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A Metric on the Manifold of Immersions and Its Riemannian Curvature

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Abstract. E. BINZ [1] considered two canonical Riemannian metrics on the space of embeddings of a closed $(n-1)$ dimensional manifold into $\mathbb{R}^n$, and computed the geodesic sprays. Here we consider the space of immersions $\mathrm{Imm}(M, N)$ where M is without boundary, and we compute the covariant derivative (in the form of its connector) and the Riemannian curvature of one of these metrics, the non trivial one. The setting is close to that used by P. MICHOR [2], and we refer the reader to this paper for notation.

1. Some Formulas

1.1. We consider an m-dimensional smooth Riemannian manifold M with metric g and covariant derivative ∇. We denote by ∇^0 an arbitrary covariant derivative.

We define two 1, 2 tensors S^0, Tor^0 and one 0,3 tensor $\nabla^0 g$ as usual

$$\begin{aligned} S^0(u,v) &= \nabla_u v - \nabla^0_u v, \\ \mathrm{Tor}^0(u,v) &= \nabla^0_u v - \nabla^0_v u - [u,v], \qquad (1.1.1.) \\ (\nabla^0_u g)(v,w) &= u(g(v,w)) - g(\nabla^0_u v, w) - g(v, \nabla^0_u w). \end{aligned}$$

The following formula is standard

$$\begin{aligned} g(u, S^0(v,w)) = 1/2 \cdot \{ &-(\nabla^0_u g)(v,w) + (\nabla^0_v g)(w,u) + \\ &+ (\nabla^0_w g)(u,v) - g(\mathrm{Tor}^0(u,v),w) - \qquad (1.1.2) \\ &- g(\mathrm{Tor}^0(v,w),u) + g(\mathrm{Tor}^0(w,u),v)\}. \end{aligned}$$

We define two 1, 3 tensors R, R^0 and one 0,4 tensor $\nabla^{0^2} g$ as usual.

$$\begin{aligned} R(u,v,w) &= \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w, \\ R^0(u,v,w) &= \nabla^0_u \nabla^0_v w - \nabla^0_v \nabla^0_u w - \nabla^0_{[u,v]} w, \qquad (1.1.3.) \end{aligned}$$

$$(\nabla^{0^2}_{u,v}\, g)(w,x) = u(\nabla^0_v\, g)(w,x) - (\nabla^0{}_{\nabla^0_u v}\, g)(w,x) - \\ - (\nabla^0_v\, g)(\nabla^0_u\, w, x) - (\nabla^0_v\, g)(w, \nabla^0_u\, x).$$

The following formula is standard

$$\begin{aligned} & g(R(u,v,w) - R^0(u,v,w), x) = \\ & \quad = g(S^0(u,w), S^0(v,x)) - g(S^0(v,w), S^0(u,x)) + \\ & \quad + 1/2 \cdot \{(\nabla^{0^2}_{u,v}\, g)(w,x) - (\nabla^{0^2}_{v,u}\, g)(w,x) + \\ & \quad + (\nabla^{0^2}_{u,w}\, g)(v,x) - (\nabla^{0^2}_{v,w}\, g)(u,x) + \\ & \quad + (\nabla^{0^2}_{v,x}\, g)(u,w) - (\nabla^{0^2}_{u,x}\, g)(v,w)\}. \end{aligned} \tag{1.1.4.}$$

1.2. Let e denote an arbitrary immersion $N \to M$. Every map $u: N \to TM$ with $\pi \circ u = e$ could be written as

$$u = Te \circ \omega(u) + \varrho(u). \tag{1.2.1.}$$

Here $\omega(u)$ denotes the tangent part a vectorfield on N, and $\varrho(u)$ denotes the normal part with

$$g \circ (Te, \varrho(u)) = 0. \tag{1.2.2.}$$

The following formulas are standard

$$g(u,v) = g^{\perp}(\varrho\, u, \varrho\, v) + e^* g(\omega\, u, \omega\, v), \tag{1.2.3.}$$

$$\begin{aligned} \nabla_u v = {} & Te \circ \nabla^e_u\, \omega(v) + \sigma(u, \omega(v)) - \\ & - Te \circ \sigma^*(u, \varrho(v)) + \nabla^{\perp}_u\, \varrho(v), \end{aligned} \tag{1.2.4.}$$

