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**Autor:** Sjölin, Per

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## Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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# Norm Inequalities for Spherical Means

By

**Per Sjölin,** Stockholm

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**Abstract.** The regularity and integrability of spherical means of functions in $L^p(\mathbb{R}^n)$, $n \geqslant 2$, are studied. Sharp results are obtained.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we set

$$F_x(t) = \int_{S^{n-1}} f(x - ty)\, d\sigma(y),\ x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

where $\sigma$ denotes the surface measure on $S^{n-1}$. We shall study the regularity of the function $F_x$. Let the space $\Lambda_\alpha(\mathbb{R})$, $\alpha > 0$, be defined as in E. M. STEIN [5], i. e. a function $g \in L^\infty(\mathbb{R})$ belongs to $\Lambda_\alpha(\mathbb{R})$ if its Poisson integral $u(t, y)$, $t \in \mathbb{R}$, $y > 0$, satisfies

$$\|g\|_{\Lambda_\alpha} = \|g\|_\infty + \sup_{t,y} \left| y^{k-\alpha} \frac{\partial^k u}{\partial y^k}(t, y) \right| < \infty,$$

where $k$ is the smallest integer greater than $\alpha$. In [4] we obtained the following results.

**Theorem A.** *Let $\varphi \in C_0^\infty(\mathbb{R})$ and assume that* $\operatorname{supp} \varphi \subset ]0, \infty[$. *If $n \geqslant 2$, $n/(n - 1/2) \leqslant p \leqslant 2$ and $\alpha = n/p' - 1/2$, then*

$$\Big(\int_{\mathbb{R}^n} \|\varphi F_x\|^2_{L^2_\alpha} dx\Big)^{1/2} \leqslant C_p \|f\|_p\,. \tag{1}$$

*Here $f \in L^p(\mathbb{R}^n)$ and $C_p$ depends on $\varphi$ but not on $f$.*

$L^2_\alpha(\mathbb{R})$, $\alpha \geqslant 0$, is defined as the space of all $g \in L^2(\mathbb{R})$ with $\|g\|_{L^2_\alpha} = \Big(\int_{\mathbb{R}} |\hat{g}(s)|^2 (1 + s^2)^\alpha ds\Big)^{1/2} < \infty$, where $\hat{g}(s) = \int e^{-ist} g(t)\, dt$ is the Fourier transform of $g$.

**Theorem B.** *Let $\varphi$ be as in Theorem A and assume $n \geqslant 2$. Then the following holds.*

*If* $1 \leqslant p < n/(n-1)$ *and* $q = 1/(1 - n/p')$, *then*

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{L^q}^2 \, dx\right)^{1/2} \leqslant C_p \|f\|_p \,. \tag{2}$$

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{\mathrm{BMO}}^2 \, dx\right)^{1/2} \leqslant C \|f\|_{n/(n-1)} \,. \tag{3}$$

*If* $n/(n-1) < p \leqslant 2$ *and* $\beta = n/p' - 1$, *then*

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{\Lambda_\beta}^2 \, dx\right)^{1/2} \leqslant C_p \|f\|_p \,. \tag{4}$$

It is also proved in [4] that the above values of $\alpha$, $q$ and $\beta$ are best possible. We shall here extend the results in Theorem B and prove the following theorem.

**Theorem 1.** *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *and assume that* $\operatorname{supp}\varphi \subset ]0, \infty[$. *If* $n \geqslant 2$ *then the following holds for* $f \in L^p(\mathbb{R}^n)$.

*If* $1 \leqslant p < n/(n-1)$, $q = 1/(1 - n/p')$ *and* $p \leqslant r \leqslant p'$, *then*

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{L^q}^r \, dx\right)^{1/r} \leqslant C_{p,r} \|f\|_p \tag{5}$$

(*where the left hand side is interpreted in the usual way for* $r = \infty$).

*If* $p = n/(n-1)$ *and* $p \leqslant r \leqslant p'$, *then*

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{\mathrm{BMO}}^r \, dx\right)^{1/r} \leqslant C_r \|f\|_p \,. \tag{6}$$

*If* $n/(n-1) < p \leqslant 2$, $\beta = n/p' - 1$ *and* $p \leqslant r \leqslant p'$, *then*

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{\Lambda_\beta}^r \, dx\right)^{1/r} \leqslant C_{p,r} \|f\|_p \,. \tag{7}$$

We shall also prove that is not possible to have $r < p$ or $r > p'$ in (5), (6) or (7).

