_\varkappa \Phi_u(X, 0). \tag{29}$$

On the other hand, if $Z \in \mathbf{L}(G)$ is not too large then

$$\begin{aligned}(\exp Z)\, \Phi_u(X, Y) &= (\exp Z)(\exp X)(\exp Y)\, u = \\ &= (\exp X)\{(\exp -X)(\exp Z)(\exp X)\}(\exp Y)\, u = \\ &= (\exp X)(\exp e^{-\operatorname{ad} X} Z)(\exp Y)\, u = \\ &= (\exp X)(\exp (e^{-\operatorname{ad} X} Z * Y))\, u.\end{aligned} \tag{30}$$

Since $Z \in I$ and I is an ideal, we know that $e^{-\operatorname{ad} X} Z \in I$ for all X. By our choice of B_u the Campbell-Hausdorff product $Z * Y$ is defined for all $Z \in B'$; moreover, $Z * Y \in I$. Thus, for all sufficiently small $Z \in B'$, we find

$$(\exp Z)\, \Phi_u(X, Y) = \Phi_u(X, e^{-\operatorname{ad} X} Z * Y) \in \Phi_u(\{X\} \times B_u). \tag{31}$$

In view of (29) and the definition of M_X this means that $e^{-\operatorname{ad} X} Z * Y \in M_X$ and it follows that an entire neighborhood of Y in I is contained in M_X. This shows that M_X is also open in B_u.

Thus $\{X\} \times M_X$ is open and closed in the open submanifold $(\{X\} \times B_u) \cap \Phi_u^{-1}(S)$ of the open cell $\{X\} \times B_u$, and hence contains the connected component of (X, Y) in $(\{X\} \times B_u) \cap \Phi_u^{-1}(S)$. Since $\Phi_u(\{X\} \times B_u) \subseteq S$ and since B_u is connected, we have $M_X = B_u$. Thus the left side of (28) is contained in the right side.

In order to prove the converse, assume that $v \in \varkappa(\Phi_u(X, 0)) \cap V$. We write v in the form $v = \Phi_u(X'', Y'')$ with $X'' \in A_u$ and $Y'' \in B_u$. By what we have just seen, $v \sim_\varkappa \Phi(X'', 0)$. By the transitivity of $\varkappa$ we now

know $\Phi_u(X,0) \sim_\varkappa \Phi_u(X'',0)$. By Lemma 13, in view of our choice of A_u, this relation implies $\Phi_u(X,0) = \Phi_u(X'',0)$, and since Φ_u is a diffeomorphism, $X = X''$. Thus $v = \Phi_u(X, Y'') \in \Phi_u(\{X\} \times B_u)$. This completes the proof of Theorem 21.

For a proof of Corollary 22, we note that the connected component of $\Phi_u(X, Y)$ in $\varkappa(\Phi_u(X, Y)) \cap V$ is the maximal integral manifold through $\Phi_u(X, Y)$ of the distribution given by I on the manifold $V \cap S$. But in V, the maximal integral manifold through $\Phi_u(X, Y)$ in V is $\Phi_u(\{X\} \times B_u)$ as (31) shows. Hence this component is contained in $\Phi_u(\{X\} \times M_X)$, and thus in the image under Φ_u of the component of $(\{X\} \times B_u) \cap \Phi_u^{-1}(S)$. □

Let us remark, that the stronger assumption $u \in U \cap S$ of the theorem yields the much stronger conclusion (28). The conclusion of the Corollary is actually only an assertion about $\varkappa'$.

In the situation of the Foliation Theorem we now assume, in addition, that the analytic subgroup N generated by $\exp I$ is closed. Let $\varkappa_N$ denote the congruence on S whose cosets are $\varkappa_N(s) = sN \cap S$. From Lemma 14 we know that for *the open set U of the* Foliation Theorem *we have* $\varkappa' = \varkappa'_N$ *with* $\varkappa' = (\varkappa \mid U \cap S)_0$ *and* $k'_N = (\varkappa_N \mid U \cap S)_0$.

