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A Limit Case of the Hörmander Multiplier Theorem

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Abstract: We prove that Fourier multipliers in a localized Besov-space $\mathscr{V}(B^2_{n/2,1})$ define bounded operators from the Hardy-space H^1 into the Lorentz-space $L^{1,r}, r > 2$.

The Hörmander theorem on Fourier multipliers [H] states that a multiplier transformation T_m defined for Schwartz-functions $f \in \mathscr{S}$ by $T_m f = \mathscr{F}^{-1}[m\hat{f}]$ extends to a bounded operator in L^p, $1 < p < \infty$, if the norm $\sup_{t>0} \| \varphi\, m(t\cdot) \|_{\mathscr{L}^2_\alpha}$, $\alpha > \frac{n}{2}$, is bounded; also T_m is of weak type $(1,1)$. Here φ denotes a radial nonzero C^∞-function with compact support in $\mathbb{R}^n \setminus \{0\}$, and $\mathscr{L}^2_\alpha$ is the standard Bessel-potential-space (see [St]).

There are generalizations of the Hörmander criterion [BS, Ca, Sg], where only weak smoothness of $\varphi\, m(t\cdot)$ with respect to the M^p-multiplier-norm is required, uniformly in $t > 0$. In this paper we examine another limit case; we replace $\mathscr{L}^2_\alpha$, $\alpha > \frac{n}{2}$, by the Besov-space $B^2_{n/2,1}$ and derive a Lorentz-space inequality for functions in the Hardy-space H^1. Note that Bernstein's theorem states that $B^2_{n/2,1} \subset \mathscr{F}L^1$, but there is no additional smoothness with respect to the L^∞-norm.

1. Theorem. *For $f \in H^1$, $r > 2$ we have the inequality*

$$\| T_m f \|_{L^{1,r}} \leqslant c \sup_{t>0} \| \varphi\, m(t\cdot) \|_{B^2_{n/2,1}} \, \| f \|_{H^1} .$$

Here L^{pq} is the Lorentz-space, defined by the quasi-norms

$$\| f \|_{L^{pq}} = \Big(\int_0^\infty [t^{1/p} f^*(t)]^q \, dt/t \Big)^{1/q}$$

$(0 < p, q < \infty)$; f^* denotes the nonincreasing rearrangement of f. c will always be an abstract constant which takes different values in different estimates.

2. Remarks. In order to compare the statement of the theorem with known criteria it is convenient to have the notion of localized spaces $\mathscr{V}(X)$, see e.g. [CS]. Given a function space X of locally integrable functions in $\mathbb{R}^n \setminus \{0\}$ we define a localization space $\mathscr{V}(X)$ with norm $\sup\{\|\varphi m(t\cdot)\|_X; t > 0\}$, with the same φ as above (actually the definition is independent of the special φ for all spaces considered here).

a) Sharp estimates for multiplier transformations mapping H^1 into L^1 are given in BAERNSTEIN and SAWYER [BS]. An inspection of their counterexample shows that there is $f \in H^1$, $m \in \mathscr{V}(B^2_{n/2,1})$ such that $T_m f \notin L^{1,s}$ for $s < 2$.

However, they showed that, setting $T(m,f) = T_m(f)$, $T: \mathscr{V}(B^2_{n(1/p-1/2),q}) \times H^p \to H^p$ is bounded for $p < 1$, and a suitable interpolation argument shows the boundedness of $T: \mathscr{V}(B^q_{n/2,p}) \times H^1 \to H^1$ if $q > 2$, $p < 1$. There is no counterpart of our theorem valid for all L^1-functions. MIYACHI [Mi] gives an example of a multiplier m which is not of weak type (1, 1) but maps H^1 into H^1; one may check that $m \in \mathscr{V}(B^q_{n/2,r})$ for every $q, r > 0$.