*Proof of Theorem 1.* First assume $f \in L^1(\mathbb{R}^n)$. We shall prove that $\varphi F_x$ is a strongly measurable function of $x$ with values in $L^1(\mathbb{R})$. Since $L^1(\mathbb{R})$ is separable it is sufficient to prove that $\varphi F_x$ is weakly measurable (see E. HILLE and R. S. PHILLIPS [3], p. 73). Therefore take $g \in L^\infty(\mathbb{R})$ and set

$$H(x) = \int_{\mathbb{R}} \varphi(t) F_x(t) g(t) \, dt, \; x \in \mathbb{R}^n \,.$$

Then

$$|H(x) - H(y)| \leqslant C \int_{\mathbb{R}} |\varphi(t)| \, |F_x(t) - F_y(t)| \, dt \leqslant$$

$$\leqslant C \int_{\mathbb{R}^n} |f(x-z) - f(y-z)|\, dz$$

and it follows that $H$ is continuous and measurable.

Now suppose $0 < a < b < \infty$ and $\operatorname{supp} \varphi \subset [a, b]$. We then have

$$\|\varphi F_x\|_{L^1} = \int_{\mathbb{R}} |\varphi(t) F_x(t)|\, dt = \int_a^b |\varphi(t) \int_{S^{n-1}} f(x - ty)\, d\sigma(y)|\, dt \leqslant$$

$$\leqslant C \int_{a \leqslant |y| \leqslant b} |f(x-y)|\, dy \leqslant C k * |f|(x)\,,$$

where $k(x) = 1$, $a \leqslant |x| \leqslant b$, and $k(x) = 0$ otherwise. Hence $k \in L^r(\mathbb{R}^n)$, $1 \leqslant r \leqslant \infty$, and it follows that

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|_{L^1}^r\, dx\right)^{1/r} \leqslant C_r \|f\|_1,\ 1 \leqslant r < \infty, \tag{8}$$

and

$$\|\ \|\varphi F_x\|_{L^1}\|_\infty \leqslant C \|f\|_1\,. \tag{9}$$

We introduce the Besov spaces $B^s_{pq}(\mathbb{R})$ with norm $\|\ \|_{B^s_{pq}}$, $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$, defined as in J. BERGH and J. LÖFSTRÖM [1], pp. 140—141. Since $L^1 \subset B^0_{1\infty}$ it is clear that $\varphi F_x$ is a function of $x$ with values in $B^0_{1\infty}$ if $f \in L^1(\mathbb{R}^n)$. We shall also prove that $\varphi F_x$ can be approximated in sup-norm by simple functions on $\mathbb{R}^n$ with values in $B^0_{1\infty}$. To prove this we shall use the fact that

$$\|\varphi F_x - \varphi F_{x+h}\|_{B^0_{1\infty}} \leqslant C \|\varphi F_x - \varphi F_{x+h}\|_{L^1} \leqslant C \|f - f_h\|_1,\ x, h \in \mathbb{R}^n, \tag{10}$$

where $f_h$ denotes a translate of $f$. For $k = 1, 2, 3, \ldots$ we partition the set $\{x \in \mathbb{R}^n;\ |x_i| \leqslant 2^k,\ i = 1, 2, \ldots, n\}$ into cubes $Q^k_j$, $j = 1, 2, \ldots, N$, with side length $2^{-k}$ and centers $x^1, \ldots, x^N$ ($N$ depends on $k$). Set

$$H_k(x, t) = \sum_1^N \chi_{Q^k_j}(x)\, \varphi(t)\, F_{x^j}(t)\,.$$

For $x \in Q^k_j$ we have

$$\|\varphi F_x - H_k(x, \cdot)\|_{B^0_{1\infty}} = \|\varphi F_x - \varphi F_{x^j}\|_{B^0_{1\infty}} \leqslant C \|f - f_{x - x^j}\|_1$$

and for $x \in \mathbb{R}^n \setminus (\bigcup_{j=1}^N Q^k_j)$ it is clear that

$$\|\varphi F_x - H_k(x, \cdot)\|_{B^0_{1\infty}} = \|\varphi F_x\|_{B^0_{1\infty}} \leqslant C \|\varphi F_x\|_{L^1} \leqslant C \int_{|y| > 2^{k-1}} |f(y)|\, dy$$

($k$ large).

