Werk

Titel: The Bott Obstruction to the Existence of Nice Polarizations.

Autor: VAISMAN, Izu

Jahr: 1981

PURL: https://resolver.sub.uni-goettingen.de/purl?362162050_0092|log21

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Mh. Math. 92, 231—238 (1981)

Monatshefte für
Mathematik
© by Springer-Verlag 1981

The Bott Obstruction to the Existence of Nice Polarizations

By

Izu Vaisman, Haifa

(Received 7 January 1981)

Abstract. In this note, we prove and discuss the following theorem: if the symplectic or almost symplectic manifold M admits a nice polarization F with $\dim(F \cap \bar{F}) = k \neq 0$, we must have Chern$^q M = 0$ and Pont$^q M = 0$, for $q > 2n - k$, and, in particular, the Euler class of M vanishes.

1. Introduction

In this note we will be referring to a $2n$-dimensional either symplectic or almost symplectic manifold M with fundamental (nondegenerate) 2-form ω, which possesses a (in general complex) polarization F, good to be used in geometric quantization (when possible). This means that F is a Nirenberg integrable subbundle of the complexified tangent bundle $T^c M$, which is Lagrangian (maximally isotropic) with respect to ω, and for which the real integrable distribution D defined by $F \cap \bar{F} = D^c$ (the bar means complex conjugation) yields a k-dimensional regular foliation of M with a Hausdorff quotient manifold M/D. The reader can find more details about this structure in any book or paper on geometric quantization, e. g. [2], [9], [13], etc. It is important to note that usually the structure is defined for symplectic manifolds only ($d\omega = 0$). But here we are refering to such aspects of the problem which do not involve the closedness of ω. This makes our results valid on any almost symplectic manifold. Under the above mentioned hypothesis, we shall say that F is a *nice polarization* of M with the *real index* k.

Let us recall that every almost symplectic manifold has almost complex structures whose Chern classes are well defined. They will be called the *Chern classes* of M, and will be denoted by $c_i(M)$ (e. g. [8]). They generate the graded *Chern ring* Chern* M. Let us also denote by Pont* M the graded ring generated by the Pontryagin classes of M,

0026—9255/81/0092/0231/$01.60

and by $e(M)$ the Euler class of M. All the characteristic classes considered in this note are real characteristic classes.

Using a well known method of BOTT [4, 6] we shall prove here

Theorem. *If the symplectic or almost symplectic manifold M^{2n} admits a nice polarization F of real index $k \neq 0$, we must have* $\text{Chern}^q M = 0$ *for* $q > 2n - k$. *In particular, we must have* $\text{Pont}^q M = 0$ *for* $q > 2n - k$, *and* $e(M) = 0$.

The second part of this theorem follows from the first part since $\text{Pont}^q M \subset \text{Chern}^q M$ and $e(M) = c_n(M) \in \text{Chern}^{2n} M$, by some well known theorems about characteristic classes (e. g. Theorem 4.5.1 and Formula 4.11 (16) of [5]). So we shall have to prove the first part only.

On the other hand, the second part of Theorem implies

Corollary. *A compact symplectic (or almost symplectic) manifold with non-vanishing Euler-Poincaré characteristic has no nice polarizations with $k \neq 0$. In particular, a compact Kähler manifold of positive sectional curvature has no nice polarizations with $k \neq 0$.*

The second part of this corollary follows from the first part since, by a result of [3], the hypotheses assumed imply the fact that the manifold has a positive Euler—Poincaré characteristic. Note also that the first part yields a new proof of SIMMS' result [9] concerning the non existence of nice polarizations with $k \neq 0$ on $S^2 \times S^2$ with the canonical symplectic structure.

2. Proof of Theorem

The proof is very simple. However we must make it explicit. Let us consider the following index conventions:

$$a, b, \ldots = 1, \ldots, k;\ \alpha, \beta, \ldots = k+1, \ldots, n;$$
$$\bar{\alpha}, \bar{\beta}, \ldots = n+1, \ldots, 2n-k;\ u, v, \ldots = 2n-k+1, \ldots, 2n.$$

By the Nirenberg—Frobenius-theorem [7], the manifold M of Theorem has an atlas $(y^a, z^\alpha, z^{\bar{\alpha}} = \bar{z}^{\bar{\alpha}-h}, y^u)$ $(h = n - k)$, consisting of the real coordinates y^a, y^u, and of the complex coordinates $z^\alpha, \bar{z}^\alpha$, such that F has the local bases $\{\partial/\partial y^a, \partial/\partial \bar{z}^\alpha\}$. The transition functions of this atlas have the local form [13]

$$\tilde{y}^a = \tilde{y}^a(y^b, z^\alpha, \bar{z}^\alpha, y^u),\ \tilde{z}^\alpha = \tilde{z}^\alpha(z^\beta, y^u),\ \tilde{y}^u = \tilde{y}^u(y^v). \tag{1}$$

