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Topological Dynamics of Transformations Induced on the Space of Probability Measures

By

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Dedicated to Prof. N. Hofreiter on his seventieth birthday

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Abstract

Let T be a continuous transformation of a compact metric space X. T induces in a natural way a transformation T_M on the space $M(X)$ of probability measures on X, and a transformation T_K on the space $K(X)$ of closed subsets of X. This note investigates which of the topological properties of $T:X \to X$ (like distality, transitivity, mixing property etc. ...) are "inherited" by $T_M:M(X) \to M(X)$ and $T_K:K(X) \to K(X)$.

I. Let X be the state space of some system and $M(X)$ the space of probability measures on X. The elements of $M(X)$ can be viewed as statistical states, representing imperfect knowledge of the system. The elements of X are imbedded in $M(X)$ as the pure states.

Let T be a transformation of X and T_M the corresponding transformation of $M(X)$. In classical statistical mechanics one studies $(M(X), T_M)$, which can be viewed as a system where the dynamics is deterministic and the configurations are stochastic.

Suppose that X is compact metric and T is continuous: then $M(X)$, provided with the weak topology, is again compact metric and T_M is continuous. It is natural to ask which of the topological properties of T carry over to T_M. (The answer is that roughly half of them do.)

Section II contains the notation and some preliminaries. In Section III we shall consider a list of standard properties of topological dynamics (all independent of the choice of the metric on X): unique ergodicity, minimality, expansiveness, topological transitivity, weak and strong mixing, the property of being an F-flow, the specification property, the property of having an

invariant measure positive on all open sets, of having positive topological entropy, disjointness, uniform almost periodicity and distality. We shall also consider the analogous problem for the transformation T_K induced by T on the space of closed subsets of X: the answers are in each case the same.

II. Let X be a compact metric space with metric d.

Let $M(X)$ denote the space of Borel probability measures on X provided with the *Prohorov metric* p defined by
$p(\mu,\nu) = \inf\{\varepsilon : \mu(A) \leqslant \nu(A^\varepsilon) + \varepsilon$ and $\nu(A) \leqslant \mu(A^\varepsilon) + \varepsilon$ for all Borel sets $A \subset X\}$ for $\mu,\nu \in M(X)$.

Here $A^\varepsilon = \{x \in X : d(x,A) < \varepsilon\}$. As V. STRASSEN showed in [13], one has $p(\mu,\nu) = \inf\{\varepsilon : \mu(A) \leqslant \nu(A^\varepsilon) + \varepsilon$ for all Borel sets $A \subset X\}$.

The induced topology is just the *weak topology* for measures. It turns $M(X)$ into a compact space (cf. [9, p. 45]).

The following assertions are equivalent (cf. [1, pp. 11 and 238]):

1) $p(\mu_n,\mu) \to 0$

2) $\int f d\mu_n \to \int f d\mu$ for all continuous f on X

3) $\liminf \mu_n(G) \geqslant \mu(G)$ for all open $G \subset X$.

For $x \in X$ let $\pi(x) \in M(X)$ denote the corresponding *point measure* defined by $\pi(x)(A) = 1$ if $x \in A$, $= 0$ otherwise. π is a homeomorphism from X onto a closed subspace of $M(X)$ (cf. [9, p. 42]). It is easy to see that $M(X)$ is convex and that the point measures are just the extremal points of $M(X)$. It follows that the convex combinations of point measures (i. e. the measures with finite support) are dense in $M(X)$.

Let $K(X)$ denote the space of closed subsets of X provided with the *Hausdorff metric* h defined by

$$h(C,D) = \max\{\max_{x \in C} \min_{y \in D} d(x,y), \max_{y \in D} \min_{x \in C} d(x,y)\} \text{ for } C, D \in K(X).$$

This metric turns $K(X)$ into a compact space (cf. [7, p. 47]). It is easy to see that the finite sets are dense in $K(X)$.

Now let T be a homeomorphism from X onto itself. T induces a map $T_M : M(X) \to M(X)$ defined by

$$(T_M \mu)(A) = \mu(T^{-1}A) \quad (\mu \in M(X), A \subset X \text{ Borel}).$$

It is easy to check that T_M is a homeomorphism from $M(X)$ into itself, sending the point measure $\pi(x)$ into the point measure $\pi(Tx)$.

