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Titel: On Minimal Vanishing Algebras.

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Jahr: 1970

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Monatshefte für Mathematik 74, 348—350 (1970)
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On Minimal Vanishing Algebras

By

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(*Received August 17, 1969*)

I. Let G be a locally compact group with left invariant Haar measure m. For any measurable subset S of G, define L_S to be the closed subspace of $L^1(G)$ consisting of all functions which vanish (a. e.) on the complement of S. When L_S forms an algebra, it is called a *vanishing algebra*. A vanishing algebra L_S is said to be *minimal* if $L_S \neq 0$ and if, whenever L_T is a vanishing algebra such that $L_T \subset L_S$, then $L_T = L_S$ or $L_T = 0$. In this note we shall prove the following theorems.

Theorem 1. *There exists a minimal vanishing algebra in $L^1(G)$ if and only if G contains a σ-compact open subgroup S which does not contain any proper measurable semigroup of positive measure. If there is a minimal vanishing algebra, it is unique and it is selfadjoint.*

As a by-product of the proof of Theorem 1, we have

Theorem 2. *If $S \subset G$ is a semigroup with $0 < m(S) < \infty$, then S is a compact open subgroup of G.*[2]

II. The notion and study of vanishing algebras was initiated by A. B. Simon [5]. Since then some further studies of vanishing algebras and generalizations of them have appeared [2], [3], [4]. We recall here one operator i and some of its properties. For a measurable subset S of G, $iS = \{x \in G$: there exists a neighborhood U of x such that $m(U/S)=0\}$. iS is open and $iS \setminus S$ is locally negligible. If L_S is a vanishing algebra, then iS is a semigroup. [See 2.]

Lemma 1. If L_S is a non-zero vanishing algebra then $iS \neq \phi$.

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[1] Supported in part by the National Science Foundation of U.S.A.
[2] When G is compact, this result is known. See [1].

Proof. Since $L_S \neq 0$, S is not locally negligible and hence there exists $A \subset S$ such that A has positive finite measure. Let

$$U = \{x \in G : \varphi_A * \varphi_A(x) > 0\}.^3$$

Since $\varphi_A * \varphi_A$ is a non-negative continuous function and since

$$\int_G \varphi_A * \varphi_A \, dm = m(A)^2 > 0,$$

U is a non-empty open set. Since L_S is a vanishing algebra, $m(U \setminus S)=0$. Thus $U \subset iS$ and iS is non-empty.

Corollary 2. If L_S is a minimal vanishing algebra, then $L_S = L_{iS}$.

Proof of Theorem 1. Suppose L_S is a minimal vanishing algebra. By Corollary 2, we may assume that S is an open semigroup. For every $x \in S$, xS is an open semigroup contained in S. Since L_{xS} is then a non-zero vanishing algebra contained in L_S, $xS = S$ locally almost everywhere. It follows that $\overline{S} = \overline{xS} = x\overline{S}$. Now, for any $y \in S$, $y \in \overline{S} = y^3\overline{S}$ implies $y^{-1} \in y\overline{S}$, which in turn implies $S^{-1} \cap yS \neq \phi$ and hence $y^{-1} \in S$. Thus S is a subgroup. To show that S is σ-compact, let V be any open subset of S with compact closure. Then $V^\infty = \bigcup_{n=1}^\infty V^n$ is a semigroup contained in S. Since L_S is minimal, we have $L_S = L_{V^\infty}$. It follows then that $S = V^\infty$, which is σ-compact[4]. The minimality of L_S clearly implies that S cannot contain any proper measurable sub-semigroup of positive measure.

Conversely, if S is a σ-compact open subgroup of G containing no proper measurable semigroup of positive measure, then L_S is a minimal vanishing algebra. For if $0 \neq L_T \subset L_S$, then T is a semigroup a. e. [3], and hence $L_T = L_S$.

Finally, if L_{S_1} and L_{S_2} are minimal vanishing algebras, then they both are equal to $L_{iS_1 \cap iS_2}$. Thus a minimal vanishing algebra, if it exists, is unique.

The self-adjointness of a minimal vanishing algebra follows from the fact that the underlying set is a subgroup.

Proof of Theorem 2. Let S be a sub-semigroup of G with $0 < m(S) < \infty$. Let $T =$ interior of S, then T is non-empty (see [1] or Lemma 1) and T is a semigroup. Since $m(xT) = m(T)$, we have that for any $x \in T$,

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[3] φ_A denotes the characteristic function of A.

[4] In fact, S is connected. If S is abelian, then S is further compact.

$x\overline{T} = \overline{T}$. Arguing exactly as in the proof of Theorem 1, we see that T is an open subgroup. Clearly then $T = S$. That S is compact follows from the fact that the Haar measure m on G, when restricted to S, gives a Haar measure on S, and $m(S) < \infty$.

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