Werk

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A Remark on the Strong Law of Large Numbers

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Abstract

The strong law of large numbers for independent and identically distributed random variables X_i, $i=1,2,3,\ldots$ with finite expectation $E|X_1|$ can be stated as, for any $\varepsilon>0$, the number of integers n such that $|n^{-1}\sum_{i=1}^{n} X_i - EX_1|>\varepsilon$, N_ε is finite a. s. It is known that $EN_\varepsilon<\infty$ iff $EX_1^2<\infty$ and that $\varepsilon^2 EN_\varepsilon \to \operatorname{var} X_1$ as $\varepsilon\to 0$, if $EX_1^2<\infty$. Here we consider the asymptotic behaviour of $EN_\varepsilon(n)$ as $n\to\infty$, where $N_\varepsilon(n)$ is the number of integers $k\leqslant n$ such that $|k^{-1}\sum_{i=1}^{k} X_i - EX_1|>\varepsilon$ and $EX_1^2=\infty$.

1. Introduction

Let X_i, $i=1,2,3,\ldots$, be a sequence of independent and identically distributed random variables with distribution function $F(x)$. Let $S_n=\sum_{i=1}^{n} X_i$. If $E|X_1|<\infty$, define for $\varepsilon>0$ $A_n=\{|n^{-1}S_n - EX_1|>\varepsilon\}$. Then the strong law of large numbers can be formulated in the form

$$N_\varepsilon(\infty):=\sum_{n=1}^{\infty} I_{A_n}<\infty \text{ a. s. for all } \varepsilon>0,$$

where I_{A_n} is the indicator function for the set A_n. Erdös [1] showed that $EN_\varepsilon(\infty)<\infty$ iff $EX_1^2<\infty$. Heyde [3] was concerned with the behaviour of $EN_\varepsilon(\infty)$ as $\varepsilon\to 0$: if $\operatorname{var} X_1=\sigma^2<\infty$ then $\varepsilon^2 EN_\varepsilon(\infty)\to\sigma^2$ as $\varepsilon\to 0$. This result follows also from the paper of Müller [4]. In this note we shall analyze the behavior of $EN_\varepsilon(n):=E\sum_{k=1}^{n} I_{A_k}$ as $n\to\infty$, if $\sigma^2=\infty$. We obtain the following

Theorem: *If* $1-F(x)+F(-x)\sim cx^{-\alpha}$ *for* $x\to\infty$, $1<\alpha\leqslant 2$, $c>0$, *then*

if $\alpha<2$: $EN_\varepsilon(n)\sim c_\varepsilon n^{2-\alpha}$ *as* $n\to\infty$ *with* $c_\varepsilon=\varepsilon^{-\alpha}c/(2-\alpha)$

if $\alpha=2$: $EN_\varepsilon(n)\sim c_\varepsilon \log n$ *as* $n\to\infty$ *with* $c_\varepsilon=\varepsilon^{-2}c$.

2. Proof of the Theorem

We suppose without loss of generality $EX_1=0$. Let $\varepsilon>0$. We have $EN_\varepsilon(n)=\sum_{k=1}^{n}P(|S_k|>k\varepsilon)$.

a) Upper estimation: Let $\theta>0$ and define for i, $1\leqslant i\leqslant k$

$$U_i=\begin{cases} X_i & \text{if } |X_i|\leqslant \varepsilon k(1-3\theta) \\ 0 & \text{if } |X_i|>\varepsilon k(1-3\theta)\end{cases} \quad \text{and} \quad V_i=X_i-U_i.$$

So $$P(|\sum X_i|>\varepsilon k)\leqslant P(|\sum U_i|>\varepsilon k)+P(\sum V_i\neq 0).$$

$$P(\sum V_i\neq 0)\leqslant k\,P\big(|X_1|>\varepsilon k(1-3\theta)\big)\leqslant c(1-3\theta)^{-\alpha}\varepsilon^{-\alpha}k^{1-\alpha}+ \\ +o(k^{1-\alpha}).$$

Define $U_i'=U_i-EU_i$. We know that $EU_i\to EX_i=0$ as $k\to\infty$. Therefore

$$P(|\sum U_i|>\varepsilon k)\leqslant P(|\sum U_i'|+k|EU_1|>\varepsilon k)\leqslant \\ \leqslant P\big(|\sum U_i'|>\varepsilon k-o(k)\big)\leqslant P\big(|\sum U_i'|>\varepsilon k(1-\theta)\big)$$

for k sufficiently large. By the definition of U_i' and U_i we get $|U_i'|\leqslant|U_i|+|EU_i|\leqslant\varepsilon k(1-3\theta)+o(1)\leqslant\varepsilon k(1-2\theta)$ for k large.