Similarly T induces a map $T_K: K(X) \to K(X)$ defined by

$$T_K(C) = T\,C \;(C \in K(X)).$$

Again T_K is a homeomorphism from $K(X)$ into itself, and

$$T_K(\{x\}) = \{T(x)\} \text{ for } x \in X.$$

Let T_i be a homeomorphism from X_i into itself $(i = 1, 2)$. Then one defines a homeomorphism $T_1 \times T_2$ from $X_1 \times X_2$ into itself by

$$T_1 \times T_2 (x_1, x_2) = (T_1 x_1, T_2 x_2),\ (x_i \in X_i).$$

If $X = X_i$ and $T = T_i$ for $i = 1, 2$, one writes $T^{(2)}$ for $T_1 \times T_2$. Similarly one defines $T^{(n)}$ as homeomorphism from $X \times \ldots \times X$ (n times) into itself.

The homeomorphism $S: Y \to Y$ is said to be a factor of a $T: X \to X$ if there exists a continuous φ from X onto Y such that $S \circ \varphi = \varphi \circ T$. φ is called a *homomorphism* from (X, T) onto (Y, S). φ induces in an obvious way a homomorphism from $(M(X), T_M)$ onto $(M(Y), S_M)$ and from $(K(X), T_K)$ onto $(K(Y), S_K)$.

Define $M_n(X) = \{\mu = (1/n)(\pi(x_1) + \ldots + \pi(x_n)),\ x_i \in X$ not necessarily distinct$\}$ for $n = 1, 2, \ldots$.

Lemma 1: $M_n(X)$ is closed in $M(X)$ and invariant under T_M. $\bigcup_{n \geqslant 1} M_n(X)$ is dense in $M(X)$.

Proof: Assume $\mu_k = (1/n)(\pi(x_1^k) + \ldots + \pi(x_n^k)) \in M_n(X)$ for $k = 1, 2, \ldots$ and $\mu_k \to \mu$. There exists a subsequence $k_j \uparrow \infty$ and points $x_1', \ldots, x_n'$ in X such that $x_i^{k_j} \to x'$ for $k_j \to \infty$ $(i = 1, \ldots, n)$. Clearly then $\mu_{k_j} \to \mu' = (1/n)(\pi(x_1') + \ldots + \pi(x_n'))$, hence $\mu = \mu' \in M_n(X)$. The density of $\bigcup_{n \geqslant 1} M_n(X)$ is a trivial consequence of the density of measures with finite support.

Define $K_n(X) = \{C \in K(X) : \text{card } C \leqslant n\}$ for $n = 1, 2, \ldots$. Clearly one has again:

Lemma 2: $K_n(X)$ is closed in $K(X)$ and invariant under T_K. $\bigcup_{n \geqslant 1} K_n(X)$ is dense in $K(X)$.

Lemma 3: The restriction of T_M to $M_n(X)$ is a factor of $T^{(n)}$, as is the restriction of T_K to $K_n(X)$. The corresponding homomorphism is at most $n!$ to one.

Proof: Let φ (resp. $\bar{\varphi}$) denote the map sending $(x_1, \ldots, x_n)$ to $(1/n)\sum_{j=1}^{n} \pi(x_j)$ (resp. to $\{x_1, \ldots, x_n\}$). φ and $\bar{\varphi}$ are obviously conti-

nuous and onto. One has $T_M \circ \varphi = \varphi \circ T$ and $T_K \circ \tilde{\varphi} = \tilde{\varphi} \circ T$. A measure $\mu = (1/n)\sum_{j=1}^{n} \pi(x_j) \in M_n(X)$ has at most $n!$ preimages under φ, namely the permutations of $\{x_1, \ldots, x_n\}$ (recall that the x_i need not all be distinct). Similarly $\{x_1, \ldots, x_n\} \in K_n(X)$ has at most $n!$ preimages under $\tilde{\varphi}$.

III. Let $M_I(X) = \{\mu \in M(X) : T_M \mu = \mu\}$ be the set of all T-invariant measures. It is well known that $M_I(X)$ is nonempty, compact and convex (cf. [8]). If $M_I(X)$ consists of a single point, then T is said to be *uniquely ergodic.*

If T is uniquely ergodic, then T_M need not be. Indeed let T be the permutation on the two point space $X = \{x_1, x_2\}$. The unique invariant measure is obviously $\frac{1}{2}(\pi(x_1) + \pi(x_2))$. But any element of $M(X)$ is of the Form $\alpha\pi(x_1) + (1-\alpha)\pi(x_2)$ with $0 \leqslant \alpha \leqslant 1$ and hence, if $\alpha \neq \frac{1}{2}$, it is a periodic point of T_M of period 2. Now for any periodic orbit there exists an invariant measure concentrated on that orbit. Hence T_M is not uniquely ergodic. Also $K(X)$ consists of two distinct periodic orbits and hence T_K is not uniquely ergodic.