It follows that $(H_k)_1^\infty$ approximates $\varphi F_x$ in sup-norm. We conclude that if $f \in L^1(\mathbb{R}^n)$ then $\varphi F_x$ is a strongly measurable function of $x$ with

values in $B^0_{1\infty}(\mathbb{R})$ and since $\|g\|_{B^0_{1\infty}} \leqslant C\|g\|_1$ it follows from (8) and (9) that

$$\left(\int_{\mathbb{R}^n} \|\varphi F_x\|^r_{B^0_{1\infty}} dx\right)^{1/r} \leqslant C_r \|f\|_1, \ 1 \leqslant r < \infty,$$

and

$$\| \ \|\varphi F_x\|_{B^0_{1\infty}}\|_\infty \leqslant C\|f\|_1.$$

We then set $Tf(x,t) = \varphi(t) F_x(t)$ for $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant 2$. We have proved that we have a bounded mapping

$$T: L^1 \to L^r(B^0_{1\infty}), \ 1 \leqslant r < \infty \ , \tag{11}$$

where $L^r(B^s_{pq}) = L^r(\mathbb{R}^n, dx; B^s_{pq})$ is defined in the same way as the spaces $L_p(A)$ are defined in [1], p. 107, for general Banach spaces A.

It also follows from the above argument that the mapping

$$T: L^1 \to L^\infty_0 (B^0_{1\infty}) \tag{12}$$

is bounded, where $L^\infty_0 (B^s_{pq}) = L^\infty_0 (\mathbb{R}^n, dx; B^s_{pq})$ is defined in the same way as $L^0_\infty (A)$ is defined in [1], p. 107. Thus $L^\infty_0 (B^s_{pq})$ is the completion in sup-norm of all simple function on $\mathbb{R}^n$ with values in $B^s_{pq}$.

According to Theorem A we also have a bounded mapping

$$T: L^2 \to L^2(L^2_{n/2-1/2}) = L^2(B^{n/2-1/2}_{22}) \ . \tag{13}$$

In fact $Tf(x,t)$ is strongly measurable with values in $B^{n/2-1/2}_{22}$ if $f \in L^2(\mathbb{R}^n)$ To see this assume that $f_k \to f$ in $L^2(\mathbb{R}^n)$ where $f_k \in C^\infty_0 (\mathbb{R}^n)$, $k = 1, 2, 3, \ldots$ . Let $g$ belong to the dual space $(L^2_{n/2-1/2})^*$ of $L^2_{n/2-1/2}$. It is sufficient to prove that $g(\varphi F_x)$ is a measurable function of $x$.

To prove this we shall approximate with $g(\varphi F^k_x)$, which is measurably for every $k$. Here

$$F^k_x(t) = \int_{S^{n-1}} f_k(x - ty)\, d\sigma(y) \ .$$

We have

$$|g(\varphi F_x) - g(\varphi F^k_x)| \leqslant \|g\|_{(L^2_{n/2-1/2})^*} \|\varphi F_x - \varphi F^k_x\|_{L^2_{n/2-1/2}}$$

and Theorem A implies that

$$\int_{\mathbb{R}^n} \|\varphi F_x - \varphi F^k_x\|^2_{L^2_{n/2-1/2}} dx \leqslant C\|f - f_k\|^2_2 \ .$$

We conclude that there exists a subsequence $(k_j)_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \|\varphi F_x - \varphi F^{k_j}_x\|_{L^2_{n/2-1/2}} = 0$$

for almost every $x$ and hence $g(\varphi F_x)$ is measurable.

We now interpolate between (12) and (13). According to [1], Thm. 5.1.2, p. 107, and Thm. 6.4.5, p. 153, it follows that the mapping $T: L^p \to L^{q^*}(B^{s^*}_{p^* q^*})$ is bounded, $1 < p < 2$, where

$$s^* = \theta\left(\frac{n}{2} - \frac{1}{2}\right)$$

$$\frac{1}{p} = \frac{1}{p^*} = 1 - \theta + \frac{\theta}{2}, \quad \frac{1}{q^*} = \frac{\theta}{2}.$$

One finds that $\theta = 2/p'$ and $p^* = p$, $q^* = p'$ and $s^* = (n-1)/p'$. It follows that the mapping $T: L^p \to L^{p'}(B^{(n-1)/p'}_{pp'})$ is bounded, $1 < p < 2$.

Now assume that $n/(n-1) < p \leqslant 2$. According to [1], Thm. 6.5.1, p. 153, we have

$$B^{(n-1)/p'}_{pp'} \subset B^{n/p'-1}_{\infty\infty} = \Lambda_{n/p'-1}$$

with norm inequality and (7) follows in the case $r = p'$.

