The Coefficients of cosh x/cos x

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1. Put

$$\frac{\cosh x}{\cos x} = \sum_{p=0}^{\infty} S_{2n} \frac{x^{2n}}{(2\,n)!}. \tag{1}$$

Gandhi [3] conjectured that the positive integers S_{2n} are divisible by 2^n. The writer [2] proved the truth of this conjecture and indeed that

$$S'_{2n} = 2^{-n} S_n \equiv (-1)^{\frac{1}{2} n(n-1)} \pmod{4}. \tag{2}$$

Salié [5] in a recent paper gave another proof of (2) and at the same time showed that

$$S'_{2n} \equiv (-1)^{\frac{1}{2} n(n-1)} + 4\,\chi_1(n) + 8\,\chi_2(n) \pmod{16}, \tag{3}$$

where

$$\chi_k(n) = \begin{cases} 0 & (n \equiv 0, 1, \ldots, 2^k - 1 \pmod{2^{k+1}}) \\ 1 & (n \equiv 2^k, \ldots, 2^{k+1} - 1 \pmod{2^{k+1}}). \end{cases}$$

Since

$$2\,\chi_1(n) = 1 - (-1)^{\frac{1}{2} n(n-1)},$$

(3) is equivalent to

$$S'_{2n} \equiv 2 - (-1)^{\frac{1}{2} n(n-1)} + 8\,\chi_2(n) \pmod{16}. \tag{4}$$

The purpose of the present note is to prove the following result:

$$(-1)^{\frac{1}{2} n(n-1)} S'_n = 1 + (-1)^n - \frac{1}{2n+1} \sum_{s=0}^{n} (-1)^{ns + \frac{1}{2} s(s-1)} \binom{2n+1}{2s} 2^{3s} B_{2s}, \tag{5}$$

where B_n is the Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

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Since $2\,B_{2s}$ is integral (mod 2), it is evident that (5) yields congruences (mod 2^r), where r is arbitrary. In particular (5) implies both (1) and (3).

We also obtain an expansion similar to (5) for the coefficients β_{2n} defined by

$$\frac{\sinh x}{\sin x} = \sum_{n=0}^{\infty} \beta_{2n} \frac{x^{2n}}{(2\,n)!}.$$

2. Since

$$\frac{1}{\cos x} = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2\,n)!},$$

where the E_{2n} are the Euler numbers in the even suffix notation, it follows at once from (1) that

$$S_{2n} = \sum_{r=0}^{n} (-1)^r \binom{2\,n}{2\,r} E_{2r}. \tag{6}$$

Now we recall that, in the notation of *Nörlund* [4, Ch. 2], if

$$C_n = 2^n E_n(0),$$

then

$$E^n = (C+1)^n,\ C^n = (E-1)^n \tag{7}$$

and

$$nC_{n-1} = 2^n(1-2^n)\,B_n. \tag{8}$$

Now if $f(x)$ is an arbitrary polynomial, it follows from (8) that

$$f'(C) = f(2\,B) - f(4\,B)$$

and in particular

$$n(C+1)^{n-1} = (2\,B+1)^n - (4\,B+1)^n. \tag{9}$$

Since

$$(2\,B+1)^n = 2^n\,B_n(\tfrac{1}{2}) = 0 \qquad (n \text{ odd}),$$

it follows from (7) and (9) that

$$f'(E) = -f(4\,B+1), \tag{10}$$

where $f(x)$ is any odd polynomial.