The atlas is adapted to the foliation D, which has the local bases

$\{\partial/\partial y^a\}$, and the local equations $z^\alpha = \text{const.}$, $\bar{z}^\alpha = \text{const.}$, $y^u = \text{const.}$ Moreover, if D is regular $(z^\alpha, \bar{z}^\alpha, y^u)$ are local coordinates on the quotient manifold M/D. It is also worthwhile noting that $y^u = \text{const.}$ define another foliation $E \supset D$ such that $E^c = F + \bar{F}$. Finally, because of the isotropy property of F, the fundamental form of M can be written as [13]

$$\omega = A_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta + \theta_u \wedge dy^u, \tag{2}$$

where $\bar{A}_{\alpha\beta} = -A_{\beta\alpha}$ is a nondegenerate matrix, and

$$\theta_u = B_{ua} dy^a + B_{u\alpha} dz^\alpha + \bar{B}_{u\alpha} d\bar{z}^\alpha + B_{uv} dy^v. \tag{3}$$

Here, B_{ua} and B_{uv} are real coefficients, and it is easy to verify that the matrix B_{ua} is nondegenerate and satisfies the transition relation

$$\tilde{B}_{ua} = \frac{\partial y^v}{\partial \tilde{y}^u} \frac{\partial y^b}{\partial \tilde{y}^a} B_{vb}. \tag{4}$$

Now, every function $f: M \to R$ has a *symplectic gradient*, which is the vector field $\text{sg} f$ defined by

$$i(\text{sg} f)\,\omega = df.$$

In particular, using (2) and (3), we obtain in a standard manner

$$\text{sg}\, y^u = \xi^{ua} \frac{\partial}{\partial y^a}, \tag{5}$$

where ξ^{ua} is the inverse matrix of B_{ua}, and, therefore, because of (4), it satisfies the transition relations

$$\tilde{\xi}^{ua} = \frac{\partial \tilde{y}^u}{\partial y^v} \frac{\partial \tilde{y}^a}{\partial y^b} \xi^{vb}. \tag{6}$$

It follows that the foliation D has the local tangent bases $\{\text{sg}\, y^u\}$, for which the transition relations are

$$\text{sg}\, \tilde{y}^v = \frac{\partial \tilde{y}^v}{\partial y^u} \text{sg}\, y^u. \tag{7}$$

It is worthwhile noting that the matrix of (7) is locally constant on the leaves of D (because of (1)), which yields

Proposition. *For an arbitrary polarization F, the leaves of D are locally affine spaces. Moreover, if F is nice (i.e. D is regular) the leaves of D are parallelizable manifolds.*

Indeed, because of (7), the formulas $d(\mathrm{sg}\, y^u) = 0$ define an affine connection of vanishing torsion and curvature on every leaf of D. In the case of a regular D, y^u are the lifts of a part of the local coordinates of M/D, and, hence, they are defined on open subsets of M which are unions of leaves of D. Therefore the fields $\mathrm{sg}\, y^u$ are defined over entire leaves of D, which makes these leaves parallelizable. This proposition is a slight generalization of theorems by ARNOLD—AVEZ [1], A. WEINSTEIN [15], and ŚNIATYCKI—TOPOROWSKI [10]. The proof of the results of [10] is essentially the same as above, but it refers only to a "very nice" F, i. e., one with both foliations D and E regular, and to a symplectic manifold M.

Now, let us come back to our Theorem, and consider the transverse bundle $Q = TM/E$. This can be identified with a direct complement of E (i. e., $TM = E \oplus Q$), and, as such, it has local bases Y_u consisting of local tangent vector fields of M with transition relations

$$\tilde{Y}_u = \frac{\partial y^v}{\partial \tilde{y}^u} Y_v . \tag{8}$$

A comparison of (7) and (8) yields that D is isomorphic to the dual bundle of Q.

Let us also consider an arbitrary Euclidian metric on the bundle Q, defined by the local matrices (h_{uv}) with transition relations

$$\tilde{h}_{uv} = \frac{\partial y^s}{\partial \tilde{y}^u} \frac{\partial y^w}{\partial \tilde{y}^v} h_{sw} . \tag{9}$$

Then, the local vector fields

$$T_u = h_{uv}(\mathrm{sg}\, y^v) \tag{10}$$

define new local bases of D with transition relations

$$\tilde{T}_u = \frac{\partial y^v}{\partial \tilde{y}^u} T_v , \tag{11}$$

and, therefore, provide us with an isomorphism between D and Q.