Then assume that $p = n/(n-1)$. Then $T$ is a bounded mapping from $L^p$ to $L^{p'}(B^{1/p}_{pp'})$. According to T. M. FLETT [2], p. 424, we have

$$B^{\delta}_{pr}(\mathbb{R}) \subset L^r(\mathbb{R}),\ 1 \leqslant p < r < \infty,\ \delta = \frac{1}{p} - \frac{1}{r}, \tag{14}$$

and

$$B^{\alpha+\delta}_{pr}(\mathbb{R}) \subset L^r_\alpha(\mathbb{R}),\ 1 \leqslant p < r < \infty,\ \delta = \frac{1}{p} - \frac{1}{r},\ \alpha > 0, \tag{15}$$

where $L^r_\alpha$ denotes the space of Bessel potentials of order $\alpha$ of functions in $L^r$. Choosing $\delta = 1/p - 1/p'$ and $\alpha = 1/p - \delta = 1/p'$ in (15) we obtain $B^{1/p}_{pp'} \subset L^{p'}_{1/p'}$. But $L^{p'}_{1/p'} \subset \mathrm{BMO}$ ([5], p. 164) and hence (6) follows in the case $r = p'$.

We shall finally study the case $1 < p < n/(n-1)$. If $p' \leqslant q$ and $1/q = 1 - n/p' = 1/p - (n-1)/p'$ then (14) yields

$$B^{(n-1)/p'}_{pp'} \subset B^{(n-1)/p'}_{pq} \subset L^q.$$

The inequality $p' \leqslant q$ holds if $(n+1)/n \leqslant p < n/(n-1)$ and we therefore obtain (5) in the case $r = p'$ for $(n+1)/n \leqslant p < n/(n-1)$. The inequality (5) in the case $r = p'$ for $1 < p < (n+1)/n$ then follows from interpolation ([1]), p. 107) between (9) and the case $p = (n+1)/n$, $r = p'$ of (5). Thus (5), (6) and (7) are proved in the case $r = p'$.

To treat the case $r = p$ we modify the above argument. We interpolate between (11) with $r = 1$ and (13) and obtain the bounded-

ness of $T$ from $L^p$ to $L^p(B_{pp'}^{(n-1)/p'})$, $1 < p < 2$. Arguing as above we then obtain (5), (6) and (7) in the case $r = p$. Since (5), (6) and (7) hold for $r = p$ and $r = p'$ they clearly hold also for $p \leqslant r \leqslant p'$ and the proof of Theorem 1 is complete.

We shall now prove that it is not possible to have $r < p$ in Theorem 1. We choose $f(x) = 0$, $|x| \leqslant 1$, and $f(x) = |x|^{-\gamma}$, $|x| > 1$, and it is clear that $f \in L^p(\mathbb{R}^n)$ if $\gamma > n/p$.

In the case $1 \leqslant p < n/(n-1)$ we observe that

$$\|\varphi F_x\|_{L^q} \geqslant c\,|x|^{-\gamma}\,,$$

if $\varphi \not\equiv 0$ and $|x|$ is large, where $c$ denotes a positive constant. We conclude that (5) does not hold with $r < p$.

In the case $n/(n-1) < p \leqslant 2$ we see that

$$\|\varphi F_x\|_{\Lambda_\beta} \geqslant \|\varphi F_x\|_{L^\infty} \geqslant c\,|x|^{-\gamma},$$

if $|x|$ is large, and hence (7) holds only if $r \geqslant p$.

To study the case $p = n/(n-1)$ we set $\chi(x) = 1$, $|x| \leqslant 1/10$, and $\chi(x) = 0$ otherwise and

$$f(x) = |x|^{-\gamma} \sum_{\substack{m \in \mathbb{Z}^n \\ m \neq 0}} \chi(x - m)\,.$$

If $\varphi(t) = 1$, $0.2 \leqslant t \leqslant 0.3$, and $\varphi(t) = 0$ if $t \leqslant 0.15$ or $t \geqslant 0.35$, we observe that

$$\|\varphi F_x\|_{\mathrm{BMO}} \geqslant c\,|x|^{-\gamma}$$

if $|x|$ is large and $0.2 \leqslant \operatorname{dist}(x, \mathbb{Z}^n) \leqslant 0.3$. We conclude that $r \geqslant p$ is a necessary condition on $r$ in (6).

