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On Injective Rational Coefficient Systems

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Abstract. Let G be a finite group. By a rational coefficient system for G we mean a contravariant functor from the category of canonical orbits of G and G-maps into the category of $\mathbb{Q}$-vector spaces. In this paper we study injective objects in the category of rational coefficient systems for G.

1. Introduction

Let G be a finite group. We shall denote by O_G the category of canonical orbits of G [1]. The objects of O_G are the orbit spaces G/H, H a subgroup of G, and the morphisms are the equivariant maps between them, where G acts on G/H by left multiplication. Thus any morphism $\hat{a}: G/H \to G/K$ in O_G is of the form $gH \mapsto gaK$ for some $a \in G$ with the property $a^{-1}Ha \leqslant K$ (here $\leqslant$ denotes inclusion of a subgroup). By a *rational coefficient system for G* we shall mean a contravariant functor from O_G into the category of $\mathbb{Q}$-vector spaces. The rational coefficient systems for G and the natural transformations between them form an abelian category which we shall denote by Vec_G. By the same arguments as in [1], the category Vec_G possesses sufficiently many projectives and injectives. In [6] there is given a complete description of projective covers in Vec_G. The aim of this paper is to study injective objects in Vec_G and to give a complete description of injective envelopes. Our attempt to give such a description has been motivated by problems from equivariant homotopy theory. Namely, in [3] we have given a sufficient condition for the equivariant splitting of a rational G-space into a product of Eilenberg—MacLaine G-spaces which involves, among other things, injective dimensions of rational coefficient systems determined by the homotopy groups of the space considered. Generally, there are spectral sequences in equivariant homotopy theory which involve Ext^* groups for rational coefficient systems [1], and it is sometimes

convenient to deal with minimal injective resultions in order to study these groups.

2. Injective Rational Coefficient Systems

Recall that the category Vec_G is abelian and a morphism $f: M \to N$ in Vec_G is a monomorphism (respectively, an epimorphism) if and only if $f(G/H)$: $M(G/H) \to N(G/H)$ is a monomorphism (respectively, an epimorphism) of $\mathbb{Q}$-vector spaces for each $H \leqslant G$ [4]. Given a G-set T, by $\mathbb{Q}(T^H)$ we shall denote the vector space over $\mathbb{Q}$ with basis T^H, where T^H is the H-fixed point set of T. The set $(G/K)^H$ has the usual left action of the group NH/H (NH denotes the normalizer of H in G). There is also the free right action of NK/K on $(G/K)^H$ [1]. Thus if V_H is a left $\mathbb{Q}(NH/H)$-module, then we can form a left $\mathbb{Q}(NK/K)$-module $\mathrm{Hom}_{\mathbb{Q}(NH/H)}(\mathbb{Q}((G/K)^H), V_H)$.

Definition 2.1. Define a *rational coefficient system* $I(V_H)$ by $I(V_H)(G/K) = \mathrm{Hom}_{\mathbb{Q}(NH/H)}(\mathbb{Q}((G/K)^H), V_H)$ on objects of O_G. Given a morphism $\hat{a}: G/K' \to G/K$ in O_G, define $I(V_H)(\hat{a})$: $I(V_H)(G/K) \to$ $\to I(V_H)(G/K')$ by $I(V_H)(\hat{a})(x) = x \circ \bar{a}$, where $x: \mathbb{Q}((G/K)^H) \to V_H$ and $\bar{a}: \mathbb{Q}((G/K')^H) \to \mathbb{Q}((G/K)^H)$ is induced by $\hat{a}$.

Proposition 2.2. *For any left $\mathbb{Q}(NH/H)$-module V_H the rational coefficient system $I(V_H)$ is injective in* Vec_G.

Proof. Let $0 \to A \xrightarrow{f} B$ be an exact sequence in Vec_G and $g: A \to I(V_H)$ be a morphism. We have to construct a morphism $h: B \to I(V_H)$ such that the diagram

$$\begin{tikzcd} 0 \to A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, "h"] \\ I(V_H) & \end{tikzcd}$$

commutes. Observe that $I(V_H)(G/H) \cong \mathrm{Hom}_{\mathbb{Q}(NH/H)}(\mathbb{Q}(NH/H), V_H) \cong$ $\cong V_H$ is an injective $\mathbb{Q}(NH/H)$-module since $\mathbb{Q}(NH/H)$ is semisimple. Thus, there exists an NH/H-map $h(G/H)$: $B(G/H) \to I(V_H)(G/H)$ such that the diagram

$$\begin{tikzcd} A(G/H) \arrow[r, "f(G/H)"] \arrow[d, "g(G/H)"'] & B(G/H) \arrow[dl, "h(G/H)"] \\ V_H & \end{tikzcd}$$

commutes. Now, we define $h(G/K)$: $B(G/K) \to I(V_H)(G/K)$ by $h(G/K)(x)(\hat{a}) = h(G/H) \circ B(\hat{a})(x)$, where $x \in B(G/K)$ and $\hat{a} \in (G/K)^H = \mathrm{Map}_G(G/H, G/K)$. It is easy to check that in this way we obtain the desired morphism $h: B \to I(V_H)$.