23. Proposition. *Let G be a Lie group, N a closed connected normal subgroup and S an open semigroup with $\mathbf{1} \in \bar{S}$. Let U be an open neighborhood of* **1**. *Then the function sending the component of u in $uN \cap U \cap S$ to uN is a local homeomorphism $\pi: (U \cap S)/\varkappa'_N \to$ $\to (U \cap S)N/N$.*

Proof. Let $u \in U \cap S$ and let E be a vector space complement for I in $\mathbf{L}(G)$. Choose an open neighborhood A_u of 0 in E and an open neighborhood B_u of 0 in I such that $\Phi_u: A_u \times B_u \to V_u$ is a diffeomorphism with the properties given in the Foliation Theorem. Assume, in addition, that $\exp A_u$ is a cross section for the orbits $N(\exp X)$ with $X \in A_u$. (The existence of such neighborhoods is secured by the fact that N is closed.) Now we apply Lemma 15 to $\varkappa_N$ and conclude that the saturation $\varkappa'_N(V_u)$ is open, whence $W_u \overset{\text{def}}{=} \varkappa'_N(V_u)/\varkappa'_N$ is an open neighborhood of $\varkappa'_N(u) \in (U \cap S)/\varkappa'_N$. The function $\alpha = \{X \mapsto$ $\mapsto \varkappa'_N((\exp X)u)\}: A_u \to W_u$ is bijective and continuous, and the function $\beta = \{X \mapsto ((\exp X)uN)\}: A_u \to V_u N/N$ is bijective, continuous and open since $(\exp A_u)u$ is a cross section for the N-orbits. Furthermore,

$\beta = (\pi/W_u) \circ \alpha$. Hence $(\pi \mid W_u)$ is bijective and open. But it is also continuous since it is the canonical map arising from the factorisation of the continuous function $v \mapsto v N : \varkappa'_N(V_u) \to V_u N/N$ through the quotient modulo its kernel relation. Hence $\pi \mid W_u : W_u \to V_u N/N$ is a homeomorphism. We have shown that some open neighborhood of every point in $(U \cap S)/\varkappa'_N$ is mapped homeomorphically onto some open neighborhood of its image, and this is the assertion of the proposition. □

We remark that the previous proposition does not refer to a previously given congruence $\varkappa$ on S, but to a closed connected normal Lie subgroup N of G and the geometric structure of $U \cap S$ via $\varkappa'_N$. The proposition shows, in particular, that $(U \cap S)\varkappa_N$ is a manifold, although in general not a Hausdorff manifold. The remark preceding the proposition shows that this gives information on the space $(U \cap S)/\varkappa' = (U \cap S)/\varkappa_N$ when N and U arise as in the Foliation Theorem.

Let us inspect the proof of the previous Proposition 23 for the case that N and U do arise as in the Foliation Theorem. Then we also have a function $\gamma = \{X \mapsto \varkappa((\exp X) u)\} : A_u \to \varkappa(W_u)/\varkappa$. By the Foliation Theorem, γ is bijective and continuous. Unlike β, the map γ will not be open in general. If we define $\pi' : (U \cap S)/\varkappa' \to \varkappa(U \cap S)/\varkappa$ by $\pi'(\varkappa'(u)) = \varkappa(u)$ (unambiguously!), then $\gamma = (\pi' \mid W_u) \circ \alpha$. This shows that $\pi' \mid W_u$ is bijective; and it is continuous for the same reason for which $\pi \mid W_u$ was continuous. But it is not open in general and $\varkappa(W_u)/\varkappa$ in general fails to be a neighborhood of $\varkappa(u)$ in $S/\varkappa$.

However, the function $(\pi' \mid W_u) \circ (\pi \mid W_u)^{-1} : V_u N/N \to \varkappa(W_u)/\varkappa$ is a continuous bijection from $V_u N/N$ onto $\varkappa(V_u)/\varkappa$.

This means that we have proved the following theorem:

24. Theorem. (The Local Factorisation Theorem) *Let G be a Lie group with an open subsemigroup S containing $\mathbf{1}$ in its closure. Let I be the ideal associated with $\varkappa$ by the* Foliation Lemma *and assume that the analytic subgroup N generated by I is closed. Then there is an open neighborhood U of $\mathbf{1}$ such that every $u \in U \cap S$ has an open neighborhood V_u such that the restriction of the quotient map $s \mapsto \varkappa(s) : S \to S/\varkappa$ to V_u factors through the continuous open map $s \mapsto N s : V_s \to V_s N/N$ and a well-defined continuous bijection $v N \mapsto \varkappa(v) : V_u N/N \to \varkappa(V_s)/\varkappa$ where*

$\varkappa(V_u) = \bigcup_{v \in V_u} \varkappa(v)$ is the $\varkappa$-saturation of V_u. If all cosets $\varkappa(v)$ for $v \in V_u$ are connected, then $\varkappa(V_u)$ is open in S and $\varkappa(V_u)/\varkappa$ is an open neighborhood of $\varkappa(u)$ in $S/\varkappa$.

