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Homomorphisms of Graphs and of Their Orientations

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With 1 Figure

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Abstract

Consider a set of graphs and all the homomorphisms among them. Change each graph into a digraph by assigning directions to its edges. Some of the homomorphisms preserve the directions and so remain as homomorphisms of the set of digraphs; others do not. We study the relationship between the original set of graph-homomorphisms and the resulting set of digraph-homomorphisms and prove that they are in a certain sense independent. This independence result no longer holds if we start with a proper class of graphs, or if we require that only one direction be given to each edge (unless each homomorphism is invertible, in which case we again prove independence). We also specialize the results to the set consisting of one graph and prove the independence of monoids (groups) of a graph and the corresponding digraph.

§ 1 Introduction

In this section we present the necessary definitions and pose the problems to be solved in the rest of the paper.

The simplest instance of the general structures we shall study is a digraph. A *digraph* X is a set V with a binary relation R; we write $X=(V,R)$. The elements of V are called the *vertices* and the elements of R, the *arcs* of X; we write $V=V(X)$ and $R=R(X)$ if X is not clear from the context. If X, Y are digraphs, a *homomorphism* $f: X\to Y$ is a mapping $f: V(X)\to V(Y)$ for which $(x,x')\in R(X)$ implies $(f(x),f(x'))\in R(Y)$. A homomorphism $X\to X$ is called an *endomorphism* of X, and a bijective endomorphism is termed an *automorphism*. Composition of mappings gives rise to the following homomorphism-related structures: the group of all automorphisms of a digraph X, denoted by $\mathrm{Aut}X$; the monoid of all

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endomorphisms of a digraph X, denoted by $\mathrm{End}\,X$; the category of all homomorphisms of a class C of digraphs, denoted by $\mathrm{Hom}\,C$. (A monoid is a semigroup with a unity; for the definition of a category see [6].)

A *graph* X is a set V with a set E of two-element subsets of V; we write $X=(V,E)$. The elements of V are called the *vertices* and the elements of E the *edges* of X; we write $V=V(X)$ and $E=E(X)$ if X is not clear from the context. For the edges of X we employ the common notation $[v,w]$ in place of $\{v,w\}$. In this paper we shall identify E with the symmetric irreflexive binary relation $\{(v,w)\colon [v,w]\in E\}$; thus each graph is a digraph. Consequently, all the preceding definitions concerning homomorphisms apply directly to graphs as well.

There are some natural ways of changing a digraph into a graph and vice versa. If $X=(V,R)$ is a digraph we define the *underlying graph* of X to be (V,E) where $E=(R\cup R^{-1})-\Delta$. (Here $R^{-1}=\{(v,w)\colon (w,v)\in R\}$ and $\Delta=\{(v,v)\colon v\in V\}$.) If X is a graph there are several possible digraphs for which X is the underlying graph (X itself is one of them). The class of all digraphs with the underlying graph X is denoted by $\mathrm{Di}\,X$. An *orientation* of the graph X is a digraph $(V,R)\in\mathrm{Di}\,X$ for which $R\cap R^{-1}=\emptyset$; the class of all orientations of X is denoted by $\mathrm{Or}\,X$.

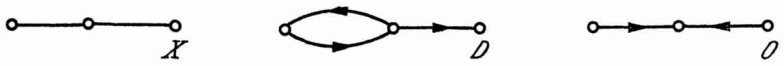

Fig. 1. A graph X and digraphs $D\in\mathrm{Di}\,X$, $O\in\mathrm{Or}\,X$

If C is a class of graphs, we define $\mathrm{Di}\,C$, $\mathrm{Or}\,C$ as follows: $\mathrm{Di}\,C$ consists of all classes $C'=\{\xi(X)\colon X\in C\}$, where ξ is a mapping such that $\xi(X)\in\mathrm{Di}\,X$ for all $X\in C$. Similarly $\mathrm{Or}\,C$ consists of all classes $C'=\{\xi(X)\colon X\in C\}$, where ξ is a mapping such that $\xi(X)\in\mathrm{Or}\,X$ for all $X\in C$.