We shall also need a direct complement $\bar{P}$ of D^c in F. Clearly, $\bar{P}$ will have some local bases $\bar{Z}_\alpha$ with transition functions

$$\tilde{\bar{Z}}_\alpha = \frac{\partial \bar{z}^\beta}{\partial \tilde{\bar{z}}^\alpha} \bar{Z}_\beta . \tag{12}$$

Then, its conjugate bundle P has the conjugate bases $Z_\alpha = \bar{\bar{Z}}_\alpha$.

Now, it is easy to see that the decomposition

$$T^c M = \text{span}\{T_u - \sqrt{-1}\, Y_u, Z_\alpha\} \oplus \text{span}\{T_u + \sqrt{-1}\, Y_u, \bar{Z}_\alpha\} \quad (13)$$

yields an almost complex structure on M, and that, if we denote by Θ the vector bundle with the local bases $\{S_u = T_u - \sqrt{-1}\, Y_u, Z_\alpha\}$, the Chern classes of Θ are precisely the Chern classes of the manifold M.

Obviously, the transition relations for the bases $\{S_u, Z_\alpha\}$ are

$$\tilde{S}_u = \frac{\partial y^v}{\partial \tilde{y}^u} S_v, \quad \tilde{Z}_\alpha = \frac{\partial z^\beta}{\partial \tilde{z}^\alpha} Z_\beta, \quad (14)$$

and, using (1), we see the fundamental fact that the transition matrices are locally constant on the leaves of D. Then, Θ is a foliate bundle in the sense of the foliation theory (e. g. [6]) with a foliate structure fixed by (14), and we could apply the general theorems of [6] in order to deduce the conclusion desired.

However, let us be a bit more precise. Since in our case, D is regular, and with a Hausdorff quotient manifold, we can write $\Theta = \pi^*(\Theta')$, where $\pi: M \to M/D$ is the natural projection, and Θ' is a complex vector bundle on M/D with the same transition relations (14). (This doesn't mean that the fields S_u, Z_α are lifts of fields of M/D!). Let us take a connection Γ' on Θ', and lift it to a connection Γ on Θ. Then, the local curvature forms of Γ contain the differentials dz^α, $d\bar{z}^\alpha$, dy^u only, and the corresponding representatives of the elements of $\text{Chern}^q \Theta$ contain q of the forms dz^α, $d\bar{z}^\alpha$, dy^u, whose total number is $2n - k$. (See, for instance, [4] for the representation of the Chern classes by differential forms.) Hence, if $q > 2n - k$ any differential form which represents an element of $\text{Chern}^q \Theta$ vanishes. This ends the proof.

3. Supplementary Remarks

The proven theorem allows us to follow the usual pattern of foliation theory [4, 6], and to construct *secondary characteristic classes* of an almost symplectic or symplectic manifold M endowed with a nice polarization F of real index $k \neq 0$. This construction can be done for either the complex bundle Θ or the real bundle TM with the foliate structure fixed by (14). For example, a construction for TM could proceed as follows.

Define a differential graded algebra

$$\mathrm{WPol}_n^k = R[x_1, \ldots, x_{n+1}]/(\text{ideal deg} > 2n - k) \otimes \Lambda(y_1, y_3, \ldots, y_{2l+1}) \quad (l = [n/2]),$$

where the first factor is a truncated polynomial algebra in x_i, and $\deg x_i = 2i (i = 1, 2, \ldots, n + 1)$ (note that this is the degree used also in the definition of the ideal of the truncation!) and the second factor is an exterior algebra in y_{2j-1}, and $\deg y_{2j-1} = 4j - 3$ $(j = 1, 2, \ldots, l + 1)$; a differential d will be defined by $dx_i = 0$, $dy_{2j-1} = x_{2j-1}$.

Then, define a homomorphism $\lambda: \mathrm{WPol}_n^k \to \Lambda(M)$, ($\Lambda(M)$ is the algebra of the exterior forms on M, endowed with the usual graduation and the exterior differential) in the following manner: let ∇^0 be a connection on TM which is the pull-back of a connection of the corresponding bundle on M/D, and let ∇^1 be a metric connection on M; compute $\lambda(x_i)$ as the representative with respect to ∇^0 of the Chern class $c_i(T^c M)$, and $\lambda(y_{2j-1})$ as the result of the integration over $t \in [0, 1]$ of the representative with respect to $t\nabla^0 + (1 - t)\nabla^1$ of c_{2j-1} (lift $T^c M$ to $M \times R$); finally, extend λ by linearity.

It is shown in [4] that λ induces a homomorphism λ^* at the cohomology level, and the images of the cohomology classes of WPol_n^k by λ^* in $H_{DR}(M)$ (de Rham cohomology) are called the *secondary characteristic classes of* (M, F), except for those of them which are usual characteristic classes. As in [4] it follows that these classes are independent of the choice of ∇^0 and ∇^1, and they are invariant by a homotopic deformation of F via nice polarizations of the fixed real index k.