We remark that the saturation $\varkappa(V_u)$ in general fails to be open in S.

The Foliation Theorem also enables us to find a canonical form for the open set $S \cap U$ which has played such a crucial role from the Foliation Lemma onward.

25. *Definition.* Let G be a Lie group, S an open subsemigroup with $\mathbf{1} \in \bar{S}$, and let I be the ideal constructed in the Local Foliation Theorem. We fix once and for all a vector space complement E for I in $\mathbf{L}(G)$. Then we define *the stratified domain* $U(S)$ to be the set of all points $s \in S$ such that for suitable open cell neighborhoods A_s of 0 in E and B_s in I the function $\Phi_s = ((X, Y) \mapsto (\exp X)(\exp Y)\, s)$ maps $A_s \times B_s$ diffeomorphically onto an open neighborhood V of s such that for each $v = \Phi(X, Y) \in V$ the intersection of the congruence class $\varkappa(v)$ with V is exactly $\Phi_s(\{X\} \times B_s)$. The function Φ_u is called a *local canonical coordinate system at u.*

The proof of the next Proposition is omitted. (It is found in [3].)

26. Proposition. *$U(S)$ is an open subset of S and there are open neighborhoods U of $\mathbf{1}$ in G such that $U \cap S = U(S)$.*

27. *Remark.* In all results such as the Foliation Lemma, the Foliation Theorem and the Local Factorisation Theorem we may replace the open set $U \cap S$ of reference by the stratified domain $U(S)$.

Examples of Congruences on Open Subsemigroups of Lie Groups

28. *Example.* (Congruences on subsemigroups of the plane $G = \mathbb{R}^2$) In the following set of examples all semigroups S occurring will be subsemigroups of the additive group $\mathbb{R}^2$ and all congruences will be subordinated to the subgroup $N = \mathbb{R} \times \{0\}$; that is, are contained in the set $\{(u, v) \in G \times G \mid u - v \in N\}$.

Definition. We define congruences $\varkappa_N$, $\varkappa_1$, $\varkappa_2$, and $\varkappa_\varepsilon$, for $\varepsilon > 0$, on each of the semigroups S defined in the sequel. Let (a, b), (a', b') be two elements in S, say $a < a'$. Then (a, b), (a', b') in S are said to be congruent with respect to

(i) $\varkappa_N$ if $b = b'$;

(ii) $\varkappa_1$ if $b = b'$ and $[a, a'] \times \{b\} \subset S$;

(iii) $\varkappa_2$ if $b = b'$ and $[a, a'] \times \{b\} \subset \bar{S}$;

(iv) $\varkappa_\varepsilon$ if $b = b'$ and $[a, a'] \times \{b\} \subset S + ([-\varepsilon, \varepsilon] \times \{0\})$.

Note that $\varkappa_2$ is the intersection of the congruences $\varkappa_\varepsilon$ with $\varepsilon > 0$. The congruence classes with respect to these congruences are always closed (but the congruence relations themselves are not always closed).

(ai) S is the interior of the set $\mathbb{R}^+ \times \mathbb{R}^+ \setminus (\omega \times \{0\}) + D$, where D is the closed disc $D = \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + y^2 \leqslant \frac{1}{4}\}$ and ω denotes the set $\{0, 1, 2, \ldots\}$. In this case, the congruence $\varkappa_2$ is closed in $S \times S$ and $S/\varkappa_2$ is a Hausdorff k-space, but not locally compact. (In fact, $S/\varkappa_2$ is locally compact at all points except the $\varkappa$-class corresponding to $(0, 1)$.) The space $S/\varkappa_2$ can be thought of as the union of $\mathbb{R}^+ \setminus \{0\}$ with countably many copies of the interval $]0, 1]$ which meet at 1. Note that $\varkappa_1$ is not closed in $S \times S$.