Proposition 2.3. *Every injective object M in* Vec_G *can be written as* $\bigoplus_{(H)} I(V_H)$, *where the sum is over all conjugacy classes of subgroups of G.*

Proof. In each conjugacy class (H) we fix just one subgroup H of G. The space $V_H = \bigcap_{H' < H} \mathrm{Ker}\, M(\hat{e}_{H',H})$, where $H' < H$ denotes $H' \leqslant H$ and $H' \neq H$, and $\hat{e}_{H',H}: G/H' \to G/H$ is the projection, is a $\mathbb{Q}(NH/H)$-submodule of $M(G/H)$. Since $\mathbb{Q}(NH/H)$ is semisimple, there is an NH/H-equivariant projection $p_H: M(G/H) \to V_H$. Put $V_{\{e\}} = M(G/\{e\})$, where $\{e\}$ denotes the trivial subgroup of G. Consider the morphism $\varphi: M \to \bigoplus_{(H)} I(V_H)$ defined by $\varphi(G/K)(x)(\hat{a}) = p_H \circ M(\hat{a})(x)$, where $x \in M(G/K)$ and $\hat{a} \in \mathrm{Map}_G(G/H, G/K) \cong (G/K)^H$. First we shall show that φ is a monomorphism. Let $0 \neq x \in M(G/K)$ and let H be a minimal subgroup of G such that $M(\hat{a})(x) \neq 0$ for a morphism $\hat{a}: G/H \to G/K$. Then $M(\hat{a})(x) \in V_H$. Thus $\varphi(G/K)(x)(\hat{a}) \neq 0$. Hence φ is a monomorphism.

Since M is injective, there is a morphism $f: \bigoplus_{(H)} I(V_H) \to M$ such that $f \circ \varphi = \mathrm{id}_M$. We shall show that f is also a monomorphism. It is clear that $f(G/\{e\})$ is an isomorphism. Suppose that $f(G/K')$ is a monomorphism for each $K' < K$. Consider $f(G/K)$. Let $x \in \mathrm{Ker}\, f(G/K)$. By the assumption, $x \in \mathrm{Ker}\,(\bigoplus_{(H)} I(V_H)(\hat{e}_{K',K}))$ for each $K' < K$. Thus, it follows that it is enough to consider only the case in which $x \in I(V_H)(G/K)$, $H < K$, and x is equal to zero outside possibly an only NH/H-orbit in $(G/K)^H$, say the orbit of some gK. Then we have $g^{-1}Hg < K$. Thus, by our assumption,

$$x(gK) = I(V_H)(\hat{e}_{g^{-1}Hg,K})(x)(g(g^{-1}Hg)) = 0.$$

Hence $x = 0$. This shows that $f(G/K)$ is a monomorphism. Thus, by an inductive argument, f is a monomorphism.

There is a simple necessary condition for a rational coefficient system M for G to be projective. Namely, each map $M(G/K) \to M(G/K')$ corresponding to a morphism $G/K' \to G/K$ of O_G must

be injective [6]. Since, in general case, a map $(G/K')^H \to (G/K)^H$ induced by a morphism of O_G need not be either surjective or injective, there is no similar global necessary condition for injectivity. Nevertheless, we can give some necessary condition for injectivity which, as we shall see later, is also sufficient in the case that G is cyclic of prime power order.

Let $\hat{a}: G/K' \to G/K$ be a morphism of O_G and $g \in aKa^{-1} \cap NK'$. Then the diagram

$$\begin{tikzcd} G/K' \arrow[r, "\hat{a}"] \arrow[d, "\hat{g}"'] & G/K \\ G/K' \arrow[ur, "\hat{a}"'] & \end{tikzcd}$$

commutes. Hence the image of the map $M(\hat{a}): M(G/K) \to M(G/K')$ is always contained in the fixed point space $M(G/K')^{(aKa^{-1} \cap NK')/K'}$.