A trivial example is offered by the classical symplectic structure of a cotangent bundle $M = T^* N$ with the polarization F tangent to the fibers. In this case TM is isomorphic to the direct sum of the lifts of the tangent and cotangent bundles of N, and we can construct a metric connection on TM, as a sum of a metric connection with its dual on N. Obviously this connection can be used both in place of ∇^0 and ∇^1 above, whence we obtain that all the secondary characteristic classes of $(T^* N, F)$ are zero.

A similar result holds for another interesting case namely, THURSTON's symplectic non-Kähler manifold M^4 [11]. This manifold is easily seen to have a natural nice real Lagrangian foliation F, and a global parallelism which yields the foliate structure corresponding to (14). This implies that all the characteristic classes of (M^4, F) vanish.

The problem of finding examples of nice polarizations with non-vanishing secondary characteristic classes is still open.

Remark. Though not related with out present subject, it is interesting to note that THURSTON's manifold M^4 possesses also a locally conformal symplectic structure [12]. Indeed, M^4 can be seen as R^4/G, where G is the group of the transformations

$$(x^1, x^2, x^3, x^4) \mapsto (x^1 + p, x^2 + q, x^3 + qx^4 + m, x^4 + n),$$

where p, q, m, n are integers. Therefore, we find on M^4 the global 1-forms dx^1, dx^2, $\theta = dx^3 - x^2 dx^4$, dx^4, the symplectic form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ (the Lagrangian foliation which we spoke about is $dx^2 = 0$, $dx^4 = 0$), and the nondegenerate 2-form $\Omega = dx^1 \wedge \theta - d\theta$. Since $d\Omega = -dx^1 \wedge \Omega$, and since dx^1 is closed but not exact, Ω is locally conformal symplectic, but not globally conformal symplectic [12]. I. e., compact locally conformal symplectic manifolds can have a symplectic structure, in contrast to the compact locally conformal Kähler manifolds which cannot have Kähler metrics [14].

References

[1] ARNOLD, V. I., AVEZ, A.: Problèmes Ergodiques de la Mécanique Classique. Paris: Gauthier-Villars. 1967.

[2] BLATTNER, R. J.: Quantization and representation theory. In: Harmonic Analysis on Homogeneous Spaces, pp. 147—165. A. M. S. Proc. Symp. Pure Math. XXVI, Williamstown 1972. Providence, R. I.: Amer. Math. Soc. 1974.

[3] BLOCK, S., GIESIEKER, D.: The positivity of the Chern classes of an ample vector bundle. Invent. Math. **12**, 112—117 (1971).

[4] BOTT, R.: Lectures on characteristic classes and foliations. In: Lectures on Algebraic and Differential Topology. By R. BOTT, S. GITLER, I. M. JONES. pp. 1—94. Lect. Notes Math. 279. New York–Berlin–Heidelberg: Springer. 1972.

[5] HIRZEBRUCH, F.: Topological Methods in Algebraic Geometry. Berlin–Heidelberg–New York: Springer. 1966.

[6] KAMBER, F. W., TONDEUR, PH.: Foliated Bundles and Characteristic Classes. Lect. Notes Math. 493. Berlin–Heidelberg–New York: Springer. 1975.

[7] NIRENBERG, L.: A complex Frobenius theorem. Sem. Anal. Funct. **1**, 172—189 (1958).

[8] RAWNSLEY, J. H.: On the pairing of polarizations. Comm. Math. Phys. **58**, 1—8 (1978).

[9] SIMMS, J. D.: Geometric quantization of energy levels in the Kepler problem. Symp. Math. **14**, 125—137 (1974).

[10] ŚNIATYCKI, J., TOPOROWSKI, S.: On representation spaces in geometric quantization. Int. J. Theor. Phys. **16**, 615—633 (1977).

[11] THURSTON, W. P.: Some simple examples of symplectic manifolds. Proc. Amer. Math. Soc. **55**, 467—468 (1976).

[12] VAISMAN, I.: On locally conformal almost Kähler manifolds. Isr. J. Math. **24**, 338—351 (1976).

[13] VAISMAN, I.: A coordinatewise formulation of geometric quantization.

Ann. Inst. H. Poincaré **31**, 5—24 (1979).

[14] VAISMAN, I.: On locally and globally conformal Kähler manifolds. Trans. Amer. Math. Soc. **262**, 533—541 (1980).

[15] WEINSTEIN, A.: Symplectic manifolds and their Lagrangian submanifolds. Adv. Math. **6**, 329—346 (1971).

Prof. I. VAISMAN
Department of Mathematics
University of Haifa
Mount Carmel
Haifa 31999, Israel