We shall finally prove that it is not possible to have $r > p'$ in (5), (6) and (7). First choose $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi \geqslant 0$ and $\psi(t) = 1$, $-1 \leqslant t \leqslant 1$, and $\psi(t) = 0$, $|t| \geqslant 2$. Then set

$$f_0(x) = \psi\left(\frac{x_1}{3\varepsilon}\right)\psi\left(\frac{x_2}{3\sqrt{\varepsilon}}\right)\cdots\psi\left(\frac{x_n}{3\sqrt{\varepsilon}}\right),$$

where $0 < \varepsilon < 1/10$, and also

$$f(x) = f_0(x - e_1)\,,$$

where $e_1 = (1, 0, \ldots, 0)$.

To treat the case $1 < p < n/(n-1)$ we observe that if $|x_1| \leqslant 1/100$ and $|x_i| \leqslant \sqrt{\varepsilon}/100$, $i = 2, 3, \ldots, n$, then

$$F_x(t) \geqslant c\,\varepsilon^{(n-1)/2}$$

for $t$ in a set of measure larger than $\varepsilon/100$, and hence

$$\|F_x\|_{L^q} \geqslant c\,\varepsilon^{(n-1)/2+1/q}\,.$$

We choose $\varphi \geqslant 0$ and $\varphi(t) = 1$ for $1/2 \leqslant t \leqslant 3/2$ and conclude that

$$\|\varphi F_x\|_{L^q} = \|F_x\|_{L^q}$$

for the above values of $x$. If (5) holds we therefore obtain

$$c\,\varepsilon^{(n-1)/2+1/q}\,\varepsilon^{(n-1)/2r} \leqslant \left(\int \|\varphi F_x\|_{L^q}^r\,dx\right)^{1/r} \leqslant$$
$$\leqslant C\,\|f\|_p < C\,\varepsilon^{(1+(n-1)/2)/p}.$$

Letting $\varepsilon$ tend to 0 and invoking the definition of $q$ one finds that

$$\frac{n}{2} - \frac{1}{2} + 1 - n + \frac{n}{p} + \frac{n-1}{2r} \geqslant \frac{1}{p} + \frac{n}{2p} - \frac{1}{2p}\,.$$

Hence

$$\frac{n-1}{r} \geqslant n - 1 + \frac{1}{p}(-n+1)$$

and $1/r \geqslant 1 - 1/p$, which yields $r \leqslant p'$.

In the case $p = n/(n-1)$ we choose $f$ and $\varphi$ as above and observe that

$$\|\varphi F_x\|_{\mathrm{BMO}} = \|F_x\|_{\mathrm{BMO}} \geqslant c\,\varepsilon^{(n-1)/2},$$

if $|x_1| \leqslant 1/100$ and $|x_i| \leqslant \sqrt{\varepsilon}/100$, $i = 2, 3, \ldots, n$. If (6) holds we obtain

$$c\,\varepsilon^{(n-1)/2}\,\varepsilon^{(n-1)/2r} \leqslant C\,\varepsilon^{(1+(n-1)/2)/p}$$

and choosing $\varepsilon$ small one finds that

$$\frac{n}{2} - \frac{1}{2} + \frac{n-1}{2r} \geqslant \frac{1}{p} + \frac{n}{2p} - \frac{1}{2p}\,.$$

Using the fact that $p = n/(n-1)$ we get $\dfrac{n-1}{2r} \geqslant \dfrac{n-1}{2n}$ and hence $r \leqslant n = p'$.

It remains to treat the case $n/(n-1) < p \leqslant 2$ and we choose $f$ and $\varphi$ as above. We shall consider values of $x$ with $|x_1| \leqslant 1/100$ and $|x_i| \leqslant \sqrt{\varepsilon}/100$, $i = 2, 3, \ldots, n$. It is clear that $F_x \in C_0^\infty(]0, \infty[)$ and we observe that $F_x(1 - x_1) \geqslant c\,\varepsilon^{(n-1)/2}$ and $F_x^{(k)}(a) = 0$, $k = 0, 1, 2, \ldots$, where $a = 1 - x_1 - 10\,\varepsilon$. It follows from Taylor's formula that

$$F_x(1-x_1) = F_x(a+10\,\varepsilon) = \frac{(10\,\varepsilon)^k}{k!} F_x^{(k)}(\xi)\,,$$

where $a < \xi < a + 10\,\varepsilon$. We conclude that

$$F_x^{(k)}(\xi) \geqslant c\,\varepsilon^{(n-1)/2-k}\,, \tag{16}$$

where $c$ depends on $k$ and $\xi$ on $\varepsilon$, $x$ and $k$. Now let $\beta = n/p' - 1$. If $\beta$ is not an integer we choose $k = [\beta]$ and set $\gamma = \beta - k$. Using (16) we then obtain

$$\|\varphi F_x\|_{\Lambda_\beta} \geqslant c\,\|(\varphi F_x)^{(k)}\|_{\Lambda_{\beta-k}} \geqslant c\,\varepsilon^{(n-1)/2-k}\,\varepsilon^{-\gamma} = c\,\varepsilon^{(n-1)/2-\beta}\,.$$