Thus given a graph X (or a class C of graphs) we pass to a digraph in $\mathrm{Di}\,X$ (or to a class of digraphs in $\mathrm{Di}\,C$) by assigning one or both directions to each edge. Similarly, we pass to a digraph in $\mathrm{Or}\,X$ (or to a class in $\mathrm{Or}\,C$) by assigning exactly one direction to each edge. In the latter case we say that we have oriented the graphs, in the former case we say that we have "dioriented" them. We shall study the possible effect of such changes on the structures

$\mathrm{Aut}X$, $\mathrm{End}X$, and $\mathrm{Hom}C$. In particular, we shall ask the following questions:

(1) Given groups G, G', when can one find a graph X and a digraph $X' \in \mathrm{Di}X$ (resp. $X' \in \mathrm{Or}X$) for which $\mathrm{Aut}X \cong G$ and $\mathrm{Aut}X' \cong G'$?

(2) Given monoids M, M', when can one find a graph X and a digraph $X' \in \mathrm{Di}X$ (resp. $X' \in \mathrm{Or}X$) for which $\mathrm{End}X \cong M$ and $\mathrm{End}X' \cong M'$?

(3) Given categories $\mathscr{C}, \mathscr{C}'$, when can one find a class C of graphs and a class $C' \in \mathrm{Di}C$ (resp. $C' \in \mathrm{Or}C$) of digraphs for which $\mathrm{Hom}C \cong \mathscr{C}$ and $\mathrm{Hom}C' \cong \mathscr{C}'$?

An obvious necessary condition for (1)—(3) is that of being a substructure (that is, G' a subgroup of G, M' a submonoid of M, $\mathscr{C}'$ a spanning subcategory of $\mathscr{C}$). In some cases we shall show this condition to be sufficient, e. g., (1), (2) with $X' \in \mathrm{Di}X$, (3) with $C' \in \mathrm{Di}C$ and $\mathscr{C}$ small. In other cases this condition will not be sufficient, e. g. (2) with orientations, (3) with orientations, or (3) with $\mathscr{C}$ big. However, in all cases where the answer is affirmative, i. e., where one can find such X and X' (resp. C and C'), *one can do so „uniformly"*. That is to say, for a *fixed* G (or M, or $\mathscr{C}$) one can find a *fixed* X (or C) such that for each G' (or M', or $\mathscr{C}'$), a desired X' (or C') exists. Thus for all considered pairs G, G' (or M, M', or $\mathscr{C}, \mathscr{C}'$) with the same G (or M, or $\mathscr{C}$), the solutions X, X' (or C, C') can have the same X. To illustrate: for any fixed group G one can find a graph X such that $\mathrm{Aut}X \cong G$ and such that for each subgroup G' of G a suitable orientation X' of X satisfies $\mathrm{Aut}X' \cong G'$.

This work extends some results of [7], and the theorems proved here were first announced (along with many related results) in [5].

§ 2 Categories

In this section we deduce the general formulations of our results which are then specialized to monoids and groups in the following sections. The reader unfamiliar with the terminology of category theory is referred to [6], or advised to proceed to § 3.

Let $\mathscr{C}$ be a category. We denote by $O_{\mathscr{C}}$ the class of objects of $\mathscr{C}$ and by $M_{\mathscr{C}}$ the class of morphisms of $\mathscr{C}$. We often abbreviate these

to O and M if the context makes $\mathscr{C}$ clear. For $a,b \in O_{\mathscr{C}}$ we denote by $M_{\mathscr{C}}(a,b)$ the set of morphisms $a \to b$ in $\mathscr{C}$; we also write $M_{\mathscr{C}}(a,*)$ for $\bigcup_{x \in O} M_{\mathscr{C}}(a,x)$, and $M_{\mathscr{C}}(*,b)$ for $\bigcup_{x \in O} M_{\mathscr{C}}(x,b)$. The category $\mathscr{C}$ is termed *small* if $O_{\mathscr{C}}$ is a set, and *big* if $O_{\mathscr{C}}$ is a proper class.