If T is expansive, then T_M need not be expansive. (T is said to be *expansive* if there exists a $\delta > 0$ such that for $x_1 \neq x_2$ there is an $n \in \mathbb{Z}$ with $d(T^n x_1, T^n x_2) > \delta$). Indeed, let T be the shift transformation on a finite alphabet (cf. [6]). This homeomorphism is easily seen to be expansive, but not uniquely ergodic, and hence $M_I(X)$, i. e. the set of fixed points of T_M, is infinite. But then T_M cannot be expansive. Also the set of fixed points of $K(X)$ (i. e. the set of subshifts) is infinite, hence T_K is not expansive. (Incidentally, if X consists of more than one point, then T_M is never expansive: if α_1 and α_2 are close, then the transforms of $\alpha_1\pi(x_1) + (1-\alpha_1)\pi(x_2)$ and $\alpha_2\pi(x_1) + (1-\alpha_2)\pi(x_2)$ will never be far apart.)

T is said to be *topologically transitive*, if there exists an $x \in X$ whose orbit $\{T^n x : n \in \mathbb{Z}\}$ is dense, or alternatively if for any two open sets $U_1, U_2 \subset X$ there is an n such that $T^n(U_1) \cap U_2 \neq \emptyset$. T is said to be *minimal* if the orbit of every $x \in X$ is dense.

If T is topologically transitive (resp. minimal) then T_M and T_K need not be topologically transitive (resp. minimal). A simple example is the following: let $X = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and T the map $x \to \theta x \pmod 1$ for some irrational θ. T is minimal and hence topologically transitive. Let $\mu_1 = \pi(x_1)$ and $\mu_2 = \frac{1}{2}(\pi(x_1) + \pi(x_2))$ with distinct $x_1, x_2 \in X$. Let U_1 and U_2 be

sufficiently small neighborhoods of μ_1 and μ_2 in $M(X)$. Then obviously $T_M^n(U_1)$ never intersects U_2. Similarly let V_1 and V_2 be sufficiently small neighborhoods of $\{x_1\}$ and $\{x_1, x_2\}$ in $K(X)$. Again $T_K^n(V_1) \cap V_2 = \emptyset$ for all $n \in \mathbb{Z}$. Hence neither T_M nor T_K are topologically transitive.

T is said to be *weakly mixing (I)*, if $T^{(2)}$ is topologically transitive, and *weakly mixing (II)*, if $T \times S$ is topologically transitive for any topologically transitive homeomorphism S of a compact metric space Y into itself (cf. [4]). T is said to be *strongly mixing*, if for any two open sets $U_1, U_2 \subset X$ there exists an N_0 such that $T^N(U_1) \cap U_2 \neq \emptyset$ for all $N \geqslant N_0$.

It is easy to check that strongly mixing implies weakly mixing (II), which implies weakly mixing (I), which implies topologically transitive.

Lemma 4: If T is strongly mixing (resp. weakly mixing (II), resp. weakly mixing (I)) then $T^{(n)}$ is strongly mixing (resp. weakly mixing (II), resp. weakly mixing (I)) for all $n > 0$.

Proof: The cases of T strongly mixing and of T weakly mixing (II) are trivial. The case of T weakly mixing (I) has been proved by FURSTENBERG in [4, Prop. II.3].

Theorem 1: *If T is strongly mixing (resp. weakly mixing (II), resp. weakly mixing (I)), then T_M and T_K are strongly mixing (resp. weakly mixing (II), resp. weakly mixing (I)).*

Proof: (a) Let V_1 and V_2 be two arbitrary open sets in $M(X)$. For some n sufficiently large, there exist points

$$\mu_1 = (1/n) \sum_{j=1}^{n} \pi(x_j^1) \in M_n(X) \cap V_1 \text{ and}$$

$$\mu_2 = (1/n) \sum_{j=1}^{n} \pi(x_j^2) \in M_n(X) \cap V_2 .$$

For $i = 1, 2$ and $j = 1, \ldots, n$ choose neighborhoods U_j^i of x_j^i such that if $y_j^i \in U_j^i$ then $(1/n) \sum_{j=1}^{n} \pi(y_j^i) \in V_i$.