b) STEIN's proof of the Hörmander criterion can be modified to obtain the following weighted norm inequality (see [St], [Ch])

$$\int g(T_m f)^2 w(x)\,dx \leqslant c\|m\|^2_{\mathscr{V}(B^q_{n/q,1})} \int g(f)^2 (\mathscr{M}|w|^r)^{1/r}\,dx \qquad (1)$$

where $1/q = 1/2r < 1/2$; $\mathscr{M}$ is the Hardy—Littlewood-maximal operator and $g(f) = (\sum_k |\eta_k * f|^2)^{1/2}$, with

$$\eta_k = \mathscr{F}^{-1}[\varphi(2^k\cdot)],\ \varphi \in C_0^\infty(\mathbb{R}^n),\ \operatorname{supp}\varphi \subset \{\tfrac{1}{2} < |x| < 2\}, \qquad (2)$$
$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}x) = 1 \text{ for all } x \neq 0\,.$$

This constitutes an endpoint result in the case $1 < p \leqslant 2$:

If $r = (p'/2)'$ and $\|m\|_{\mathscr{V}(B^q_{n/q,1})} = A$ we obtain

$$\|T_m f\|^2_{p'} \leqslant cA^2 \|g^2(f)\|_{L^{(p'/2),1}} \sup_{\|w\|_r = 1} \|(\mathscr{M}(w^r))^{1/r}\|_{L^{r'\infty}} \leqslant$$
$$\leqslant cA^2 \|g(f)\|^2_{L^{p'2}} \leqslant c'A^2 \|f\|^2_{L^{p'2}}$$

by duality, Littlewood—Paley-theory and a multiplication theorem in Lorentz-spaces ([Hu]). Again by duality we conclude that

$$T: \mathscr{V}(B^q_{n/q,1}) \times L^p \to L^{p^2} \text{ is bounded, if } 1/q = 1/p - 1/2 .$$

This proof breaks for $f \in H^1$, and our theorem can be seen as a substitute for $p = 1$. We don't know whether L^{1r}, $r > 2$ can be replaced by L^{12}.[1]

3. Some technical tools. We need a version of Sagher's theorem on analytic families of operators in Lorentz spaces, in a vector-valued setting. Let (A_0, A_1), (B_0, B_1) be interpolation couples of Banach spaces and denote bei $A_\theta = [A_0, A_1]_\theta$ the complex interpolation space. For $z \in S = \{z; 0 \leqslant \operatorname{Re} z \leqslant 1\}$ let (T_z) be a family of operators mapping simple functions with values in $A_0 \cap A_1$ into measurable functions with values in $B_0 + B_1$. Assume that for each measurable $E \subset \mathbb{R}^n$, and each fixed $y \in \mathbb{R}^n$ the function $z \to T_z(\chi_E)(y)$ is analytic in the interior of S, bounded and continuous in S (the boundedness may be replaced by an assumption of "admissible growth").

Lemma 1. *If for all simple functions with values in* $A_0 \cap A_1$

$$\| T_{j+i\tau} f \|_{L^{\eta_j q_j}(B_j)} \leqslant \gamma_j \| f \|_{L^{p_j q_j}(A_j)}, \; j = 0, 1 ,$$

then for

$$\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}\right) = (1-\theta)\left(\frac{1}{p_0}, \frac{1}{q_0}, \frac{1}{r_0}, \frac{1}{s_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1}, \frac{1}{s_1}\right) \in (0, \infty)^4$$

there is a unique extension T_θ *satisfying*

$$\| T_\theta f \|_{L^{rs}(B_\theta)} \leqslant c\, \gamma_0^{1-\theta} \gamma_1^{\theta} \| f \|_{L^{pq}(A_\theta)}, \text{ for all } f \in L^{pq}(A_\theta) .$$

The proof in [Sa] can be used to prove also Lemma 1, with very few modifications. In the proof of our multiplier result we are led to consider a scale of function spaces Y^q which we presently define. Let η_k be as in the definition of $g(f)$, $\eta = \eta_0$. Let $\Psi \in C_0^\infty$ be supported in $\{|x| < 1\}$, $\Psi(x) = 1$ if $|x| \leqslant 3/4$.

$$\Psi_l(x) = \Psi\left(\frac{x}{2^l}\right) - \Psi\left(\frac{x}{2^{l-1}}\right), \; l \geqslant 1, \; \Psi_0 = \Psi . \tag{3}$$

[1] After this paper had been submitted, the author established the $H^1 - L^{12}$-boundedness.