We shall say that $\mathscr{C}'$ is a *full* subcategory of $\mathscr{C}$ if $O_{\mathscr{C}'} \subset O_{\mathscr{C}}$ and $M_{\mathscr{C}'}(a,b) = M_{\mathscr{C}}(a,b)$ for all $a,b \in O_{\mathscr{C}'}$; we shall say that $\mathscr{C}'$ is a *spanning* subcategory of $\mathscr{C}$ if $O_{\mathscr{C}'} = O_{\mathscr{C}}$ and $M_{\mathscr{C}'} \subset M_{\mathscr{C}}$. A *full embedding* $\mathscr{C} \to \mathscr{D}$ is an isomorphism of $\mathscr{C}$ onto a full subcategory of $\mathscr{D}$ (i. e., a full faithful covariant functor); a *spanning embedding* $\mathscr{C} \to \mathscr{D}$ is an isomorphism of $\mathscr{C}$ onto a spanning subcategory of $\mathscr{D}$ (i. e., a faithful covariant functor onto $O_{\mathscr{D}}$).

Let $\mathscr{R}$ denote the category of all digraphs and their homomorphisms, and $\mathscr{G}$ the category of all graphs and their homomorphisms. Clearly $\mathscr{G}$ is a full subcategory of $\mathscr{R}$. We shall need more general categories introduced below. Let I be a fixed set. The relational category $I\mathscr{R}$ is defined as follows: objects of $I\mathscr{R}$ are pairs $(V,(R_i)_{i \in I})$, where V is a set and $(R_i)_{i \in I}$ a family of binary relations on V. A mapping $f\colon V \to W$ is a morphism $(V,(R_i)_{i \in I}) \to (W,(S_i)_{i \in I})$ if and only if $(x,x') \in R_i$ implies $(f(x),f(x')) \in S_i$. Note that $I\mathscr{R} = \mathscr{R}$ when $|I| = 1$. We extend the terminology of $\mathscr{R}$ to $I\mathscr{R}$: the elements of V are called *vertices*, the elements of R_i, *arcs of relation* R_i, and so on.

In this section, the symbols $I\mathscr{R}$, $\mathscr{R}$, or $\mathscr{G}$, with a subscript will always denote a full subcategory of $I\mathscr{R}$, $\mathscr{R}$, or $\mathscr{G}$ respectively. Given a category $\mathscr{G}_0$ we define the class $\mathrm{Di}\,\mathscr{G}_0$ of *full* subcategories of $\mathscr{R}$ as follows: $\mathscr{R}_0 \in \mathrm{Di}\,\mathscr{G}_0$ if and only if $O_{\mathscr{R}_0} \in \mathrm{Di}\,O_{\mathscr{G}_0}$. Thus $\mathrm{Di}\,\mathscr{G}_0$ consist of all categories obtained from $\mathscr{G}_0$ by selecting one or both directions for each edge ("diorienting"), and by retaining only those homomorphisms that preserve these directions. The class $\mathrm{Or}\,\mathscr{G}_0$ is defined analogously.

For any $\mathscr{R}_0 \in \mathrm{Di}\,\mathscr{G}_0$ we define the *symmetrization functor* $\gamma\colon \mathscr{R}_0 \to \mathscr{G}_0$ as follows: $\gamma(X)$ is the underlying graph of $X \in O_{\mathscr{R}_0}$ and $\gamma(f) = f$ for $f \in M_{\mathscr{R}_0}$. Each symmetrization functor is a spanning embedding. Consequently, we know that any $\mathscr{R}_0 \in \mathrm{Di}\,\mathscr{G}_0$ is isomorphic to a spanning subcategory of $\mathscr{G}_0$. Conversely, Theorem 1 asserts that for some $\mathscr{G}_1 \cong \mathscr{G}_0$ we also know that each spanning subcategory of $\mathscr{G}_1$ is isomorphic to some $\mathscr{R}_1 \in \mathrm{Di}\,\mathscr{G}_1$.