We take

$$f(x) = \frac{1}{2\,n+1} \sum_{r=0}^{r} (-1)^r \binom{2\,n+1}{2\,r+1} x^{2\,r+1} \tag{11}$$

so that

$$f'(E) = \sum_{r=0}^{n} (-1)^r \binom{2\,n}{2\,r} E_{2r} = S_{2n},$$

by (6). Thus

$$S_{2n} = -\frac{1}{2n+1}\sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r+1}(4B+1)^{2r+1}. \tag{12}$$

Now

$$\sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r+1}(4B+1)^{2r+1} = \sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r+1}\sum_{s=0}^{2r+1}\binom{2r+1}{s}4^s B_s$$

$$= \sum_{s=0}^{2n+1}\binom{2n+1}{s}4^s B_s \sum_{r}(-1)^r\binom{2n-s+1}{2n-2r}.$$

Since

$$\sum_{r=s-1}^{2n}\binom{2n-s+1}{2n-r}x^r = x^{s-1}(1+x)^{2n-s+1},$$

it is evident that

$$\sum_{r}\binom{2n-s+1}{2n-2r}(-1)^r = \frac{i^{s-1}}{2}\{(1+i)^{2n-s+1} + (-1)^{s-1}(1-i)^{2n-s+1}\}.$$

In particular we have

$$\sum_{r=0}^{n}(-1)^r\binom{2n}{2r} = \tfrac{1}{2}\{(1-i)^{2n} + (1-i)^{2n}\}$$

$$= 2^{n-1}i^n(1+(-1)^n)$$

$$= \begin{cases}(-1)^{n/2}2^n & (n \text{ even})\\ 0 & (n \text{ odd}),\end{cases}$$

$$\sum_{r=s}^{n}(-1)^r\binom{2n-2s+1}{2n-2r} = \frac{i^{2s-1}}{2}\{(1+i)^{2n-2s+1} - (1-i)^{2n-2s+1}\}$$

$$= \frac{i^{2s-1}}{2}(1-i)^{2n-2s+1}((-1)^{n-s}i - 1)$$

$$= -2^{n-s-1}i^{n+s}(i+1)(i-(-1)^{n+s})$$

$$= (-1)^{\frac{1}{2}(n+s)(n+s-1)}2^{n-s}.$$

It follows that

$$\sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r+1}(4B+1)^{2r+1}$$

$$= -2(2+1)\sum_{r=0}^{n}(-1)^r\binom{2n}{2r} + \sum_{s=0}^{n}\binom{2n+1}{2s}4^{2s}B_{2s}\sum_{r=s}^{n}(-1)^r\binom{2n-2s+1}{2n-2r}$$

$$= -2^{n+1}(2n+1)g(n) + \sum_{s=0}^{n}\binom{2n+1}{2s}4^{2s}B_{2s}\cdot(-1)^{\frac{1}{2}(n+s)(n+s-1)}2^{n-s}$$

$$= -2^{n+1}(2n+1)g(n) + 2^n\sum_{s=0}^{n}(-1)^{\frac{1}{2}(n+s)(n+s-1)}\binom{2n+1}{2s}2^{3s}B_{2s},$$

$$g(n) = \begin{cases} (-1)^{n/2} & (n \text{ even}) \\ 0 & (n \text{ odd}). \end{cases}$$

Therefore (12) becomes

$$S'_{2n} = 2\, g(n) - \frac{1}{2\,n-1} \sum_{s=0}^{n} (-1)^{\frac{1}{2}(n+s)(n+s-1} \binom{2\,n+1}{2s} 2^{3s} B_{2s}. \tag{13}$$

Multiplying by $(-1)^{\frac{1}{2}n(n-1)}$ we get

$$(-1)^{\frac{1}{2}n(n-1)} S'_{2n} = 1 + (-1)^n - \frac{1}{2\,n+1} \sum_{s=0}^{n} (-1)^{ns + \frac{1}{2}s(s-1)} \binom{2n+1}{2s} 2^{3s} B_{2s}. \tag{14}$$