For a tempered distribution g let

$$A_q^{kl}(g) = 2^{ln/q'} \| [\eta * 2^{kn} g(2^k \cdot)] \Psi_l \|_q .$$

Y^q is defined as the space of all $g \in S'$ for which

$$\| g \|_{Y^q} = \sup_{k \in \mathbb{Z}} \sum_{l=0}^{\infty} A_q^{kl}(g)$$

is finite. Denote by Y_0^q the subspace of Y^q of those g with $\lim_{|k| \to \infty} \sum_{l=0}^{\infty} A_q^{kl} = 0$. Note that by Hölder's inequality $Y^{q_1} \subset Y^{q_0}$ if $1 \leqslant q_0 \leqslant q_1 \leqslant \infty$; further, by Plancherel's theorem,

$$\| g \|_{Y^2} = \sup_k \sum_{l=0}^{\infty} 2^{ln/2} \| [\varphi \hat{g}(2^{-k} \cdot)] * \Psi_l \|_2 \approx \| g \|_{\mathcal{F}(B^2_{n/2,1})} . \tag{4}$$

Lemma 2. $[Y_0^{q_0}, Y_0^{q_1}]_\theta = Y_0^q$, *if* $\dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1} = \dfrac{1}{q}$, $1 \leqslant q_0, q_1 \leqslant \infty$.

Proof. Denote by l_0^∞ the space of all bounded (a_k) satisfying $\lim_{|k| \to \infty} a_k = 0$. Then we have the interpolation formula

$$[l_0^\infty(l^1(L^{q_0})), l_0^\infty(l^1(L^{q_1}))]_\theta = l_0^\infty(l^1(L^q)) ,$$

and the lemma follows by showing that Y_0^q is a retract in $l_0^\infty(l^1(L^q))$: Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ supported in $\{1/4 < |\xi| < 4\}$, $\tilde{\varphi}(\xi) = 1$, if $1 \leqslant |\xi| \leqslant 2$, $\tilde{\eta}_k = F^{-1}[\tilde{\varphi}(2^k \cdot)]$; further let $\tilde{\Psi}_l = \Psi_{l+2} - \Psi_{l-2}$, $l \geqslant 2$, $\tilde{\Psi}_0 = \tilde{\Psi}_1 = \Psi\left(\frac{\cdot}{4}\right)$. Then $\eta_k * \tilde{\eta}_k = \eta_k$, $\Psi_l * \tilde{\Psi}_l = \Psi_l$. Define for $g \in Y^q$, $k \in \mathbb{Z}$, $l \in \mathbb{N} \cup \{0\}$,

$$[Rg]_{k,l} = 2^{ln/q'} (\eta * 2^{kn} g(2^k \cdot)) \Psi_l$$

and for $G \in l^\infty(l^1(L^q))$

$$S(G) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} 2^{-ln/q'} 2^{-kn} [(G_{k,l} \tilde{\Psi}_l) * \tilde{\eta}](2^{-k} \cdot) .$$

Then $S \circ R$ is the identity in Y^q and R: $Y^q \to l^\infty(l^1(L^q))$ an isometry. Further

$$\| S(G) \|_{Y^q} \leqslant$$

$$\leqslant \sup_{\mu \in \mathbb{Z}} \sum_{\nu=0}^{\infty} \sum_{|k-\mu| \leqslant 3} \sum_{l=0}^{\infty} 2^{(\nu-l)n/q'} \| [\eta * \tilde{\eta}(2^{\mu-k} \cdot) * (G_{k,l} \tilde{\Psi}_l)(2^{\mu-k} \cdot)] \Psi_\nu \|_q .$$

The boundedness of S follows since for $\eta_0 \in S$, $h \in L^q$ the expression $\| [\eta_0 * (h \tilde{\Psi}_l)] \Psi_\nu \|_q$ is dominated by $c \| h \|_q$, if $| l - \nu | \leqslant 5$; by $C_N 2^{-\nu N} \| h \|_q$, for every $N \in \mathbb{N}$, if $\nu > l + 5$; by $C_N 2^{-lN} \| h \|_{q'}$ if $0 \leqslant \nu < l - 5$. □