If $\beta$ is an integer we set $k = \beta - 1$ and then have

$$\|\varphi F_x\|_{\Lambda_\beta} \geqslant c\,\|(\varphi F_x)^{(k)}\|_{\Lambda_1} \geqslant c\,\varepsilon^{(n-1)/2-k}\,\varepsilon^{-1} = c\,\varepsilon^{(n-1)/2-\beta}\,.$$

If (7) holds we obtain

$$c\,\varepsilon^{(n-1)/2-\beta}\,\varepsilon^{(n-1)/2r} \leqslant \Big(\int_{\mathbb{R}^n} \|\varphi F_x\|_{\Lambda_\beta}^r\,dx\Big)^{1/r} \leqslant C\,\varepsilon^{(n+1)/2p}\,.$$

Letting $\varepsilon \to 0$ one finds that

$$\frac{n-1}{2} - \frac{n}{p'} + 1 + \frac{n-1}{2r} \geqslant \frac{n+1}{2p}\,.$$

It follows that

$$\frac{n-1}{r} \geqslant n - 1 + \frac{1}{p}(-n+1)$$

and hence $1/r \geqslant 1/p'$ and $r \leqslant p'$.

*Remark*. The referee has suggested the following extension of Theorem 1. The $L^r$ norms in (5), (6) and (7) can be replaced by the norm in the amalgam space $l^p(L^{p'})$. This turns out to be a consequence of Theorem 1. The norms in these spaces are defined in the following way. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ set

$$E_m = \{x \in \mathbb{R}^n; m_1 \leqslant x_1 \leqslant m_1 + 1, \ldots, m_n \leqslant x_n \leqslant m_n + 1\}\,.$$

Then

$$\|f\|_{l^p(L^q)} = \Big(\sum_m \Big(\int_{E^m} |f(x)|^q\,dx\Big)^{p/q}\Big)^{1/p}$$

if $f$ is a measurable function on $\mathbb{R}^n$.

We shall show how (5) can be extended. Let $p$ and $q$ satisfy the conditions in (5). Let $\tilde{E}_m$ denote the cube $k\,E_m$ with the same center as

$E_m$ and side length $k$. If $k$ is large enough (depending on the support of $\varphi$) the values of $\varphi(t) F_x(t)$ for $x \in E_m$, $t \geqslant 0$, do not change if we replace $f$ by $f\chi_{\tilde{E}_m}$, where $\chi_{\tilde{E}_m}$ is the characteristic function of $\tilde{E}_m$. The inequality (5) therefore yields

$$\left(\int_{E_m} \|\varphi F_x\|_{L^q}^{p'} dx\right)^{1/p'} \leqslant C_p\left(\left(\int_{\tilde{E}_m} |f|^p dx\right)^{1/p}.$$

Summing over $m$ we obtain

$$\sum_m \left(\int_{E_m} \|\varphi F_x\|_{L^q}^{p'} dx\right)^{p/p'} \leqslant C_p \sum_m \int_{\tilde{E}_m} |f|^p dx \leqslant C_p \int |f|^p dx,$$

which gives the extension mentioned above. The inequalities (6) and (7) can be extended in the same way.

## References

[1] BERGH, J., LÖFSTRÖM, J.: Interpolation Spaces. Berlin–Heidelberg–New York: 1976.

[2] FLETT, T. M.: Temperatures, Bessel potentials and Lipschitz spaces. Proc. London Math. Soc. **22**, 385—451 (1971).

[3] HILLE, E., PHILLIPS, R. S.: Functional Analysis and Semi-groups. Providence: Amer. Math. Soc. Coll. Publ. 1957.

[4] SJÖLIN, P.: Regularity and integrability of spherical means. Mh. Math. **96**, 277—291 (1983).

[5] STEIN, E. M.: Singular integrals and differentiability properties of functions. Princeton: Princeton Univ. Press 1970.

P. SJÖLIN
Department of Mathematics
University of Stockholm
Box 6701
113 85 Stockholm, Sweden