If T is strongly mixing, so is $T^{(n)}$. Hence there exists an N_0 such that for each $N \geqslant N_0$

$$T^{(n)N}(U_1^1 \times \ldots \times U_n^1) \cap (U_1^2 \times \ldots \times U_n^2) \neq \emptyset .$$

For each $N \geqslant N_0$ and $j = 1, \ldots, n$ pick some $y_j^2 \in T^N(U_j^1) \cap U_j^2$. One has $\nu_2 = (1/n) \sum_{j=1}^{n} \pi(y_j^2) \in V_2$ and $T_M^{-N} \nu_2 = (1/n) \sum_{j=1}^{n} \pi(T^{-N} y_j^2) \in V_1$. Hence $T_M^N(V_1) \cap V_2 \neq \emptyset$ for all $N \geqslant N_0$, and so T_M is strongly mixing.

Suppose T is weakly mixing (II). Let S be some homeomorphism of a compact metric space Y onto itself.

Let O_1 and O_2 be two open sets in $M(X)\times Y$. There exist open sets V_1, V_2 in $M(X)$ and U_1, U_2 in Y such that $V_1\times U_1\subset O_1$ and $V_2\times U_2\subset O_2$. Since $T^{(n)}$ is weakly mixing (II), $T^{(n)}\times S$ is topologically transitive, and hence there exists an N such that

$$(T^{(n)}\times S)^N(U_1^1\times\ldots\times U_n^1\times U_1)\cap(U_1^2\times\ldots\times U_n^2\times U_2)\neq\emptyset.$$

Choose $y_2\in S^N(U_1)\cap U_2$ and $y_j^2\in T^N(U_j^1)\cap U_j^2$ for $j=1,\ldots,n$. With $\nu_2=(1/n)\sum_{j=1}^n\pi(y_j^2)$, $\nu_1=T_M^{-N}(\nu_2)$ and $y_1=S^{-N}(y_2)$ one obtains $(\nu_1, y_1)\in V_1\times U_1$ and $T^N(\nu_1, y_1)=(\nu_2, y_2)\in V_2\times U_2$. Hence $(T_M\times S)^N(O_1)\cap O_2\neq\emptyset$. $T_M\times S$ is topologically transitive and thus T_M is weakly mixing (II).

Suppose T is weakly mixing (I). Letting $T^{(m)}$ play the role of S, one shows just as above that $T_M\times T^{(m)}$ is topologically transitive for $m=1,2,\ldots$. By letting T_M play the role of S one shows then that $T_M\times T_M$ is topologically transitive and hence that T_M is weakly mixing (I).

(b) The proofs for T_K are similar, putting $K_n(X)$ instead of $M_n(X)$.

One may also ask whether one has inheritance the other way round, from T_M to T.

Proposition 1: If T_M is topologically transitive (resp. weakly mixing (I), resp. weakly mixing (II), resp. strongly mixing) then so is T.

Proof: Assume that T_M is topologically transitive. Let U_1, U_2 be two open sets in X. Write $V_i=\{\mu\in M(X):\mu(U_i)>\frac{9}{10}\}$ $(i=1,2)$. If $\mu_n\to\mu$ and $\mu_n\in M(X)\setminus V_1$ then $\mu_n(U_i)\leqslant\frac{9}{10}$ and hence $\mu(U_i)\leqslant\frac{9}{10}$, i. e. $\mu\in M(X)\setminus V_1$. Thus V_1 and V_2 are open and there exists an N with $T_M^N(V_1)\cap V_2\neq\emptyset$. This obviously implies that $T^N(U_1)\cap U_2\neq\neq\emptyset$. One applies the same method to the other cases.

Proposition 2: If T_K is topologically transitive (resp. weakly mixing (I), resp. weakly mixing (II), resp. strongly mixing) then so is T.

Proof: Assume that T_K is topologically transitive.

Let U_1, U_2 be two open sets in X. Write $V_i=\{C\in K(X):C\subset U_i\}$ for $i=1,2$. It is easy to see that the V_i's are open subsets of $K(X)$. Hence there exists an N such that $T_K^N(V_1)\cap V_2\neq\emptyset$. Obviously then $T^N(U_1)\cap U_2\neq\emptyset$.

Again one proves the other cases in the same way.

$T: X \to X$ is said to be an *F-flow* (cf. [4]) if the periodic points are dense and $T^k: X \to X$ is topologically transitive for $k = 1, 2, \ldots$. Ergodic automorphisms of the torus and shift transformations are examples of F-flows.