We may regard (14) as a 2-adic expansion of $(-1)^{\frac{1}{2}n(n-1)}) S'_{2n}$. Since, by the *Staudt-Clausen* theorem, $2\,B_{2s}$ is integral (mod 2), the term corresponding to the value s on the right of (14) is divisible by at least 2^{3s-1}. Thus for example (14) implies

$$(-1)^{\frac{1}{2}n(n-1)} S'_{2n} \equiv 1 + (-1)^n - \frac{1}{2\,n+1} \equiv 1 \pmod 4,$$

which is identical with (2). Next we have

$$(-1)^{\frac{1}{2}n(n-1)} S'_{2n} \equiv 1 + (-1)^n - \frac{1}{2\,n+1} - (-1)^n \frac{4\,n}{3} \pmod{32}. \tag{15}$$

To see that (15) implies (3) or (4) it suffices to show that

$$2(-1)^{\frac{1}{2}n(n-1)} - 1 + 8\,\chi_2(n) \equiv 1 + (-1)^n - \frac{1}{2\,n+1} - (-1)^n \frac{4\,n}{3} \pmod{16},$$

or what is the same thing

$$8\,\chi_2(n) \equiv 2(1 - (-1)^{\frac{1}{2}n(n-1)}) - (-1)^n (4\,n) \qquad \pmod{16}. \tag{16}$$

For $n \equiv 0, 1, 2, 3 \pmod 8$ we find that the right member of (16) is congruent to $0 \pmod{16}$, while for $n \equiv 4, 5, 6, 7 \pmod 8$, it is congruent to $8 \pmod{16}$. This evidently implies the truth of (16).

3. The following corollary of (14) may be noted. For brevity put

$$S''_{2n} = (-1)^{\frac{1}{2}n(n-1)} (2\,n+1)\, S'_{2n}. \tag{17}$$

Then it is evident that for $r \geq 1$, $b \geq 1$,

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} S''_{2n+4bj} = - \sum_{s=0}^{n+2br} (-1)^{ns + \frac{1}{2}s(s-1)} 2^{3s} B_{2s} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{2n+4bj}{2s}.$$

Since $\binom{x}{2s}$ is a polynomial in x of degree $2\,s$ we have

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{2n+4bj}{2s} = 0 \qquad (r > 2s),$$

so that

$$\sum_{=0}^{2r-1} (-1)^j \binom{2r-1}{j} S''_{2n+4bj} \equiv 0 \qquad (\mathrm{mod}\ 2^{3r-1}). \tag{18}$$

This result can be improved.

4. Let

$$\frac{\sinh x}{\sin x} = \sum_{n=0}^{\infty} \beta_{2n} \frac{x^{2n}}{(2n)!}, \tag{19}$$

so that the β_{2n} are positive rational numbers with odd denominator. Some properties of β_{2n} are discussed in [1]. *Gandhi* [3] has conjectured that the numerator of β_{2n} is divisible by 2^n. If we put [4, Ch. 2]

$$\frac{x}{\sin x} = \sum_{n=0}^{\infty} (-1)^n D_{2n} \frac{x^{2n}}{(2n)!},$$

so that

$$D_{2n} = (2B+1)^{2n} = (2 - 2^{2n}) B_{2n}, \tag{20}$$

then it follows from (19) that

$$\beta_{2n} = \frac{1}{2n+1} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{2s} D_{2s}. \tag{21}$$

Moreover it follows easily from (1) and (19) that

$$2^{2n} \beta_{2n} = \sum_{r=0}^{n} \binom{2n}{2r} \beta_{2r} S_{2n-2r} \tag{22}$$

and

$$S_{2n} = \sum_{r=0}^{n} (-1)^r \binom{2n}{2r} 2^{2n-2r} \beta_{2r} \beta_{2n-2r}. \tag{23}$$

Differentiating (19) we get

$$\sum_{n=1}^{\infty} \beta_{2n} \frac{x^{2n-1}}{(2n-1)!} = \frac{\sinh x}{\sin x} (\coth x - \cot x).$$