$$\begin{aligned} R u v = {} & Te \circ \{R^e u v - \sigma^*(u, \sigma(v,.)) + \sigma^*(v, \sigma(u,.))\}\, \omega + \\ & + \{\nabla_u \sigma(v,.) - \nabla_v \sigma(u,.)\}\, \omega + Te \circ \{\nabla_u \sigma^*(v,.) + \nabla_v \sigma^*(u,.)\}\, \varrho + \\ & + \{R^{\perp} u v - \sigma(u, \sigma^*(v,.)) + \sigma(v, \sigma^*(u,.))\}\, \varrho, \end{aligned} \tag{1.2.5}$$

$$\begin{aligned} & g \circ (\sigma(u,v), w) = e^* g(v, \sigma^*(u,w)), \\ & \nabla^e e^* g = 0, \nabla^{\perp} g^{\perp} = 0, \mathrm{Tor}^e = 0. \end{aligned} \tag{1.2.6.}$$

1.3. Let vol denote the volumdensity defined by g

$$g_m(X_1 \wedge X_2 \wedge \ldots \wedge X_k, Y_1 \wedge Y_2 \wedge \ldots \wedge Y_k) = \det((g(X_i, Y_j))_{i,j}) \tag{1.3.1.}$$

$$\mathrm{vol} = \sqrt{g_m}. \tag{1.3.2.}$$

Recall the well known formulas

$$\begin{aligned} &\operatorname{div}(X)\cdot \mathrm{vol} = L_X \mathrm{vol}\,,\\ &\operatorname{div}(X) = \operatorname{tr}(\nabla X)\,,\\ &g(\operatorname{grd}(f), X) = X(f)\,,\\ &\textstyle\int f\cdot \operatorname{div}(X)\cdot \mathrm{vol} + \int g(\operatorname{grd}(f), X)\cdot \mathrm{vol} = 0\,. \end{aligned} \tag{1.3.3.}$$

1.4. In the following we make use of natural mappings on higher tangent bundles. Recall the following 6 mappings

$$\begin{array}{ll} \pi_M : TM \to M & (a,b) \to a \\ 0_M : M \to TM & a \to (a,0) \\ \cdot_M : \mathbb{R}\times TM \to TM & \lambda, (a,b) \to (a, \lambda\cdot b) \\ +_M : TM \times_M TM \to TM & (a,b),(a,b') \to (a, b+b') \\ V_M : TM \times_M TM \to T^2M & (a,b),(a,d) \to (a,b,0,d) \\ \varkappa_M : T^2M \to T^2M & (a,b,c,d) \to (a,c,b,d)\,. \end{array}$$

We need these mappings together with their derivatives. The following notation will be useful.

$$\gamma_{k+1}^{k+l+1} \equiv T^l \gamma_{(T^k M)}\,, \quad (\pi_2^4 = T^2 \pi_{TM})\,.$$

1.5. A covariant derivative may be written as

$$\nabla_u^0 v = k^0 \circ Tv \circ u, \quad (k^0 : T^2M \to TM)\,. \tag{1.5.1.}$$

k^0 is called the *connector* of ∇^0. k^0 is linear in both vector bundle structures of T^2M ($k^0 \circ (+\,^2_1) = (+\,^1_1)\circ(k^0, k^0)\ldots$). Note the following formulas

$$\begin{array}{ll} k^0 \circ V_1^1 = p_2 & TM \times_M TM \to TM \\ S^0 \circ (\pi_1^2, \pi_2^2) = k - k^0 & T^2M \to TM \\ \mathrm{Tor}^0 \circ (\pi_2^2, \pi_1^2) = k^0 \circ \varkappa_1^1 - k^0 & T^2M \to TM \\ Dg = g\circ(k^0, \pi_2^2) + g\circ(\pi_2^2, k) + & \\ \quad + \nabla^0 g \circ (\pi_1^2, \pi_2^2, \pi_2^2) & T(TM\times_M TM) \to \mathbb{R} \end{array} \tag{1.5.2}$$