Lemma 4:

(a) If T is an F-flow, then T^k is weakly mixing (I) for $k = 1, 2, \ldots$.

(b) If T is an F-flow, then $T^{(n)}$ is an F-flow for $n = 1, 2 \ldots$.

Proof: (a) Let $k > 0$ be given. One has to show that for any four open sets U_1, U_2, V_1, V_2 in X, there is an N with $T^{kN}(U_1) \cap U_2 \neq \neq \emptyset$ and $T^{kN}(V_1) \cap V_2 \neq \emptyset$.

Since T^k is transitive there exists an $x \in U_1$ and an $l \in N$ with $T^{kl}(x) \in U_2$. Since periodic points are dense, there is a $u \in U_1$ with $T^{kl}(u) \in U_2$ and $T^{kp}(u) = u$ for some $p \in \mathbb{N}$. Hence $T^n(U_1) \cap U_2 \neq \emptyset$ for $n = k(l + jp)$, where $j = 0, 1, 2, \ldots$. Let $T^{kl} V_1 = \tilde{V}_1$. Since T^{kp} is topologically transitive, there exists a $j_0 \in \mathbb{N}$ such that $T^{kpj_0}(\tilde{V}_1) \cap V_2 \neq \emptyset$. Hence with $N = l + j_0 p$ one obtains $T^{kN}(V_1) \cap V_2 \neq \emptyset$ and $T^{kN}(U_1) \cap U_2 \neq \emptyset$.

(b) If $T_i: X_i \to X_i$ $(i = 1, 2)$ are F-flows, then the product $T_1 \times T_2$ has obviously periodic points dense. Also T_1^k and T_2^k are weakly mixing (I), hence $(T_1 \times T_2)^k$ is weakly mixing (I) for $k = 1, 2, \ldots$. Thus $T_1 \times T_2$ is an F-flow. This implies (b).

There seems to be no known example of an F-flow which is not strongly mixing.

Proposition 3: If T is an F-flow then so are T_M and T_K.

Proof: If the periodic points of T are dense in X, the periodic points of T_M (resp. T_K) are dense in $M(X)$ (resp. $K(X)$). T^k is weakly mixing (I), hence T_M^k and T_K^k are weakly mixing (I) and a fortiori topologically transitive, for $k = 1, 2, \ldots$. Hence T_M and T_K are F-flows.

A property still stronger than that of being an F-flow and being strongly mixing is the specification property. $T: X \to X$ is said to satisfy the *periodic specification property (PSP)* if for any $\varepsilon > 0$ there exists an integer $M'(\varepsilon) > 0$ such that for any two points x_1 and x_2 in X and any integers $a_1 \leqslant b_1 < a_2 \leqslant b_2$ with $a_2 - b_1 \geqslant M'(\varepsilon)$ and any integer $p \geqslant b_2 - a_1 + M'(\varepsilon)$, there exists a $z \in X$ with $T^p(z) = z$ and such that

$$d(T^j z, T^j x_1) \leqslant \varepsilon \quad \text{for} \quad a_1 \leqslant j \leqslant b_1$$

and

$$d(T^j z, T^j x_2) \leqslant \varepsilon \quad \text{for} \quad a_2 \leqslant j \leqslant b_2 .$$

Roughly speaking, this means that any two finite pieces of orbit can be approximated up to ε by some periodic orbit, provided the period is large enough. This property seems fairly technical, but is satisfied by many important transformations like shift systems, Anosov diffeomorphisms etc. . . . For more details we refer to [11].

Theorem 2: *If T satisfies PSP, then so do T_M and T_K.*

Proof: (a) Let $\varepsilon > 0$ be given and let $M'(\varepsilon/2)$ be as in the definition of PSP. Let $\mu_1, \mu_2 \in M(X)$ be given, as well as integers $a_1 \leqslant b_1 < a_2 \leqslant b_2$ and p with $a_2 - b_1 > M'(\varepsilon/2)$ and $p > b_2 - a_1 + M'(\varepsilon/2)$.