Theorem 1. *Let $\mathscr{C}$ be a small category. There exists a category $\mathscr{G}_{\mathscr{C}}$ isomorphic to $\mathscr{C}$ such that each spanning subcategory $\mathscr{C}'$ of $\mathscr{C}$ is isomorphic to some $\mathscr{R}_{\mathscr{C}'} \in \mathrm{Di}(\mathscr{G}_{\mathscr{C}})$.*

(*Furthermore, there exists an isomorphism* $\varphi: \mathscr{C} \to \mathscr{G}_{\mathscr{C}}$ *such that, for any* $\mathscr{C}'$, *a suitable restriction* (*cf. below*) φ' *of* φ *is an isomorphism* $\mathscr{C}' \to \mathscr{R}_{\mathscr{C}'}$.)

Proof. We shall construct isomorphisms φ_1, φ_2 such that for any spanning embedding α there exist isomorphisms φ_1', φ_2' and spanning embeddings β, γ for which the diagram

$$\begin{array}{ccccc} \mathscr{C} = (0, M) & \xrightarrow{\varphi_1} & (M\mathscr{R})_{\mathscr{C}} & \xrightarrow{\varphi_2} & \mathscr{G}_{\mathscr{C}} \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ \mathscr{C}' = (0', M') & \xrightarrow[\varphi_1']{} & ((M \dot\cup M')\mathscr{R})_{\mathscr{C}'} & \xrightarrow[\varphi_2']{} & \mathscr{R}_{\mathscr{C}'} \end{array}$$

commutes, i. e., $\varphi_2' \circ \varphi_1' = \varphi'$ is a restriction of $\varphi = \varphi_2 \circ \varphi_1$.

The definition of φ_1 (cf. [1]): For $a \in O$ we put $\varphi_1(a) = (M(*,a), (R_n)_{n \in M})$, where $R_n = \{(f,g): g = f \circ n\}$ for each $n \in M$. For $m \in M(a,b)$ we let $\varphi_1(m)$ be the mapping $M(*,a) \to M(*,b)$ defined by $\varphi_1(m)(f) = m \circ f$. Note that $\varphi_1(m)$ is a morphism of $M\mathscr{R}$, since $(f,g) \in R_n$ implies $g = f \circ n$ and so $(\varphi_1(m)(f), \varphi_1(m)(g)) = (m \circ f, m \circ g) \in R_n$. The functor φ_1 is faithful, as $\varphi_1(m)(1_a) = m \neq m' = \varphi_1(m')(1_a)$ for distinct $m, m' \in M(a,b)$. Moreover, φ_1 is a full functor because any morphism $\xi: \varphi_1(a) \to \varphi_1(b)$ in $M\mathscr{R}$ satisfies $(\xi(1_a), \xi(n)) \in R_n$ for all $n \in M(*,a)$; thus $\xi(n) = \xi(1_a) \circ n$ for all $n \in M(*,a)$, i. e., $\xi = \varphi_1(\xi(1_a))$. Consequently, φ_1 is a full embedding $\mathscr{C} \to M\mathscr{R}$, i. e., an isomorphism $\mathscr{C} \to M\mathscr{R}_{\mathscr{C}}$ for some full subcategory $M\mathscr{R}_{\mathscr{C}}$ of $M\mathscr{R}$.

The definition of φ_2 (cf. e. g. [2, 3, 4, 5]): We shall use the colimit construction ([3, 4]) in which each arc of a fixed relation R_n is replaced by a copy of the same graph. According to [4, Theorem 1] there exists a family $((S_n; s_n, t_n))_{n \in M}$ of graphs that can be used for such replacements; each arc of relation R_n is replaced by a copy of S_n identifying s_n with its beginning and t_n with its end. (A minor technical difficulty arises because of loops in some relations R_n; we will discuss this at the end of the present proof, cf. Remark 2.) Moreover, we may assume that each S_n contains two adjacent vertices p_n, q_n. Let one such pair p_n, q_n be fixed in each $S_n, n \in M$.

Let $\alpha: \mathscr{C}' \to \mathscr{C}$ be a given spanning embedding.

The definition of φ_1': For $a \in O'$ we put

$$\varphi_1'(a) = (M_{\mathscr{C}}(*, \alpha(a)), (R_n)_{n \in M}, (R_k')_{k \in M'}),$$

where $R_n = \{(f,g): g = f \circ n \text{ in } \mathscr{C}\}$ for $n \in M$,

$R_k' = \{(\alpha(f'), \alpha(g')): g' = f' \circ k \text{ in } \mathscr{C}'\}$ for $k \in M'$.