In the next lemma we state some characterizations of H^p-spaces and intermediate spaces via Littlewood—Paley-functions. These are easy consequences of deep results in FEFFERMAN and STEIN [FS 2] and UCHIYAMA [U 1, U 2]. Denote by $\mathcal{N}(f)$ the grand maximal function,

$$\mathcal{N}(f) = \sup \{ | \Phi * f(x) |, \ \| \Phi \|_N^* \leqslant 1 \}$$

where $\| \Phi \|_N^* = \sum_{|\alpha| \leqslant N} \sup (1 + |x|)^N | D^\alpha \Phi(x) |$, for sufficiently large $N = N(p)$. Then the Fefferman—Rivière—Sagher interpolation theorem states that $[H^{p_0}, L^\infty]_{\theta, q} = H^{pq}$, $(1 - \theta)/p_0 = 1/p$, with $\| f \|_{H^{pq}} = \| \mathcal{N} f \|_{L^{pq}}$. An analogous result holds in the vector-valued setting.

Lemma 3. *For $0 < p, q < \infty$ we have $\| f \|_{H^{pq}} \leqslant c \| g(f) \|_{L^{pq}}$.*

Proof. Let $F \in H^p(l^2)$, then we obtain by interpolation the inequality

$$\| \sum_k \tilde{\eta}_k * F^k \|_{H^{pq}} \leqslant c \| F \|_{H^{pq}(l^2)}$$

which is proved in [U 2] in the case $p = q$ using the atomic decomposition. Let $\mathcal{M}$ be the Hardy—Littlewood-maximal-function and $\mathcal{M}_s(f) = \mathcal{M}(|f|^s)^{1/s}$. For a fixed $\Phi \in S$, $\hat{\Phi}(0) = 1$ let $\mathcal{N}_\Phi^+ F = \sup_{t > 0} | t^n \Phi(t \cdot) * F |_{l^2}$ be the "radial" maximal function. Then we need the estimates

$$\mathcal{N}(F) \leqslant c \mathcal{M}_s(\mathcal{N}_\Phi^+(F)), \text{ for some } s < p,$$

proved in [U 1] for scalar functions, and

$$\mathcal{N}_\Phi^+(\{\eta_k * f\}) \leqslant c \mathcal{M}_s(g(f))$$

(see [U 2]). The assertion of the lemma follows by setting $F = \{\eta_k * f\}$, since $\mathcal{M}_s$ is bounded in L^{pq}, $p > s$.

4. Proof of the Theorem. For $F \in L^1(l^2)$, $h \in Y^2$ we define an operator $\tilde{T}$ by

$$[\tilde{T} F]_k = \eta_k * h * F^k .$$

We shall prove

$$\| \tilde{T} F\|_{L^{1,r}(l^2)} \leqslant c \| h\|_{Y^2} \| F\|_{L^1(l^2)}\,, \tag{5}$$

$r > 2$, and setting $h = \mathscr{F}^{-1}[m]$, $F = \{\tilde{\eta}_k * f\}$, the assertion follows by (4) and Lemma 3. The main point is a weak-type estimate:

Proposition. *For any $q \in [1, \infty)$ we have*

$$|\{x, |\tilde{T} F|_{l^q} > \alpha\}| \leqslant c\, \alpha^{-1} \| h\|_{Y^q} \| F\|_{L^1(l^q)}\,.$$

Let us first conclude the proof of the theorem. Let $\theta = \frac{1}{2}$ and q such that $1 - \theta + \theta/q = 1/r$, i.e., $q = r/(r-2)$. Fix $h \in Y^2$. By Lemma 2 we may find for each $z \in S = \{0 \leqslant \operatorname{Re} z \leqslant 1\}$ an $h_z \in Y^1$, such that $h_{1/2} = h$, and such that the function $z \to h_z$ is bounded and continuous in S, analytic in the interior of S and the inequality

$$\sup_{\tau \in \mathbb{R}} \max \{\| h_{i\tau}\|_{Y^1}, \| h_{1+i\tau}\|_{Y^q}\} \leqslant 2 \| h\|_{Y^2}$$

holds. For every $N > 0$ we set

$$[\tilde{T}_{z,N} F]_k = \eta_k * h_z * F^k, \quad \text{if} \quad |k| < N \quad \text{and} \quad = 0, \quad \text{if} \quad |k| \geqslant N\,.$$