Proposition 2.4. *A necessary condition that a rational coefficient system M for G be injective is that for any morphism $\hat{a}: G/K' \to \to G/K$ of O_G such that $aKa^{-1} \leqslant NK'$, the induced map $M(\hat{a})': M(G/K) \to M(G/K')^{(aKa^{-1})/K'}$ is surjective.*

Proof. By Proposition 2.3, it is enough to consider only the case in which $M = I(V_H)$ for a $\mathbb{Q}(NH/H)$-module V_H. Thus suppose that $\hat{a}: G/K' \to G/K$ is a G-map, $aKa^{-1} \leqslant NK'$, and $f' \in \in I(V_H)(G/K')^{(aKa^{-1})/K'}$. Then for each $k \in K$ and $gK' \in (G/K')^H$ the following holds: $f'(ga^{-1}K') = f'(ga^{-1}aka^{-1}K') = f'(gka^{-1}K')$. Thus the formula

$$f(gK) = \begin{cases} f'(ga^{-1}K') & \text{if } ga^{-1}K' \in (G/K')^H \\ 0 & \text{otherwise,} \end{cases}$$

defines an element f of $I(V_H)(G/K)$ such that $I(V_H)(\hat{a})(f) = f'$.

3. Injective Envelopes

Definition 3.1. An *injective envelope* of a rational coefficient system M is a monomorphism $\varphi: M \to I$, where I is injective in Vec_G, with the following property: For any injective I' in Vec_G and any monomorphism $\varphi': M \to I'$ there exists a monomorphism $f: I \to I'$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & I \arrow[d, "f"] \\ & I' \end{tikzcd}$$

commutes.

Proposition 3.2. *Let M be a rational coefficient system for G. Then the map $\varphi\colon M \to \bigoplus_{(H)} I(V_H)$ of section 2 is an injective envelope of M.*

Proof. We have seen that the map φ is a monomorphism for any rational coefficient system M for G.

Suppose that I' is injective in Vec_G and $\varphi'\colon M \to I'$ is a monomorphism. Then there is a morphism $f\colon I = \bigoplus_{(H)} I(V_H) \to I'$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi'"] \arrow[d, "\varphi"'] & I \arrow[dl, "f"] \\ I' & \end{tikzcd}$$

commutes. According to Proposition 2.3, we can write $I' = \bigoplus_{(H)} I(W_H)$. Moreover, we can choose an isomorphism $I' \cong \bigoplus_{(H)} I(W_H)$ in such a way that $\varphi'(V_H) \subset W_H$ for $V_H \subset M(G/H)$. Since $\varphi\colon M(G/H) \supset \supset V_H \xrightarrow{\cong} V_H \subset \bigoplus_{(H)} I(V_H)(G/H)$, we see that $f(G/H)$ maps V_H monomorphically into W_H. Thus $f(G/H)$ is a monomorphism for each subgroup H of G.

Remark 3.3. The existence and uniqueness up to isomorphism of an injective envelope of a rational coefficient system for G follows from general theorems of category theory (see [4]). Nevertheless, our approach gives a very simple construction.

Proposition 3.4. *Let $d(G)$ denote the largest integer n such that there is a sequence of proper inclusions of subgroups $\{e\} = G_0 < < G_1 < \ldots < G_n = G$. Then every rational coefficient system for G has an injective resolution of length $d(G)$.*

Proof. Let $M \xrightarrow{i_0} M^0 \xrightarrow{i_1} M^1 \to \ldots$ be an injective resolution of M such that M^0 is an injective envelope of M and M^k is an injective envelope of $\mathrm{Coker}(i_{k-1})$, $k \geqslant 1$. By our construction of an injec-

tive envelope, $M^0(G/\{e\}) = M(G/\{e\})$ and therefore $M^1(G/\{e\}) = 0$. Similarly, if K is a minimal proper subgroup of G then $\operatorname{Coker}(i_0(G/K)) = M^1(G/K)$, and therefore $M^2(G/K) = 0$. Continuing this we obtain that $M^k = 0$ for $k > d(G)$.

Remark 3.5. By Proposition 7.24 of [2], the statement of Proposition 3.4 is equivalent to the one of Lemma 3.4 of [5] which says that $\operatorname{Ext}^k(A, B) = 0$ for any rational coefficient systems for G, if $k > d(G)$.

4. Injective Rational Coefficient Systems for $G = \mathbb{Z}_{p^k}$

Let $\mathbb{Z}_{p^k}$ denote the cyclic group of order p^k, where p is prime and k is a positive integer. Throughout this section we shall assume that $G = \mathbb{Z}_{p^k}$. A rational coefficient system M for $\mathbb{Z}_{p^k}$ can be written as follows:

$$M_0 \xrightarrow{m_0} M_1 \to \ldots \to M_{k-1} \xrightarrow{m_{k-1}} M_k,$$

where $M_i = M(G/\mathbb{Z}_{p^{k-i}})$ and the map m_i is induced by the projection $G/\mathbb{Z}_{p^{k-i-1}} \to G/\mathbb{Z}_{p^{k-i}}$, $0 \leqslant i < k$.