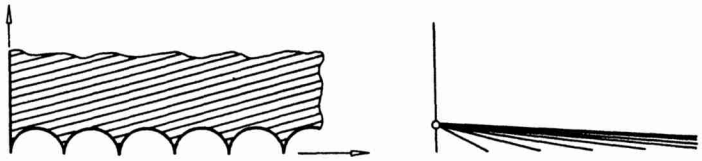

Fig. 1

(aii) Replacing in (ai) the set ω by some finite cardinal $\{0, 1, 2, \ldots, k\}$ we get an open subsemigroup S of $\mathbb{R} \times \mathbb{R}$ such that $S/\varkappa_2$ is locally compact; the quotient space $S/\varkappa_2$ is the union of finitely many copies of $]0, 1]$, meeting at a point.

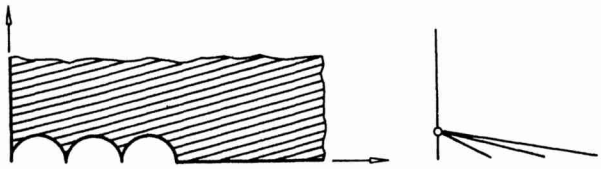

Fig. 2

(aiii) Still more sophisticated examples can be obtained by subtracting discs of variable diameter (or isoscele triangles, or other

fancy shapes — as long as the difference set is a semigroup). For instance, we may define S as the interior of the set $\mathbb{R}^+ \times \mathbb{R}^+ \setminus \bigcup \{D_n \mid n \in \omega\}$; where

$$D_n = \{(x,y) \in \mathbb{R}^2 \mid (x - 2 + 3(\tfrac{1}{2})^n)^2 + y^2 < (\tfrac{1}{2})^{2n}\} .$$

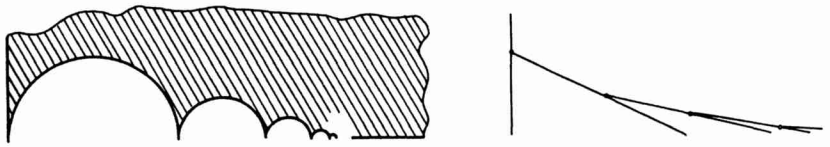

Fig. 3

(b) The next set of examples provides congruences where in every neighborhood U of $(0,0)$ there is an element s of S such that $\varkappa(s) \cap U$ is disconnected.

(bi) For every $n \in \mathbb{N}$ define $S_n = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid 1/n < nx < y\}$. Then each S_n is a semigroup and if $m < n$ then $S_m + S_n \subset S_m$. (Indeed, let $(x,y) \in S_m$, $(x',y') \in S_n$. Then $1/m < m(x + x') < mx + nx' < y + y'$, hence $(x + x', y + y') \in S_m$.) Thus the union $S \overset{\text{def}}{=} \bigcup \{S_n \mid n \in \mathbb{N}\}$ is an open semigroup; obviously, $(0,0) \in \bar{S}$ (for instance, $(0,0) = \lim\left(\frac{n+1}{n^3}, \frac{1}{n}\right)$.)

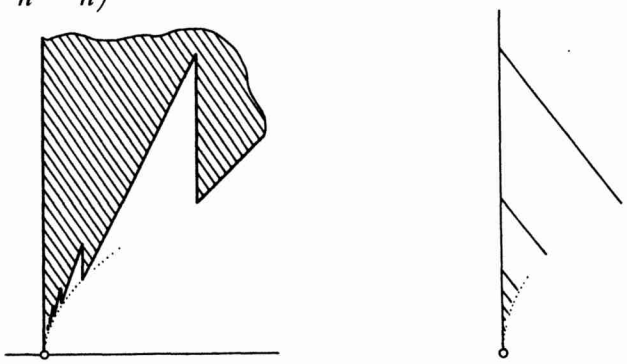

Fig. 4

(bii) We now modify the example in (bi) so that near the identity of the resulting semigroup every $\varkappa_N$-class has infinitely many connected components. Also, the closure $\bar{S}$ of S will contain the y-axis, but $\bar{S}$ will be locally disconnected at every point $(0,y) \in \bar{S}$. In fact,

every connected subset of $\bar{S}$ which meets the y-axis either contains $\{0\} \times \mathbb{R}^+ \cap \bar{S}$ or is contained in $\{0\} \times \mathbb{R}^+ \cap \bar{S}$.