Since T_M is continuous and $M(X)$ is compact, there exists an $\eta > 0$ such that if $p(\mu, \nu) < \eta$ then $p(T_M^j \mu, T_M^j \nu) < \varepsilon/2$ for $a_1 \leqslant j \leqslant b_2$. For some $n \in \mathbb{N}$ there exist $\nu_1 \in M_n(X)$ and $\nu_2 \in M_n(X)$ such that $p(\mu_1, \nu_1) < \eta$ and $p(\mu_2, \nu_2) < \eta$. Hence $p(T_M^j \mu_i, T_M^i \nu_i) < \varepsilon/2$ for $a_i \leqslant j \leqslant b_i$, $i = 1, 2$. Now $\nu_i = (1/n) \sum_{k=1}^{n} \pi(x_k^i)$ for $i = 1, 2$.

Since T satisfies PSP, there exist $z_k \in X$ with $T^p z_k = z_k$ and $d(T^j z_k, T^j x_k^i) \leqslant \varepsilon/2$ for $a_i \leqslant j \leqslant b_i$, $k = 1, \ldots, n$, $i = 1, 2$.

Put $\varrho = (1/n) \sum_{k=1}^{n} \pi(z_k)$. Obviously $T_M^p(\varrho) = \varrho$. Also

$$p(T_M^j \varrho, T_M^j \nu_i) = p\left((1/n) \sum_{k=1}^{n} \pi(T^j z_k), (1/n) \sum_{k=1}^{n} \pi(T^j x_k^i)\right) < \varepsilon/2$$

and hence $p(T_M^j \varrho, T_M^j \mu_i) \leqslant \varepsilon$ for $a_i \leqslant j \leqslant b_i$, $i = 1, 2$.

Hence T_M satisfies PSP: for each $\varepsilon > 0$ the corresponding integer $M'(\varepsilon)$ for T_M is just the $M'(\varepsilon/2)$ for T.

(b) The proof for T_K is analogous, with $K_n(X)$ instead of $M_n(X)$.

There exist several variants of the specification property: T is said to satisfy the *weak specification property (WSP)* if the z in the definition of PSP is not requested to be periodic (hence no condition on p) (cf. [10]). T is said to satisfy the *strong specification property (SSP)* if for any $\varepsilon > 0$ there exists an integer $M'(\varepsilon) > 0$ such that for any $k \geqslant 2$, for any points $x_1, \ldots, x_k \in X$, any integers $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \ldots < a_k \leqslant b_k$ with $a_l - b_{l-1} > M'(\varepsilon)$ for $l = 2, 3, \ldots, k$, and any p with $p > b_k - a_1 + M'(\varepsilon)$ there exists a $z \in X$ with $T^p z = z$ and $d(T^j z, T^j x) < \varepsilon$ for $a_l \leqslant j \leqslant b_l$, $l = 1, \ldots, k$ (cf. [3]). Again we can show:

Proposition 4: If T satisfies WSP (resp. SSP) then T_M and T_K satisfy WSP (resp. SSP).

With the help of the specification properties, one can study the oscillation of the time averages $(1/N) \sum_{j=0}^{N-1} T_M^j \mu$ (cf. [12]).

Proposition 5: If T admits an invariant measure which is positive on all (nonempty) open sets, then so do T_M and T_K.

Proof: (a) It is easy to see that $T^{(n)}$ admits an invariant measure positive on all open sets. This implies that the restriction of T_M to $M_n(X)$, as a factor of $T^{(n)}$, admits an invariant measure positive on all open sets of $M_n(X)$. Let $\mu^{(n)}$ denote this measure. Since $\bigcup_{n\geqslant 1} M_n(X)$ is dense, the T_M-invariant measure $\mu=\sum_{n=1}^{\infty}(1/2^n)\mu^{(n)}\in$ $\in M(M(X))$ is positive on all open sets of $M(X)$.

(b) The same argument shows that T_K admits an invariant measure positive on all open sets of $K(X)$.

Let $h_{\text{top}}(T)$ denote the *topological entropy of* T. (For a definition we refer to [2]).

Proposition 6: If $h_{\text{top}}(T)>0$, then $h_{\text{top}}(T_M)=h_{\text{top}}(T_K)=\infty$.

Proof: (a) Let $a=h_{\text{top}}(T)$. It is well known that $h_{\text{top}}(T^{(n)})=na$ (cf. [5]). Hence $h_{\text{top}}((T_M/M_n(X))=na$ (cf. [2, Th. 17]). Since $h_{\text{top}}(T_M)\geqslant h_{\text{top}}(T_M/M_n(X))$ for all n, one obtains $h_{\text{top}}(T_M)=\infty$.

(b) Similarly for T_K.