Since

$$x \coth x - x \cot x = 2 \sum_{n=0}^{\infty} \frac{(2x)^{4n+2}}{(4x+2)!} B_{4n+2},$$

we get

$$n \beta_{2n} = \sum_{2r<n} \binom{2n}{4r+2} 2^{4r+2} B_{4r+2} \beta_{2n-4r+2},$$

so that

$$2^{-n}\beta_{2n} = \sum_{2r<n} \frac{1}{2r+1}\binom{2n-1}{4r+1} 2^{2r+1} B_{4r+2}\, \frac{\beta_{2n-4r-2}}{2^{n-2r-1}}. \tag{24}$$

This evidently shows that the numerator of β_{2n} is divisible by 2^n; this can also be proved by means of (22).

If we put $\beta'_{2n} = 2^{-n}\beta_{2n}$, (22) and (23) become

$$2^{2n}\beta'_{2n} = \sum_{r=0}^{n}\binom{2n}{2r}\beta'_{2r} S'_{2n-2r} \tag{25}$$

and

$$S'_{2n} = \sum_{r=0}^{n}(-1)^r 2^{2n-2r}\beta'_{2r}\beta'_{2n-2r}, \tag{26}$$

respectively. Thus in particular

$$S'_{2n} \equiv (-1)^n \beta'_{2n} \qquad (\text{mod } 4),$$

so that

$$\beta'_{2n} \equiv (-1)^{\frac{1}{2}n(n+1)} \qquad (\text{mod } 4). \tag{27}$$

We have also from (26)

$$S'_{2n} \equiv (-1)^n\{\beta'_{2n} - 4\beta'_2\beta'_{2n-2}\} \qquad (\text{mod } 16).$$

Since

$$\beta'_2\beta'_{2n-2} \equiv -(-1)^{\frac{1}{2}n(n-1)} \qquad (\text{mod } 4),$$

we get

$$\beta'_{2n} \equiv (-1)^n S'_{2n} - 4(-1)^{\frac{1}{2}n(n+1)} \qquad (\text{mod } 16). \tag{28}$$

In the next place, we have, by (21) and (20),

$$\beta_{2n} = \frac{1}{2n+1}\sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r}\sum_{s=0}^{2r}\binom{2r}{s}2^s B_s$$

$$= \frac{1}{2n+1}\sum_{s=0}^{2n}\binom{2n+1}{s}2^s B_s \sum_r (-1)^r\binom{2n-s+1}{2r-s}.$$

Now, exactly as above,

$$\sum_{r=1}^{n}(-1)^r\binom{2n}{2r-1} = \begin{cases} 0 & (n \text{ even}) \\ (-1)^{\frac{1}{2}(n+1)}\, 2^n & (n \text{ odd}) \end{cases}$$

and

$$\sum_{r=s}^{n}(-1)^r\binom{2n-2s+1}{2r-2s} = (-1)^{\frac{1}{2}(n+s)(n+s+1)}2^{n-s}.$$

Thus

$$\beta_{2n} = -(-1)^{\frac{1}{2}n(n+1)}(1-(-1)^n)\,2^{n-1} +$$

$$+ \frac{1}{2n+1}\sum_{s=0}^{n}\binom{2n+1}{2s}2^{2s}B_{2s}\cdot(-1)^{\frac{1}{2}(n+s)(n+s+1)}2^{n-s}$$

and therefore

$$(-1)^{\frac{1}{2}n(n+1)}\beta'_{2n} = \frac{1}{2n+1} - \epsilon_n + $$

$$+ \frac{1}{2n+1}\sum_{s=1}^{n}(-1)^{ns+\frac{1}{2}s(s+1)}\binom{2n+1}{2s}2^s B_{2s}, \qquad (29)$$

where

$$\epsilon_n = \begin{cases} 0 & (n \text{ even}) \\ 1 & (n \text{ odd}). \end{cases}$$

While (29) can also be thought of as a 2 — adic expansion, the convergence is considerably slower than that of (14). For example it requires a little computation to verify that (29) implies (27).

References

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