These formulas are easy consequences of $\nabla^0 fX = f\nabla^0 X + X \cdot Df$ and (1.1.1.). Similarly we get the following

$$
\begin{aligned}
R^0(u,v,w) &= \nabla^0_u \nabla^0_v w - \nabla^0_v \nabla^0_u w - \nabla^0_{[u,v]} w = \\
&= \nabla^0_u (k^0 \circ Tw \circ v) - \nabla^0_v (k^0 \circ Tw \circ u) - k^0 \circ Tw \circ [u,v] = \\
&= k^0 \circ Tk^0 \circ T^2 w \circ Tv \circ u - k^0 \circ Tk^0 \circ T^2 w \circ Tu \circ v - \\
&\quad - k^0 \circ Tk^0 \circ T^2 w \circ V^1_1 \circ (v, [u,v]) = \\
&= k^0 \circ Tk^0 \circ T^2 w \circ Tv \circ u - k^0 \circ Tk^0 \circ T^2 w \circ Tu \circ v - \\
&\quad - k^0 \circ Tk^0 \circ T^2 w \circ Tv \circ u + k^0 \circ Tk^0 \circ T^2 w \circ \varkappa^2_2 \circ Tu \circ v = \\
&= (k^0 \circ Tk^0 \circ \varkappa^2_2 - k^0 \circ Tk^0) \circ T^2 w \circ Tu \circ v.
\end{aligned}
$$

This implies

$$
\begin{aligned}
R^0 \circ (\pi^2_1 \circ \pi^3_3, \pi^2_1 \circ \pi^3_2, \pi^2_2 \circ \pi^3_3) = k^0 \circ Tk^0 \circ \varkappa^2_2 - k^0 \circ Tk^0 \\
T^3 M \to TM.
\end{aligned}
\tag{1.5.3}
$$

1.6. The derivative of the mapping, which maps every metric to its volumdensity was computed by P. MICHOR [2]. We are interested in vol: $\mathrm{Imm}(N,M) \to \Gamma^+(\Lambda^n T^* N \otimes \mathrm{Or}(TN))$, the mapping which maps every immersion to the volumdensity of $e^* g$.

Lemma: $D\,\mathrm{vol}(u) = \{\mathrm{div}^e(\omega(u)) - g(\nu(e), u)\} \cdot \mathrm{vol}(e)$ $\quad (\pi \circ u = e)$ (1.6.1)

Proof: $e^* g = g \circ (Te, Te)$, $(\nu(e) = \mathrm{tr}(e^* g^{-1} \sigma))$.

$$
\begin{aligned}
D(g_* \circ (T,T)) &= Dg_* \circ (\varkappa^1_1{}_* \circ T, \varkappa^1_1{}_* \circ T) = \\
= g_* \circ (\pi^2_2{}_* \circ \varkappa^1_1{}_* \circ T, k_* \circ \varkappa^1_1{}_* \circ T) &+ g_* \circ (k_* \circ \varkappa^1_1{}_* \circ T, \pi^2_2{}_* \circ \varkappa^1_1{}_* \circ T) = \\
&= g_* \circ (T \circ \pi^1_1{}_*, k_* \circ T) + g_* \circ (k_* \circ T, T \circ \pi^1_1{}_*).
\end{aligned}
\tag{1.6.2.}
$$

Now use (1.3.1.), (1.3.2.), and

$$(D\sqrt{\det}(A,B) = 1/2 \cdot \mathrm{tr}(A^{-1} \cdot B) \cdot \sqrt{\det}(A)).$$

Then

$$
\begin{aligned}
D\,\mathrm{vol}(u) &= \mathrm{tr}(e^* g^{-1} \cdot (g \circ (Te, \nabla u) + g \circ (\nabla u, Te))) \cdot \mathrm{vol}(e)/2 = \\
&= \{\mathrm{tr}(\nabla^e \omega(u)) - \mathrm{tr}(e^* g^{-1} \cdot g \circ (u, \varrho(k \circ T^2 e))\} \cdot \mathrm{vol}(e) = \\
&= \{\mathrm{div}^e(\omega(u)) - g(\nu(e), u)\} \cdot \mathrm{vol}(e).
\end{aligned}
$$

Lemma: $D^2 \operatorname{vol}(w) - D \operatorname{vol}(k \circ w) =$

$$= \{- \operatorname{tr}(\omega \nabla_{\omega \nabla v} u) + \operatorname{tr}((e^* g)^{-1} \cdot g^{\perp} \circ (\varrho(\nabla u), \varrho(\nabla v)) + \tag{1.6.3.}$$
$$+ \operatorname{tr}(\omega(R(v, Te, u)) + \operatorname{tr}(\omega \nabla u) \cdot \operatorname{tr}(\omega \nabla v)\} \cdot \operatorname{vol}(e)$$
$$(w: N \to T^2 M,\ \pi_2^2 w = u,\ \pi_2^2 w = v,\ \pi_1^1 \circ u = \pi_1^1 \circ v = e)$$