(Note that $\varphi_1'(a)$ differs from $\varphi_1(\alpha(a))$ only in having some additional relations.) For $m \in M_{\mathscr{C}'}(a,b)$ we let $\varphi_1'(m)$ be the mapping $M_{\mathscr{C}}(*,\alpha(a)) \to M(*,\alpha(b))$ defined by $\varphi_1'(m)(f) = \alpha(m) \circ f$. As before, $\varphi_1'(m)$ is a morphism of $(M \dot\cup M')\mathscr{R}$, since $(f,g) \in R_n$ implies $g = f \circ n$ in $\mathscr{C}$ and so $(\varphi_1'(m)(f), \varphi_1'(m)(g)) = (\alpha(m) \circ f, (\alpha(m) \circ f) \circ n) \in R_n$ for all $n \in M$, and since $(\alpha(f'), \alpha(g')) \in R_k'$ implies $g' = f' \circ k$ in $\mathscr{C}'$ and so $(\varphi_1'(m)(\alpha(f')), \varphi_1'(m)(\alpha(g'))) = (\alpha(m \circ f'), \alpha((m \circ f') \circ k)) \in R_k'$ for all $k \in M'$. Since α is a spanning embedding, any two distinct m, m' in $M_{\mathscr{C}'}(a,b)$ satisfy $\varphi_1'(m)(1_{\alpha(a)}) = \alpha(m) \neq \alpha(m') = \varphi_1'(m')(1_{\alpha(a)})$, thus φ_1' is a faithful functor. To prove that φ_1' is full, we note that if $\xi: \varphi_1'(a) \to \varphi_1'(b)$ is a morphism in $(M \dot\cup M')\mathscr{R}$, then $(\xi(1_{\alpha(a)}), \xi(n)) \in R_n$ for all $n \in M_{\mathscr{C}}(*,\alpha(a))$ and $(\xi(\alpha(1_a)), \xi(\alpha(k))) \in R_k'$ for all $k \in M_{\mathscr{C}'}(*,a)$. The first condition implies that $\xi(n) = \xi(1_{\alpha(a)}) \circ n$ for all $n \in M_{\mathscr{C}}(*,\alpha(a))$; the second condition implies that $\xi(\alpha(1_a)) = \xi(1_{\alpha(a)}) = \alpha(f')$ for some $f' \in M_{\mathscr{C}'}(a,b)$. Consequently $\xi = \varphi_1'(f')$, and φ_1' is an isomorphism $\mathscr{C}' \to ((M \dot\cup M')\mathscr{R})_{\mathscr{C}'}$.

The definition of β: Since φ_1, φ_1' are isomorphisms and α a spanning embedding, the functor $\beta = \varphi_1' \circ \alpha \circ (\varphi_1)^{-1}$ is a spanning embedding as required. Note that $\beta(M(*,\alpha(a)), (R_n)_{n\in M}, (R_k')_{k\in M'}) = (M(*,\alpha(a)), (R_n)_{n\in M})$.

Remark 1. So far, we have seen that the effect of judiciously adding relations to the objects of $M\mathscr{R}$ is the elimination of undesired morphisms. We wish to stress, however, that the action we have performed is "doubling some of the existing relations" rather than "adding completely new ones". To be specific, we note that $R_k' \subset R_{\alpha(k)}$, i. e., arcs of the relation R_k' are only present where there "already are" arcs of the relations $R_{\alpha(k)}$. (If $(f,g) \in R_{\alpha(k)}$, i. e., $g = f \circ \alpha(k)$ in $\mathscr{C}$, then $(f,g) \in R_k'$ iff $f = \alpha(f')$ and $g = \alpha(g')$.) This observation will be crucial in defining φ_2'.