Then

$$\| \tilde{T}_{i\tau, N} F\|_{L^1(l^1)} \leqslant \sum_k \| \eta_k * h_{i\tau}\|_1 \quad \| F^k\|_1 \leqslant$$

$$\leqslant \| h_{i\tau}\|_{Y^1} \| F\|_{l^1(L^1)} \leqslant 2 \| h\|_{Y^2} \| F\|_{L^1(l^1)}$$

and, by the proposition,

$$\| \tilde{T}_{1+i\tau, N} F\|_{L^{1,\infty}(l^q)} \leqslant c \| h_{1+i\tau}\|_{Y^q} \| F\|_{L^1(l^q)} \leqslant c \| h\|_{Y^2} \| F\|_{L^1(l^q)}$$

with c being independent of N. Now we may apply Lemma 1 to obtain

$$\| \tilde{T}_{\theta, N} F\|_{L^{1,r}(l^2)} \leqslant c \| h\|_{Y^2} \| F\|_{L^1(l^2)}$$

and (5) follows by the monotone convergence theorem.

Proof of the proposition. We use a variant of Calderón—Zygmund-theory. Fix $\alpha > 0$, $F \in L^1(l^q)$ and introduce the exceptional set

$$\Omega = \Omega_{\alpha/A} = \{x; (\sum_k [\mathscr{M}(|F^k|)(x)]^q)^{1/q} > \alpha/A\}, \tag{6}$$

with $A = \| h\|_{Y^q}$. The weak-type inequality for sequences of maximal functions (see FEFFERMAN and STEIN [FS 1]) implies

$$|\Omega| \leqslant c\, A\, \alpha^{-1} \| F\|_{L^1(l^q)}\,.$$

Let $\Omega_* = \{x, \mathcal{M}\chi_\Omega \geqslant (20\,n)^{-n}\}$ then clearly $|\Omega_*| \leqslant c\,|\Omega|$. Using the Whitney covering theorem ([St]) we get as usual a decomposition $F = G_0 + \sum_j B_j$ where G_0 is supported in $\mathbb{R}^n \setminus \Omega$, and $B_j = \sum_i b_{j,i}$; $b_{j,i}$ is supported in a dyadic cube $Q_{j,i}$ with sidelength 2^j. The interiors of all $Q_{j,i}$ are pairwise disjoint, and the ten-fold-dilates of the $Q_{j,i}$ are contained in Ω_*.

Let Φ be a C^∞-bump-function supported in $\{x, |x| \leqslant \frac{1}{2}\}$ with the properties

$$\int \Phi = 1,\ \int \Phi(x)\,x^\alpha\,dx = 0,\quad 0 < |\alpha| \leqslant 4n,\ \Phi_j = 2^{-jn}\Phi(2^{-j}\cdot)\,.$$

Then we set

$$G = G_0 + \sum_j \Phi_j * B_j,\quad B = \sum_j (\delta - \Phi_j) * B_j = \sum_j \tilde{B}_j = \sum_{j,i} \tilde{b}_{j,i}$$

where $\tilde{b}_{j,i}$ is supported in a double $\tilde{Q}_{j,i}$ of $Q_{j,i}$.

Our definition of Ω implies

$$\Big(\sum_k \Big[\int |\tilde{b}_{j,i}(y)|\,dy\Big]^q\Big)^{1/q} \leqslant c\,(\alpha/A)\,|Q_{j,i}| \tag{7}$$

and $\|G\|_{L^\infty(l^q)} \leqslant c\,\alpha/A$, further

$$\|G\|_{L^1(l^q)} \leqslant c\,\|F\|_{L^1(l^q)}\,.$$

Since $\tilde{T}$ is bounded on $L^q(l^q)$, if $h \in Y_1$, it follows by Čebyšev's inequality