We define for every $n \in \mathbb{N}$ the semigroup $S_n \overset{\text{def}}{=} \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : 2^{-n} < 2^{n^2} x < y\}$. As in Example (bi) above, we have $S_m + S_n \subseteq \subseteq S_m$ whenever $m < n$. Thus

$$S \overset{\text{def}}{=} \bigcup \{S_n : n \in \mathbb{N}\}$$

is an open subsemigroup of $\mathbb{R}^2$ and $\bar{S}$ contains the identity.

(c) (An example with discrete $\varkappa$-classes) Let S be the half-plane $\mathbb{R}_+ \times \mathbb{R}$. Then we define

$$(a, b) \sim_\varkappa (a', b') \text{ if for some } n \in \mathbb{N} \; a = a' \geqslant \frac{1}{2^n} \text{ and } b \in b' + \mathbb{Z}\, 2^n .$$

29. *Example.* In our next example the analytic normal subgroup N, constructed according to the Foliation Theorem, is *not* closed. Let us start with simply connected objects. In the additive group $\mathbb{R}^3$ write S_0 for the pointed cone $\{(x, y, z) \in \mathbb{R}^3 \mid 0 < y < y + z < x\}$, and N_0 for the closed one-parameter group $\mathbb{R} \cdot (0, 1, \sqrt{2})$.

Now let H be the discrete subgroup $\{(0, y, z) \mid \{y, z\} \subset \mathbb{Z}\}$, and write G for the quotient group G_0/H. Then the image S of S_0 under the quotient map $q: G_0 \to G_0/H = G$ is an open subsemigroup of G and $\mathbf{1} \in \bar{S}$. We define $\varkappa$ to be the congruence whose congruence class at a point $v = (x, y, z)$ is given by:

$$\varkappa(q(v)) = \begin{cases} q((v + N_0) \cap S_0), & \text{if } 0 < x < 1, \\ q(\{x\} \times \mathbb{R} \times \mathbb{R}) \cap S, & \text{if } 1 \leqslant x. \end{cases}$$

It is not difficult to see that $\varkappa$ is a closed congruence on S and that $N = q(N_0)$; obviously, N is not closed in G.

Added in Proof

The theory presented in this paper can be extended so that it covers one-sided congruences as well. Once this is established, it is possible to show that a two-sided congruence creates foliations not only near the identity, but also in the vicinity of an arbitrary point of the semigroup. For details see [3], Chapter V, Section 7.

References

[1] BOURBAKI, N.: Groupes et algèbres de Lie, Chap. I–VIII. Paris: Hermann. 1971—1975.

[2] CARRUTH, J. H., HILDEBRANDT, J. A., KOCH, R. J.: The Theory of Topological Semigroups, Vol. I and II. New York: Marcel Dekker. 1983 and 1986.

[3] HILGERT, J., HOFMANN, K. H., LAWSON, J. D.: Lie Groups, Convex Cones, and Semigroups. Oxford: University Press. 1989.

[4] HOFMANN, K. H.: Topologische Halbgruppen mit dichter submonogener Unterhalbgruppe. Math. Z. **74**, 232—276 (1960).

[5] HOFMANN, K. H., MOSTERT, P. S.: Elements of Compact Semigroups. Columbus, Ohio: Merrill Books. 1966.

[6] KEIMEL, K.: Eine Exponentialfunktion für kompakte abelsche Halbgruppen. Math. Z. **97**, 7—25 (1967).

[7] KEIMEL, K.: Lokal kompakte Kegelhalbgruppen und deren Einbettung in topologische Vektorräume. Math. Z. **99**, 405—428 (1967).

[8] KEIMEL, K.: A cross section theorem for compact abelian semigroups. Semigroup Forum **1**, 254—259 (1970).

[9] KEIMEL, K.: Congruence relations on cone semigroups. Semigroup Forum **2**, 130—147 (1971).

[10] LAWSON, J. D., MADISON, B.: Quotients of k-semigroups. Semigroup Forum **9**, 1—18 (1974).

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