Proof:

$$D^2(g_* \circ (T, T)) = D(g_* \circ (k_* \circ T, T \circ \pi_1^1 *) + g_* \circ (T \circ \pi_1^1 *, k_* \circ T)) =$$
$$= D g_* \circ (T k_* \circ \varkappa_2^2 * \circ T,\ \varkappa_1^1 * \circ T \circ \pi_1^2 *) +$$
$$+ D g_* \circ (\varkappa_1^1 * \circ T \circ \pi_1^2 *, T k_* \circ \varkappa_2^2 * \circ T) =$$
$$= g_* \circ (k_* \circ T k_* \circ \varkappa_2^2 * \circ T,\ \pi_2^2 * \circ \varkappa_1^1 * \circ T \circ \pi_1^2 *) +$$
$$+ g_* \circ (\pi_2^2 * \circ T k_* \circ \varkappa_2^2 * \circ T, k_* \circ \varkappa_1^1 * \circ T \circ \pi_1^2 *) +$$
$$+ g_* \circ (k_* \circ \varkappa_1^1 * \circ T \circ \pi_1^2 *,\ \pi_2^2 * \circ T k_* \circ \varkappa_2^2 * \circ T) +$$
$$+ g_* \circ (\pi_2^2 * \circ \varkappa_1^1 * \circ T \circ \pi_1^2 *,\ k_* \circ T k_* \circ \varkappa_2^2 * \circ T) =$$
$$= g_* \circ (k_* \circ T \circ k_*,\ T \circ \pi_1^1 * \circ \pi_2^2 *) + g_* \circ (T \circ \pi_1^1 * \circ \pi_2^2 *, k_* \circ T \circ k_*) +$$
$$+ g_* \circ (R_* \circ (\pi_1^2 *,\ T \circ \pi_1^1 * \circ \pi_2^2 *,\ \pi_2^2 *),\ T \circ \pi_1^1 * \circ \pi_2^2 *) +$$
$$+ g_* \circ (T \circ \pi_1^1 * \circ \pi_2^2 *,\ R_* \circ (\pi_1^2 *, T \circ \pi_1^1 * \circ \pi_2^2 *,\ \pi_2^2)) + \tag{1.6.4.}$$
$$+ g_* \circ (k_* \circ T \circ \pi_1^2 *, k_* \circ T \circ \pi_2^2 *) + g_* \circ (k_* \circ T \circ \pi_2^2 *,\ k_* \circ T \circ \pi_1^2 *).$$

Now use

$$D^2 \sqrt{\det}(A, B, C, D) = \{\operatorname{tr}(A^{-1} \cdot D)/2 - \operatorname{tr}(A^{-1} \cdot C \cdot A^{-1} \cdot B)/2$$
$$+ \operatorname{tr}(A^{-1} \cdot B) \operatorname{tr}(A^{-1} \cdot C)/4\} \sqrt{\det(A)},$$
$$D^2 \operatorname{vol}(w) = \{\operatorname{tr}(e^* g^{-1} \cdot (g \circ (\nabla u, \nabla v) + g \circ (\nabla v, \nabla u) +$$
$$+ g \circ (\nabla(k \circ w), Te) + g \circ (Te, \nabla(k \circ w)) +$$
$$+ g \circ (R \circ (v, Te, u), Te) + g \circ (Te, R \circ (v, Te, u)))/2 -$$
$$- \operatorname{tr}(e^* g^{-1} \cdot (g \circ (\nabla u, Te) + g \circ (Te, \nabla u)) \cdot$$
$$\cdot e^* g^{-1} \cdot (g \circ (\nabla v, Te) + g \circ (Te, \nabla v)))/2 +$$
$$+ \operatorname{tr}(e^* g^{-1} \cdot (g \circ (Te, \nabla u))) \operatorname{tr}(e^* g^{-1} \cdot (g \circ (Te, \nabla v))\} \cdot \operatorname{vol}(e).$$

2. The Metric G

2.1. For $u, v \in \Gamma_c(e^* TM)$ we define

$$G(u, v) = \int g \circ (u, v) \cdot \mathrm{vol}(e).$$

G is a positive definite symmetric two form on the space of immersions $\mathrm{Imm}(N, M)$ (N without boundary). G is invariant under the action of $\mathrm{Diff}(N)$ by composition from the right. G is not complete on $\mathrm{Imm}(N, M)$ since there are curves which leave $\mathrm{Imm}(N, M)$ in a finite time. G is C_C^∞ since every mapping contained in it is C_C^∞.