The definition of φ_2': In the definition of φ_2 we used a family of graphs $((S_n; s_n, t_n))_{n\in M}$ and we distinguished a pair of vertices p_n, q_n in each $S_n, n \in M$. Let $\tilde S_n$ be an orientation of S_n, such that (p_n, q_n) is an arc of $\tilde S_n$, for each $n \in M$. Since the family $((S_n; s_n, t_n))_{n\in M}$ can be used for the colimit construction on $(M\mathscr{R})_{\mathscr{C}}$ (i. e., can be used for replacing the arcs of objects of $(M\mathscr{R})_{\mathscr{C}}$), so can the family $((\tilde S_n; s_n, t_n))_{n\in M}$. The result of such a replacement will be an orientation of the result of the corresponding replacement by $((S_n; s_n, t_n))_{n\in M}$. Let $\varphi_2'((M(*,\alpha(a)), (R_n)_{n\in M}, (R_k')_{k\in M'}))$ be the result of a replacement of arcs of $(M(*,\alpha(a)), (R_n)_{n\in M})$ by

$((\tilde{S}_n; s_n, t_n))_{n\in M}$ with the following modification: whenever $(f,g)\in R'_k$, then in addition to the arc $(p_{\alpha(k)}, q_{\alpha(k)})$ in the copy of $\tilde{S}_{\alpha(k)}$ corresponding to the arc $(f,g)\in R_{\alpha(k)}$, we *also* introduce the arc $(q_{\alpha(k)}, p_{\alpha(k)})$. The image $\varphi'_2(m)$ of a morphism m is determined by the colimit embedding — it takes the copy of $\tilde{S}_n$ corresponding to $(f,g)\in R_n$ (with the added arc where it exists), to the copy of $\tilde{S}_n$ corresponding to $(m(f), m(g))\in R_n$. Note that a morphism of $\mathscr{R}_{\mathscr{C}'}$ can take a copy of $\tilde{S}_n$ with the added arc only to a copy of $\tilde{S}_n$ with the added arc, just as a morphism of $((M \dot\cup M')\mathscr{R})_{\mathscr{C}}$, can take an arc in $R_{\alpha(k)} \cap R'_k$ only to an arc $R_{\alpha(k)} \cap R'_k$. It follows easily that φ'_2 is an isomorphism.

To end the proof we observe that $\gamma \circ \varphi'_2 = \varphi_2 \circ \beta$.

Remark 2. We return now to the difficulty concerning loops. In [4] the graphs $(S_n; s_n, t_n)$ were constructed for use in colimit embeddings on *irreflexive* relational systems (i. e., without loops), however some of our relations R_n are not irreflexive. One way to resolve this problem would be to modify the graphs S_n so that they can be used even for non-irreflexive relations; similar modifications exist, cf. [2]. A simpler solution is obtained by constructing two further isomorphisms $\varphi_{1.5}, \varphi'_{1.5}$ which take $(M\mathscr{R})_{\mathscr{C}}$, $((M \dot\cup M')\mathscr{R})_{\mathscr{C}'}$ to categories with only irreflexive relations, and then continuing as outlined earlier.

$$\begin{array}{ccccccc}
\mathscr{C} & \xrightarrow{\varphi_1} & (M\mathscr{R})_{\mathscr{C}} & \xrightarrow{\varphi_{1.5}} & (2M\mathscr{R})_{\mathscr{C}} & \xrightarrow{\varphi_2} & \mathscr{G}_{\mathscr{C}} \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma} \\
\mathscr{C}' & \xrightarrow[\varphi'_1]{} & ((M \dot\cup M')\mathscr{R})_{\mathscr{C}'} & \xrightarrow[\varphi'_{1.5}]{} & (2(M \dot\cup M')\mathscr{R})_{\mathscr{C}'} & \xrightarrow[\varphi'_2]{} & \mathscr{R}_{\mathscr{C}'}
\end{array}$$

In the above commutative diagram, φ_1, φ_2 and (given α) $\varphi'_1, \varphi'_2, \beta_1, \gamma$ have already been constructed. (We write $2S$ for the set $S \dot\cup S$.) The isomorphisms $\varphi_{1.5}, \varphi'_{1.5}$ are colimit embeddings which replace each arc of a relation R_x by two consecutive arcs, one of $R_x^{(1)}$ and one of $R_x^{(2)}$. The effect of such a replacement is the elimination of loops; the resulting categories have twice as many relations and are isomorphic to the original ones (cf. [5]). The definition of β_2 and the fact that $\varphi_{1.5} \circ \beta_1 = \beta_2 \circ \varphi'_{1.5}$ are both obvious.