If $h_{\text{top}}(T)=0$, then one can have $h_{\text{top}}(T_M)=h_{\text{top}}(T_K)=0$ (for example if X is finite). But one can also construct examples with $h_{\text{top}}(T_M)=h_{\text{top}}(T_K)=\infty$.

$T:X\to X$ and $S:Y\to Y$ are said to be *disjoint*, if whenever Δ is a closed $T\times S$-invariant subset of $X\times Y$ such that $p_x(\Delta)=X$, $p_y(\Delta)=Y$ (p_x,p_y are the projections from $X\times Y$ onto X and Y) then $\Delta=X\times Y$ (cf. [4]).

If T and S are disjoint, then neither T_M and S_M nor T_K and S_K have to be disjoint. Indeed, let T be an F-flow and S an irrational rotation of the one-dimensional torus. Then S is minimal and hence disjoint from the F-flow T (cf. [4]). If T_M and S_M were disjoint, then one of them would have to be minimal (cf. [4]), which is not the case. For the same reason T_K and S_K are not disjoint.

$A\subset\mathbb{Z}$ is said to be *relatively dense*, if there exists a finite $F\subset\mathbb{Z}$ with $A+F=\mathbb{Z}$.

The transformation T is said to be *uniformly almost periodic* (u. a. p.), if given $\varepsilon>0$ there exists a relatively dense set A such that $d(T^nx,x)<\varepsilon$ for all $n\in A$, $x\in X$.

For X compact the following characterization is well known (cf. [6, 4.38 and 11.09]):

Lemma 5: T is uniformly almost periodic iff $\{T^n:n\in Z\}$ is equicontinuous.

Proposition 7: If T is uniformly almost periodic then so are T_M and T_K.

Proof: Let T be u. a. p. According to the preceding lemma $\{T^n : n \in Z\}$ is equicontinuous and (since X is compact) even uniformly equicontinuous, i. e. for $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}$. Of course we may assume that $\delta < \varepsilon$. Let $\mu, \mu' \in M(X)$ with $p(\mu,\mu') < \delta$. For any Borel subset A of X and any $n \in \mathbb{Z}$ we have

$$\mu(T^{-n}A) \leqslant \mu'\big((T^{-n}A)^\delta\big) + \delta. \qquad (\divideontimes)$$

For $y \in (T^{-n}A)^\delta$ there is a $z \in T^{-n}A$ with $d(y,z) < \delta$. Thus by uniform equicontinuity $d(T^n y, T^n z) < \varepsilon$ and hence $y \in T^{-n}(A^\varepsilon)$. Therefore $(T^{-n}A)^\delta \subset T^{-n}(A^\varepsilon)$. Thus $(\divideontimes)$ implies $\mu(T^{-n}A) \leqslant \leqslant \mu'(T^{-n}(A^\varepsilon)) + \varepsilon$ and therefore $T_M^n \mu(A) \leqslant T_M^n \mu'(A^\varepsilon) + \varepsilon$ and $p(T_M^n \mu, T_M^n \mu') < \varepsilon$. Since this holds for all $n \in \mathbb{Z}$, T_M is u. a. p.

A similar argument proves the result for T_K.

T is said to be *distal* if $\inf_{n \in \mathbb{Z}} d(T^n x, T^n y) = 0$ implies $x = y$. Remark that a uniformly almost periodic T is distal.

Theorem 3: *The distality of T does not necessarily imply the distality of T_M and T_K.*

Proof: Let X be the closed complex unit disc and

$$T(x e^{i\varphi}) = x e^{i(\varphi + 2\pi x)} \text{ for } 0 < x \leqslant 1 \text{ and } T(0) = 0.$$

T is a distal homeomorphism. Let R denote the right half of X: $\{z \in X : \mathrm{Re}\, z \geqslant 0\}$ and λ the normed Lebesgue measure on X. Define $\mu \in M(X)$ by $\mu(B) = 2\lambda(B \cap R)$ for any Borel set B. We shall show that $T_M^n \mu$ converges weakly towards the invariant measure λ. This implies that T_M cannot be distal.