Let us identify $\mathbb{Z}_{p^k}/\mathbb{Z}_{p^{k-i}} = \mathbb{Z}_{p^i}$. Then each M_i is a left $\mathbb{Q}(\mathbb{Z}_{p^i})$-module and it follows from section 2 that the image of m_i is contained in $(M_{i+1})^{\mathbb{Z}_p}$, where $\mathbb{Z}_p$ is the subgroup of $\mathbb{Z}_{p^{i+1}}$ generated by $[p^i]$.

Proposition 4.1. *A rational coefficient system M for $\mathbb{Z}_{p^k}$ is injective if and only if each map m_i': $M_i \to (M_{i+1})^{\mathbb{Z}_p}$ determined by m_i is surjective.*

Proof. By Proposition 2.4, we have only to prove that the above condition is sufficient. Thus, assume that each map m_i' is onto, $i = 0, \ldots, k-1$. Let us denote $V_i = \operatorname{Ker}(m_i)$. We shall show that $M = \bigoplus_{i=0}^{k} I(V_i)$. We have the monomorphism φ: $M \to \bigoplus_{i=0}^{k} I(V_i)$ of section 2, and it is easy to check that

$$I(V_i)(G/\mathbb{Z}_{p^{k-j}}) = \begin{cases} (V_i)^{\mathbb{Z}_{p^{i-j}}} & \text{if } i \geqslant j \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbb{Z}_{p^{i-j}}$ is the subgroup of $\mathbb{Z}_{p^i}$ generated by $[p^j]$. Under the identification $\bigoplus_{i=0}^{k} I(V_i)(G/\mathbb{Z}_{p^{k-j}}) = \bigoplus_{i=j}^{k} (V_i)^{\mathbb{Z}_{p^{i-j}}}$ the map φ is given by $\varphi_j(x) = \sum_{i=j}^{k} p_i \circ m_{ji}(x)$, where

$$m_{ij} = \begin{cases} m_i \circ \ldots \circ m_j & \text{if } i > j \\ \mathrm{id}_{M_i} & \text{if } i = j, \end{cases}$$

and $p_i: M_i \to V_i$ is a $\mathbb{Z}_{p^i}$-equivariant projection.

It is enough to show that for fixed j, $0 \leqslant j \leqslant k$, the map $\varphi_j: M_j \to \bigoplus_{i=j}^{k} (V_i)^{\mathbb{Z}_{p^{i-j}}}$ is surjective. Clearly, every element of V_j is in the image of φ_j. Suppose that every element of $\bigoplus_{i=j}^{n} (V_i)^{\mathbb{Z}_{p^{i-j}}}$ is in the image of φ_j, where $j < n < k$. Let $y \in \bigoplus_{i=j}^{n+1} (V_i)^{\mathbb{Z}_{p^{i-j}}}$. Then $y = y_1 + y_2$, where $y_1 \in \bigoplus_{i=j}^{n} (V_i)^{\mathbb{Z}_{p^{i-j}}}$, $y_2 \in (V_{n+1})^{\mathbb{Z}_{p^{n+1-j}}}$. By the inductive assumption, $y_1 = \varphi_j(x_1)$ for some $x_1 \in M_j$. From the fact that $m_i': M_i \to (M_{i+1})^{\mathbb{Z}_p}$ is onto for $i = 0, \ldots, k-1$, it follows that $\mathrm{Im}(m_{j,n+1}) = (M_{n+1})^{\mathbb{Z}_{p^{n+1-j}}}$. Thus, there exists $x_2 \in M_j$ such that $y_2 = m_{j,n+1}(x_2)$. Then $\varphi_j(x_2) - y_2 \in \bigoplus_{i=j}^{n} (V_i)^{\mathbb{Z}_{p^{i-j}}}$. Hence $\varphi_j(x_2) - y_2 = \varphi_j(x_3)$ for some $x_3 \in M_j$. Finally, $y = \varphi_j(x_1 + x_2 - x_3)$. Thus, by an inductive argument, φ_j is surjective.

Corollary 4.2. *Every rational coefficient system for $\mathbb{Z}_{p^k}$ has an injective resolution of length* 1.

Proof. It follows from the fact that if N is a subsystem of a rational coefficient system M and both the maps $m_i': M_i \to (M_{i+1})^{\mathbb{Z}_p}$, $n_i': N_i \to (N_{i+1})^{\mathbb{Z}_p}$ are onto, $0 \leqslant i < k$, then the same holds for the quotient system M/N.

Remark 4.3. By Proposition 7.24 of [2], the statement of Corollary 4.2 is equivalent to the one of Lemma 5.7 of [7] which says that every rational coefficient system for $\mathbb{Z}_{p^k}$ has a projective of length 1.

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