Remark 3. Theorem 1 answers, in part, question (3) of the Introduction: If $\mathscr{C}$ is a *small category*, then given categories $\mathscr{C}, \mathscr{C}'$ one can find classes C, C' with $C' \in \mathrm{Di}\, C$ and $\mathrm{Hom}\, C \cong \mathscr{C}$, $\mathrm{Hom}\, C' \cong \mathscr{C}'$ if and only if $\mathscr{C}'$ is a spanning subcategory of $\mathscr{C}$. When $\mathscr{C}$ is a *big*

category this is no longer true: Let $\mathscr{I}$ be a category with a proper class O of objects and with a unique morphism $i_{a,b}$ for each $a,b \in O$; the composition is defined so that $i_{a,a} = 1_a$ and $i_{a,b} = i_{b,a}^{-1}$. Let $\mathscr{D}$ be a spanning subcategory of $\mathscr{I}$ with only the morphisms $1_a, a \in O$. Clearly, if $\mathrm{Hom}\, C \cong \mathscr{I}$ then C consists of a proper class of isomorphic copies of the same graph $X = (V,E)$ and so in any $C' \in \mathrm{Di}\, C$ we have at most a set (of cardinality 2^E) of non-isomorphic digraphs; hence $\mathrm{Hom}\, C' \not\cong \mathscr{D}$. By the same argument we conclude: If $\mathscr{C}$ is a big category with isomorphisms between all pairs of objects, and $\mathscr{C}'$ a spanning subcategory of $\mathscr{C}$ with no isomorphism between any pair of distinct objects, then there does not exist a class C of graphs and a class $C' \in \mathrm{Di}\, C$ with $\mathrm{Hom}\, C \cong \mathscr{C}$, $\mathrm{Hom}\, C' \cong \mathscr{C}'$.

We shall use the term *groupoid* to designate a category in which every morphism is an isomorphism. The terms subgroupoid, spanning groupoid, etc., have their obvious meanings.

Theorem 2. *Let $\mathscr{C}$ be a small category. There exists a category $\mathscr{G}_{\mathscr{C}}$ isomorphic to $\mathscr{C}$ such that each spanning subgroupoid $\mathscr{C}'$ of $\mathscr{C}$ is isomorphic to some $\mathscr{R}_{\mathscr{C}'} \in \mathrm{Or}(\mathscr{G}_{\mathscr{C}})$.*

Proof. We assume that $\mathscr{C}'$ is a groupoid and proceed, exactly as in the proof of Theorem 1, to obtain a class of graphs C and of digraphs C' such that $C' \in \mathrm{Di}\, C$ and $\mathrm{Hom}\, C = \mathscr{G}_{\mathscr{C}} \cong \mathscr{C}$, $\mathrm{Hom}\, \mathscr{C}' = \mathscr{R}_{\mathscr{C}'} \cong \mathscr{C}'$. The class C' is almost an orientation of C; only the added arcs (q_n, p_n) stand in the way. Observe that whenever $f: X \to Y$ is a morphism in $\mathscr{R}_{\mathscr{C}'}$ then f is invertible, and so f can take a copy of $\tilde{S}_n$ without the added arc only to a copy of $\tilde{S}_n$ without the added arc. In other words, the morphisms of $\mathscr{R}_{\mathscr{C}'}$ remain the same when we delete the arc (p_n, q_n) wherever we added the arc (q_n, p_n), for all $n \in M$. The new category is still isomorphic to $\mathscr{C}$ and it is an *orientation* of $\mathscr{G}_{\mathscr{C}}$.