Define ∇^0 by $\nabla_X^0 Y = k_* \circ TY \circ X \quad (k^0 = k_*)$

Lemma: $\mathrm{Tor}^0 = 0$, $(\nabla_X^0 G)(Y, Z) = \int g \circ (Y, Z) \cdot D\,\mathrm{vol} \circ X$

and (2.1.1.)

$$(\nabla_{X,Y}^{0^2} G)(Z, U) = \int g \circ (Y, Z) \cdot \nabla_{X,Y}^{0^2} \mathrm{vol}.$$

2.2. Theorem: *The connection of G exists, and*

$$S^0(v, w) = 1/2 \cdot \{v \cdot \mathrm{div}(\omega(w)) + w \cdot \mathrm{div}(\omega(v)) + Te \circ \mathrm{grd}(g \circ (v, w)) - - v \cdot g \circ (\nu(e), w) - w \cdot g \circ (\nu(e), v) + \nu(e) \cdot g \circ (v, w)\}. \quad (2.2.1.)$$

$$(\pi \circ v = \pi \circ w = e)$$

Proof: (1.1.2.), (2.1.1.) imply

$$G(u, S^0(v, w)) = 1/2 \cdot \int \{- g \circ (v, w) \cdot D\,\mathrm{vol}(u) + g \circ (w, u) \cdot D\,\mathrm{vol}(v) + g \circ (u, v) \cdot D\,\mathrm{vol}(w)\}.$$

(1.6.1.) implies

$$G(u, S^0(v, w)) = 1/2 \cdot \int \{g \circ (v, w) \cdot (g \circ (\nu(e), u) - \mathrm{div}(\omega(u))) + + g \circ (w, u) \cdot (\mathrm{div}(\omega(v)) - g \circ (\nu(e), v)) + + g \circ (u, v) \cdot (\mathrm{div}(\omega(w)) - g \circ (\nu(e), w))\} \cdot \mathrm{vol}(e).$$

(1.3.3.) implies

$$G(u, S^0(v, w)) = 1/2 \cdot \int g \circ (u, \{\nu(e) \cdot g \circ (v, w) + Te \circ \mathrm{grd}(g \circ (v, w)) + w \cdot \mathrm{div}(\omega(v)) - w \cdot g \circ (\nu(e), v) + + v \cdot \mathrm{div}(\omega(w)) - v \cdot g \circ (\omega(e), w)\}) \cdot \mathrm{vol}(e).$$

2.3. Theorem: *The Riemannian curvature of G satisfies*

$$
\begin{aligned}
&G(R(u, v, w) - R \circ (u, v, w), x) = \\
&= g(S^0(u, w), S^0(v, x)) - G(S^0(v, w), S^0(u, x)) + \\
&+ 1/2 \cdot \int \{ g \circ (v, x) \cdot \nabla^{02}_{u, v} \operatorname{vol} - g \circ (u, x) \cdot \nabla^{02}_{u, w} \operatorname{vol} + \\
&+ g \circ (u, w) \cdot \nabla^{02}_{v, x} \operatorname{vol} - g \circ (v, w) \cdot \nabla^{02}_{u, x} \operatorname{vol} \} .
\end{aligned}
\tag{2.3.1.}
$$

Proof: From (1.1.4.), (2.1.2.) and

$$
\begin{aligned}
&R^0 \circ (\pi_1^2 \circ \pi_3^3, \pi_1^2 \circ \pi_2^3, \pi_2^2 \circ \pi_3^3)_* = k_* \circ Tk_* \circ \varkappa_{2*}^2 - k_* \circ Tk_* = \\
&= (k \circ Tk \circ \varkappa_2^2 - k \circ Tk)_* = (R \circ (\pi_1^2 \circ \pi_3^3, \pi_1^2 \circ \pi_2^3, \pi_2^2 \circ \pi_3^3))_*
\end{aligned}
$$

we obtain

$$R^0 = R_* .$$

References

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