$$|\{x, |\tilde{T}G(x)|_{l^q} > \alpha\}| \leqslant a^{-q}\|\tilde{T}G\|^q_{L^q(l^q)} \leqslant c\,(A/\alpha)^q\|G\|^q_{L^q(l^q)} \leqslant$$

$$\leqslant c\,(A/\alpha)\|G\|_{L^1(l^q)}\,.$$

In order to estimate $|\{x, |\tilde{T}B|_{l^q} > \alpha\}|$ we make a dyadic decomposition of the kernel: Let $\tilde{\psi} \in C_0^\infty$ be supported in $\{x, |x| \leqslant \frac{1}{2}\}$ and $\tilde{\psi}(x) = 1$, if $|x| \leqslant \frac{1}{4}$. Setting $\psi(x) = \tilde{\psi}(x/2) - \tilde{\psi}(x)$, $\psi_l(x) = \psi(2^{-l}x)$, we may decompose

$$\tilde{T}B = \tilde{T}_1 B + \tilde{T}_2 B$$

where

$$[\tilde{T}_1 B]_k = \Big(\sum_{j>k} + \sum_{j\leqslant k}\Big)\sum_{l>0} (\eta_k * h)\,\psi_{j+l} * (\delta - \Phi_j) * B_j^k = I^k + II^k$$

and $\tilde{T}_2 B$ is supported in Ω_*. Since $|\Omega_*| \leqslant c\,A\,\alpha^{-1}\|F\|_{L^1(l^q)}$, it suffices to estimate $|\{x, |T_1\tilde{B}|_{l^q} > \alpha\}|$.

The term $|II|_{l^q}$ is less singular and L^1-bounds follow by routine estimates using the cancellation of $\tilde{B}_j$, for details see [Ca]: Introducing an additional Littlewood—Paley-decomposition we may write

$$(\eta_k * h)\,\psi_{j+l} * (\delta - \Phi_j) = \sum_i 2^{-in}\mathscr{F}[m_{ikjl}\tilde{\varphi}(1 - \hat{\Phi}(2^{j-i}\cdot))](2^{-i}\cdot)$$

with

$$m_{ikjl}(\xi) = \tilde{\varphi}(\xi)\int [\varphi \hat{h}](2^{k-i}y)\,2^{(j+l-i)n}\hat{\psi}(2^{j+l-i}(\xi - y))\,dy.$$

If $|i-k|>3$, one computes

$$|D^\gamma m_{ikjl}(\xi)| \leqslant c_N 2^{(j+l-i)(|\gamma|+n-N)}\min(2^{(i-k)n}, 2^{(i-k)(n-N)})\,\|\hat{h}\|_\infty,$$

if $1/4 \leqslant |\xi| \leqslant 4$. Further an application of Taylor's formula gives

$$\|\mathscr{F}[\tilde{\varphi}(1 - \hat{\Phi}(2^{j-i}\cdot))]\|_1 \leqslant c_M 2^{(j-i)M}$$

for $0 \leqslant M \leqslant 4n+1$. Summing several geometric geometric series we get

$$\|II\|_{L^1(l^q)} \leqslant \sum_k \|II^k\|_1 \leqslant$$

$$\leqslant c\sum_k \sum_{|i-k|\leqslant 3}\sum_{l>0}\sum_{j\leqslant k} 2^{(j-i)M}\|(\eta * 2^{kn}h(2^k\cdot))\,\psi_{j+l-k}\|_1\,\|B_j^k\|_1 +$$

$$+ c\sum_k \sum_{|i-k|>3}\sum_{l>0}\sum_{j\leqslant k} 2^{(j+l-i)(2n-N)}\min\{1, 2^{(j-i)M}\}\cdot$$

$$\cdot \min\{2^{(i-k)n}, 2^{(i-k)(n-N)}\}\,\|\hat{h}\|_\infty\,\|B_j^k\|_1 \leqslant$$

$$\leqslant c[\|h\|_{Y^1} + \|\hat{h}\|_\infty]\sum_j (\sum_k \|B_j^k\|_1^q)^{1/q} \leqslant cA\sum_{j,i}(\sum_k \|b_{j,i}^k\|_1^q)^{1/q}.$$