We want to use the inequality (43) in FUK and NAGAEV [2], which was obtained independently by BENNETT and HOEFFDING. For that purpose we calculate the variance of $\sum U_i'$:

$$\operatorname{var}(\sum U_i')=k\operatorname{var}(U_1')\leqslant kEU_1^2\leqslant\begin{cases}\tilde{c}\,k(\varepsilon k(1-3\theta))^{2-\alpha} & \text{if } \alpha<2\\ \tilde{c}\,k\log\varepsilon k(1-3\theta) & \text{if } \alpha=2.\end{cases}$$

Now, specializing the inequality (43) of FUK and NAGAEV, we obtain

$$P\big(|\sum U_i'|>\varepsilon k(1-\theta)\big)\leqslant 2\exp\left\{-\frac{1-\theta}{1-2\theta}\log k^{\alpha-1}+o(\log k)\right\}= \\ =o(k^{1-\alpha}) \quad \text{since } \theta>0.$$

Therefore we have

$$P(|\sum X_i| > \varepsilon k) \leqslant c(1-3\theta)^{-\alpha}\varepsilon^{-\alpha}k^{1-\alpha} + o(k^{1-\alpha}) \text{ for all } \theta > 0.$$

Thus for all $\delta > 0$ there exists θ_0 such that for all $\theta < \theta_0$

$$\sum_{k=1}^{n} P(|\sum_{i=1}^{k} X_i| > \varepsilon k) \leqslant \begin{cases} (1+\delta)c\varepsilon^{-\alpha}n^{2-\alpha}/(2-\alpha) & \text{if } \alpha < 2 \\ (1+\delta)c\varepsilon^{-2}\log n & \text{if } \alpha = 2 \end{cases}$$

and n sufficiently large.

b) Under estimation: For all $\delta > 0$ we have

$$\begin{aligned} P(|S_k| > x) &= P(S_k > x) + P(S_k < -x) \geqslant \\ &\geqslant P(\exists i \leqslant k: X_i > x(1+\delta), S_k - X_i > -\delta x) + \\ &\quad + P(\exists i \leqslant k: X_i < -x(1+\delta), S_k - X_i < \delta x) \geqslant \\ &\geqslant k P(X_k > x(1+\delta)) P(S_{k-1} > -\delta x) - \\ &\quad - \sum_{i>j}\sum P(X_i > x(1+\delta), X_j > x(1+\delta)) + \\ &\quad + k P(X_k < -x(1+\delta)) P(S_{k-1} < \delta x) - \\ &\quad - \sum_{i>j}\sum P(X_i < -x(1+\delta), X_j < -x(1+\delta)) \geqslant \\ &\geqslant k P(|X_k| > x(1+\delta)) P(|S_{k-1}| < \delta x) - \\ &\quad - \frac{k(k-1)}{2} P^2(|X_1| > x(1+\delta)). \end{aligned}$$

From the weak law of large numbers we know that for all $\theta > 0$ there exists k_0 such that for all $k \geqslant k_0$

$$P(|S_k| > \delta\varepsilon k) < \theta \text{ and } P(|S_k| < \delta\varepsilon k) > 1-\theta.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n} P(|S_k| > k\varepsilon) &\geqslant \sum_{k_0+1}^{n} k P(|X_k| > k\varepsilon(1+\delta)) P(|S_{k-1}| < \delta\varepsilon(k-1)) \\ &\quad - \sum_{k_0+1}^{n} k^2 P^2(|X_1| > k\varepsilon(1+\delta)) \geqslant \\ &\geqslant \sum_{k_0+1}^{n} (1-\theta) k P(|X_1| > k\varepsilon(1+\delta)) - \sum_{k_0+1}^{n} k^2 P^2(|X_1| > k\varepsilon(1+\delta)). \end{aligned}$$

By trivial calculations we find that for $\alpha < 2$

$$I_1 = \sum_{k_0+1}^{n} k\, P\left(|X_1| > k\varepsilon(1+\delta)\right) \sim (1+\delta)^{-\alpha}\varepsilon^{-\alpha} c\, n^{2-\alpha}/(2-\alpha) \text{ and}$$

$$I_2 = \sum_{k_0+1}^{n} k^2 P^2\left(|X_1| > k\varepsilon(1+\delta)\right) = o(n^{2-\alpha}) \text{ as } n \to \infty$$

and that for $\alpha = 2$

$$I_1 \sim (1+\delta)^{-2}\varepsilon^{-2} c \log n \quad \text{and} \quad I_2 = o(\log n) \text{ as } n \to \infty.$$

This completes the proof.

3. Generalization

If we consider for $1 - F(x) + F(-x) \sim x^{-\alpha} L(x)$ as $x \to \infty$, where $1 < \alpha \leqslant 2$ and $L(x)$ varies slowly as $x \to \infty$, we can show with the same proof that for $\varepsilon > 0$

$$\text{if } 1 < \alpha < 2 \quad E\, N_\varepsilon(n) \sim \varepsilon^{-\alpha} n^{2-\alpha} L(n)/(2-\alpha) \text{ as } n \to \infty$$

$$\text{if } \alpha = 2 \qquad E\, N_\varepsilon(n) \sim \varepsilon^{-2} \int_{n_0}^{\varepsilon n} y^{-1} L(y)\, dy \qquad \text{as } n \to \infty.$$

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