Explicitly we have to prove that for $\varepsilon > 0$ there exists a natural number n_ε, such that

$$T_M^n \mu(A) \leqslant \lambda(A^\varepsilon) + \varepsilon \text{ for all } n \geqslant n_\varepsilon \text{ and all Borel sets } A.$$

This inequality is equivalent to $2\lambda(R \cap T^{-n}A) \leqslant \lambda(A^\varepsilon) + \varepsilon$. Because of $R \cap T^{-n}A = T^{-n}(A \cap T^n R)$ and the invariance of λ this is equivalent to

$$2\lambda(A \cap T^n R) \leqslant \lambda(A^\varepsilon) + \varepsilon$$

The boundary of R consists of a diameter which is the union of the "upper" radius D and the "lower" radius E. T^n turns the two

radii into spirals winding around the origin and enclosing $T^n R$. For $0 \leqslant \varphi < 2\pi$ let F_φ be the radius $\{re^{i(\varphi+\pi/2)} : 0 \leqslant r \leqslant 1\}$. Then $F_\varphi \cap T^n D$ consists of the points with radii $x_k = (k + \varphi/2\pi)/n$ for $k = 0, 1, 2, \ldots, n-1$ and $x = 0$ whereas $F_\varphi \cap T^n E$ consists of the points with radii $y = 0$, $y_k = (k - 1/2 + \varphi/2\pi)/n$ for $k = 1, 2, \ldots, n-1$ together with $(n - 1/2 - \varphi/2\pi)/n$ in the case $0 \leqslant \varphi < \pi$ and with $(\varphi/2\pi - \frac{1}{2})/n$ in the case $\pi \leqslant \varphi < 2\pi$. Thus each radius is intersected by $T^n R$ in a finite number of line segments all having length $1/2n$ except possibly the first and the last which have length at most $1/2n$. This shows that the spiral $T^n R$ can be made arbitrarily thin by choosing n sufficiently large.

Now divide the unit disc into congruent sectors. For each sector S consider the connected components of $S \cap T^n R$. Partition all these components into two classes $\mathfrak{A}_i$ and $\mathfrak{A}_0$ of which $\mathfrak{A}_i$ (the "inner" components) consists of those components $C \subset S \cap T^n R$ (for some S) having a similar component C' of $S \setminus T^n R$ as an "outer neighbour" (i. e. adherent to C on the side of greater radii), such that C' does not touch the unit circle. We have $\lambda(C) \leqslant \lambda(C')$ for $C \in \mathfrak{A}_i$.

By choosing the sectors small enough and n large enough, the previous remark concerning the thickness of the spiral $T^n R$ allows us to achieve that the diameter of $C \cap C'$ is less than ε for each $C \in \mathfrak{A}_i$. Thus for any Borel set A, $A \cap C \neq \emptyset$ implies that A^ε contains $C \cap C'$ and $2\lambda(A \cap C) \leqslant 2\lambda(C) \leqslant \lambda(C \cup C')$.

This argument does not apply to the "outer" components $C \in \mathfrak{A}_0$, because either they do not possess such an "outer neighbour" or the "outer neighbour" $C' \subset S \setminus T^n R$ need not have measure $\lambda(C') \geqslant \lambda(C)$. But $\bigcup_{C \in \mathfrak{A}_0} C$ is contained in a concentric ring V with thickness $1/n$, so that

$$2\lambda(A \cap \bigcup_{C \in \mathfrak{A}_0} C) \leqslant 2\lambda(A \cap V) \leqslant 2\lambda(V) \leqslant 2\pi[1 - (1 - 1/n)^2]$$

and this is less than ε for n large enough. Therefore we have

$$2\lambda(A \cap T^n R) = 2\lambda(A \cap \bigcup_{C \in \mathfrak{A}_i} C) + 2\lambda(A \cap \bigcup_{C \in \mathfrak{A}_0} C) \leqslant$$
$$\leqslant \lambda\big((A \cap \bigcup_{C \in \mathfrak{A}_i} (C \cup C')\big) + \varepsilon \leqslant \lambda(A^\varepsilon) + \varepsilon .$$

The proof for T_K is similar.

The example in the last theorem shows that *pointwise almost periodicity of* T (i. e. for $x \in X$ and any $\varepsilon > 0$ there exists a relatively

dense set $A \subset \mathbb{Z}$ such that $d(T^n x, x) < \varepsilon$ for all $n \in A$) is not inherited by T_M and T_K: A distal transformation T is pointwise almost periodic (cf. [6]), but the point $\mu \in M(X)$ of the previous example is obviously not an almost periodic point of T_M. For T_K similarly.

Added in Proof: After submitting this paper we learned that theorem 3 (which was presented by the first author at a conference in Oberwolfach in 1972) has also been obtained (for a system with different structure) by Namioka in „Right topological groups, distal flows, and a fixed point theorem" Math. Syst. Theory **6**, 193—209 (1972).

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