Remark 4. Theorem 2 also answers part of question (3) of the Introduction: For a *small category* $\mathscr{C}$ and a *groupoid* $\mathscr{C}'$ one can find classes C, C' with $C' \in \mathrm{Or}\, C$ and $\mathrm{Hom}\, C \cong \mathscr{C}$, $\mathrm{Hom}\, C' \cong \mathscr{C}'$ if and only if $\mathscr{C}'$ is a spanning subcategory of $\mathscr{C}$. We note, however, that it is possible to have $\mathscr{C}' \in \mathrm{Or}\, \mathscr{C}$ without $\mathscr{C}'$ being a groupoid, cf. [5, Fig. 21]. With respect to big categories, we note that the first counterexample of Remark 3 applies equally well to the present context. Finally, we remark that it is not true that the conclusion of Theorem 2 is valid for each spanning *subcategory* of $\mathscr{C}$; a counterexample appears in the next section (Remark 5).

§ 3 Applications

In view of the fact that a monoid is a one-object category and a group a one-object groupoid, we will specialize the preceding results. (The reader unfamiliar with category theory will find an informal description of the proof of the following results, as well as some intuitive explanation of the meaning of the categorical results, in [5].)

Theorem 3. *Let M be a monoid. There exists a graph X with* $\mathrm{End}\,X \cong M$ *such that each submonoid M' of M is isomorphic to* $\mathrm{End}\,X'$ *for some* $X' \in \mathrm{Di}\,X$.
(A submonoid M' of M and the monoid M have the same unity by definition.)

Theorem 4. *Let M be a monoid. There exists a graph X with* $\mathrm{End}\,X \cong M$ *such that each subgroup G' of M is isomorphic to* $\mathrm{End}\,X'$ *for some* $X' \in \mathrm{Or}\,X$.

Corollary ([7]). *Let G be a group. There exists a graph X with* $\mathrm{Aut}\,X \cong G$ *such that each subgroup G' of G is isomorphic to* $\mathrm{Aut}\,X'$ *for some* $X' \in \mathrm{Or}\,X$.

Proof. If $\mathrm{End}\,Y$ is a group then $\mathrm{End}\,Y = \mathrm{Aut}\,Y$.

The above theorems follow directly from Theorems 1, 2. They provide some answer to questions (1), (2) of the Introduction: To question (1) the answer is "if and only if G' is a subgroup of G", according to the preceding Corollary. To question (2), stated with the alternative $X' \in \mathrm{Di}\,X$, the answer is "if and only if M' is a submonoid of M", by Theorem 3. It remains to study question (2) with the alternative $X' \in \mathrm{Or}\,X$.

Remark 5. There exists a monoid M and a submonoid M' of M such that no graphs X, X' satisfy $X' \in \mathrm{Or}\,X$, $\mathrm{End}\,X \cong M$, $\mathrm{End}\,X' \cong M'$. One such pair M, M' of infinite monoids appears in [5]. We proceed to give an example with *finite* monoids M, M': Let $M = \{1, f, g\}$, $M' = \{1, f\}$ and $f \cdot f = f \cdot g = f$, $g \cdot g = g \cdot f = g$. Assume that for a graph X and a digraph $X' \in \mathrm{Or}\,X$ we have $\mathrm{End}\,X = M$, $\mathrm{End}\,X' = M'$. We shall denote by Y the subgraph of X induced by $f(V(X))$, by Z the subgraph of X induced by $g(V(X))$; similarly we shall denote by Y' the subdigraph of X' induced by $f(V(X'))$, and by Z' the subdigraph of X' induced by $g(V(X'))$. Clearly $Y' \in \mathrm{Or}\,Y$ and $Z' \in \mathrm{Or}\,Z$. The graphs Y, Z are isomorphic, as $(f \mid Z) \circ (g \mid Y) = 1_Y$ and $(g \mid Y) \circ (f \mid Z) = 1_Z$. Since $f \mid Z' : Z' \to Y'$ is a

bijective homomorphism and since both Y', Z' are orientations, $f|Z'$ is an isomorphism. Let $h: Y' \to Z'$ be the inverse of $f|Z'$; then $h \circ f$ is an endomorphism of X'. If $h \circ f = 1_{X'}$ then f is one-to-one, contrary to the fact that $f(x) = f(g(x))$ for all $x \in V(X')$. If $h \circ f = f$ then $h = 1_{Y'}$ and $f|Z' = 1_{Z'}$, again contrary to the fact that $f(g(x)) = f(x)$ for all $x \in V(X')$. Thus the assumed situation can never occur.

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