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Titel: A Condition on Near-Rings with Identity.

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A Condition on Near-Rings with Identity [1]

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Various classes of groups cannot be the additive groups of near-rings with identity. By using the deep theorem that a group of odd order is solvable [3], CLAY and MALONE [2] show that a simple group of composite order cannot be the additive group of a near-ring with identity. Again using the result in [3], LIGH [4] extends this to: A finite non-abelian group with exactly one non-trivial proper normal subgroup cannot be the additive group of a near-ring with identity. Using only number theory techniques, CLAY and MALONE [2] and CLAY and DOI [1] respectively show that symmetric groups S_n, $n \geqslant 3$ and alternating groups A_n, $n \geqslant 4$ cannot be additive groups of near-rings with identity. We give a very easy and elementary proof of a theorem which vastly generalizes all of these results.

A (left) *near-ring* is a triple $(G, +, \cdot)$ such that $(G, +)$ is a group, $(G, \cdot)$ is a semi-group and $a(b+c) = ab + ac$ for all a, b and c in G. An element e of G is an *identity* if $ea = ae = a$ for all a in G. For a in G, $A(a)$ denotes the set $\{g \in G \mid ag = 0\}$ and aG denotes $\{ag \mid g \in G\}$. Easily $A(a)$ is a normal subgroup and aG is a subgroup of G.

A subset P of a partially ordered set is an *antichain* if no two distinct elements of P are comparable.

Theorem. *Let $(G, +)$ be a group having elements of M distinct prime orders. If each antichain of the lattice of normal subgroups of G has cardinality less than M, then $(G, +)$ cannot be the additive group of a near-ring with identity.*

Proof. Suppose $(G, +, \cdot)$ is a near-ring with identity e. Let the M elements $g_1, g_2, \ldots$ of G have distinct prime orders $p_1, p_2, \ldots$

[1] This article is adapted from a portion of the author's dissertation which was directed by JAMES R. CLAY.

respectively. If $A(g_i) \supseteq A(g_j)$, then it is easy to check that $\varphi: g_j G \to g_i G$ defined by $\varphi(g_j g) = g_i g$ for each g in G is well defined and a group morphism. But then $\varphi(g_j) = \varphi(g_j e) = g_i e = g_i$ and so from group theory p_i divides p_j and so $i = j$. Hence $\{A(g_1), A(g_2), \ldots\}$ is an antichain with cardinality M in the lattice of normal subgroups of G, a contradiction.

Corollary 1. If the group $(G, +)$ has composite order and lattice of normal subgroups linearly ordered, then $(G, +)$ cannot be the additive group of a near-ring with identity.

Since the lattices of normal subgroups of simple groups and of the symmetric groups S_n are linearly ordered, we have the following two results of CLAY and MALONE [2].

Corollary 2 (CLAY and MALONE). *If $(G, +)$ is a simple group of composite order, then $(G, +)$ cannot be the additive group of a near-ring with identity.*

Corollary 3 (CLAY and MALONE). *If $(G, +)$ is a symmetric group S_n, $n \geqslant 3$, then $(G, +)$ cannot be the additive group of a near-ring with identity.*

Corollary 4 (LIGH [4]). *A finite non-abelian group $(G, +)$ with exactly one non-trivial proper normal subgroup cannot be the additive group of a near-ring with identity.*

Proof. A non-abelian p-group has order at least p^3, hence at least 2 non-trivial proper normal subgroups (see [5] Cor. 5.5 and Ex. 5.2). Hence $(G, +)$ has composite order. Apply Corollary 1.

A *left distributive binary operation* $\cdot$ on a group $(G, +)$ satisfies $a(b+c) = ab+ac$ for all a, b and c in G. The proof of the Theorem does not use associativity, hence our results hold for left distributive binary operations on groups. Since the alternating groups A_n have lattice of normal subgroups linearly ordered, we have the following result of CLAY and DOI [1].

Corollary 5 (CLAY and DOI). *There are no left distributive binary operations with identity on the alternating groups A_n, $n \geqslant 4$.*

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