Therefore by Čebyšev's inequality and (7)

$$|\{x, |II|_{l^q} > \alpha\}| \leqslant c\alpha^{-1}|II|_{L^1(l^q)} \leqslant c\sum_{j,i}|Q_{j,i}| \leqslant cA\alpha^{-1}\|F\|_{L^1(l^q)}.$$

Let us estimate the main term $|I|_{l^q}$. By Čebyšev's inequality

$$|\{x, |I|_{l^q} > \alpha\}| \leqslant \alpha^{-q}\sum_k \|I^k\|_q^q.$$

Substitute $\lambda = j - k + l$ and apply Minkowski's inequality:

$$\|I^k\|_q^q \leqslant c[\sum_{\lambda>0}\|(\eta_k * h)\,\psi_{k+\lambda} * \sum_{0<l\leqslant\lambda}\tilde{B}^k_{k-l+\lambda}\|_q]^q.$$

We devide $\mathbb{R}^n$ into a grid of dyadic cubes $R_{k+\lambda,s}$ with disjoint interior, with sidelength $2^{k+\lambda}$, and denote by $\chi_{k+\lambda,s}$ the indicator function of

$R_{k+\lambda,s}$. Since $(\eta_k * h)\,\psi_{k+\lambda}$ is supported in a cube with sidelength $2^{k+\lambda}$ we obtain

$$\|I^k\|_q^q \leqslant c\Big[\sum_{\lambda>0} \|(\eta_k * h)\,\psi_{k+\lambda}\|_q \Big(\sum_s \Big[\int_{R_{k+\lambda,s}} \Big|\sum_{0<l<\lambda} \tilde{B}^k_{k-l+\lambda}\Big|\,dx\Big]^q\Big)^{1/q}\Big]^q \qquad (8)$$

Fix a cube $Q_{j,i}$ and let $Q^*_{j,i}$ be its double. Then there is an $x_{j,i} \in \mathbb{R}^n \setminus \Omega$, such that for all $R_{k+\lambda,s}$ with $k+\lambda \geqslant j$, which contain $Q_{j,i}$,

$$\int_{R_{k+\lambda,s}} \Big|\sum_{0<l<\lambda} \tilde{B}^k_{k+\lambda-l}\Big|\,dx \leqslant 2^{(k+\lambda)n}\,\mathcal{M}(F^k)(x_{j,i}).$$

Inserting this into (8) we obtain

$$\|I^k\|_q^q \leqslant \Big[\sum_\lambda 2^{-kn/q'} \|[\eta * 2^{kn} h(2^k\cdot)]\,\Psi_\lambda\|_q \cdot$$

$$\cdot \Big(\sum_s \Big(\int_{R_{k+\lambda,s}} \Big|\sum_{0<l<\lambda} \tilde{B}^k_{k+\lambda-l}\Big|\,dx\Big)^{q-1} \sum_{Q_{j,i}\subset R_{k+\lambda,s}} \int_{Q^*_{j,i}} |\tilde{b}^k_{j,i}|\,dx\Big)^{1/q}\Big]^q \leqslant$$

$$\leqslant c\,\|h\|_{Y^q}^q \sum_{j,i} [\mathcal{M}(F^k)(x_{j,i})]^{q-1} \int_{Q_{j,i}} |b^k_{j,i}|\,dx\,.$$

We continue with Hölder's inequality, and since $x_{j,i} \notin \Omega$, we get

$$\sum_k \|I^k\|_q^q \leqslant c\,A^q \sum_{j,i} \Big[\sum_k (\mathcal{M}(F^k)(x_{i,j})]^q\Big)^{1/q'} \Big(\sum_k \Big[\int_{Q^*_{j,i}} |\tilde{b}^k_{j,i}|\,dx\Big]^q\Big)^{1/q} \leqslant$$

$$\leqslant c\,A^q (\alpha/A)^{q-1} \sum_{j,i} |Q_{j,i}|\,\alpha/A \leqslant c\,A\,\alpha^{q-1}\,\|F\|_{L^1(l^q)}\,,$$

hence

$$|\{x, |I|_{l^q} > \alpha\}| \leqslant c\,A\,\alpha^{-1}\,\|F\|_{L^1(l^q)}\,. \quad \square$$

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