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Titel: Numerical Analogs to the Schwarz Alternating Procedure.

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Jahr: 1965

PURL: https://resolver.sub.uni-goettingen.de/purl?362160546_0007|log13

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Numerische Mathematik 7, 91—103 (1965)

Numerical Analogs to the Schwarz Alternating Procedure

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1. Introduction

The alternating procedure of H. A. SCHWARZ [7] was introduced to prove the solvability of the Dirichlet problem for Laplace's equation on the union of two overlapping plane regions, provided that the Dirichlet problem is solvable on each separately, and that their boundaries intersect at non-zero angles. Schwarz proved the convergence of his alternating procedure with the aid of the following property, which we shall call "the $q<1$ property": Let D be a plane domain bounded by a simple closed Jordan curve and let β consist of a finite number of non-adjacent proper subarcs of ∂D. Let γ be any Jordan curve lying interior to D, except that its endpoints may meet ∂D at endpoints of the arcs of β, making non-zero interior angles of intersection with these arcs. Then the supremum on γ of the harmonic measure of β is a number $q<1$. Therefore, if u and v are two harmonic functions on D, and $u \equiv v$ on $\partial D - \beta$, then

$$\max_{\gamma} |u-v| \leqq q \max_{\beta} |u-v|, \qquad q<1.$$

For a proof see [5] for example. As a consequence, the error at the nth step of the alternating procedure is $O(q^n)$.

We will be interested only in the problem of evaluating solutions already known to exist, rather than in proving existence. Schwarz's method presents some intriguing possibilities for numerical methods. Firstly, quite simple explicit solutions by classical methods are often known for simple regions such as rectangles or circles. Also, better numerical solutions, from the standpoint of the computational work involved, are often known for certain types of regions than for others. By Schwarz's method we may be able to extend these classical results and these computational advantages to more complicated regions. Secondly, the alternating procedure may involve one less dimension than the original problem since only the boundary values enter in; moreover, the convergence rate $O(q^n)$ is very good. Thus, the procedure should require little work.

In Section 2 we consider the abstract formulation of a Dirichlet problem on two overlapping regions, and for its numerical solution a corresponding abstract numerical problem, which can be solved by a numerical analog to the Schwarz alternating procedure. The success of this method depends only upon four formal hypotheses; from these follow rates of convergence, stability of the numerical method, convergence to the solution of the Dirichlet problem, etc.

Sections 3 through 5 are examples; in each case the four formal hypotheses are shown to hold. For simplicity of exposition all of our examples will concern

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Laplace's equation in the plane. However, it will be apparent as we progress that the restriction to 2-dimensions is unnecessary, and also that the linear interpolation method developed in Section 3 applies equally well in theory to uniformly elliptic equations.

As a specific example of the general method in Section 3, we present an $O(\delta^2)$ correct approximation on two overlapping discs which requires preliminary evaluation of $O(\delta^{-2})$ matrix coefficients followed by $O(\delta^{-2}|\log\delta|)$ arithmetical operations. Evaluation of the matrix coefficients by their Fourier series requires computation of $O(\delta^{-2}|\log\delta|^2)$ sine and cosine terms. In Section 4 we present an $O(h^2)$ correct finite difference approximation on two overlapping rectangles which requires $O(h^{-2}|\log h|^3)$ arithmetical operations. In Section 5 we consider the case of an overlapping disc and rectangle to show that the classical and finite difference techniques of Sections 3 and 4 can be combined at will.

This paper is taken from the author's masters thesis of several years ago [6]. The author wishes to acknowledge his indebtedness to his advisor, Professor JIM DOUGLAS, JR., of Rice University.

2. The abstract numerical alternating procedure

The usual Dirichlet problem is the following: Given a closed region R, with interior R^0 and boundary ∂R, and a function f on ∂R, find a function u on R which equals f on ∂R and which satisfies certain local conditions on R, such as a partial differential equation and additional continuity requirements. Assuming uniqueness, we denote the unique solution for boundary function f by $u(f)$.

Now consider the Dirichlet problem on the union R of two closed regions R_1, R_2. Let $\beta_i=\partial R_i\cap R_j^0$, $i\neq j$, $\alpha_i=\partial R_i-\beta_i$, $i=1,2$, and suppose that $\partial R=\alpha_1\cup\alpha_2$. Let us consider a fixed Dirichlet problem, with given boundary function f on ∂R, and let u denote its Dirichlet solution. Let f_i be the boundary function on ∂R_i which is the restriction of f to the external arc α_i; that is, $f_i=f$ on α_i, $f_i=0$ on β_i. Let g_i be the boundary function on ∂R_i which is the restriction of u to the interior arc β_i; that is, $g_i=u$ on β_i, $g_i=0$ on α_i. Let $u_i(h)$ denote the solution of the Dirichlet problem on R_i for a given boundary function h on ∂R_i. The notation is summarized in Fig. 1. It is then clear that

$$u=u_1(g_1+f_1) \quad \text{on } R_1,$$
$$u=u_2(g_2+f_2) \quad \text{on } R_2.$$

Therefore, if it is possible to solve the Dirichlet problem on R_1 and R_2 separately, then finding u is equivalent to finding the unknown boundary functions g_1 and g_2. We concentrate henceforth upon the problem of evaluating g_1 and g_2 to within arbitrary accuracy. Notice that as a particular case of the above equations we have

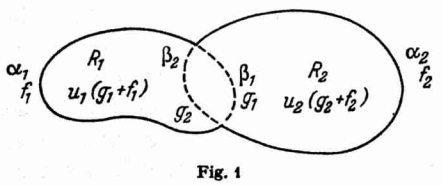

Fig. 1

$$(2.1)\qquad g_1=u_2(g_2+f_2) \quad \text{on } \beta_1,$$
$$g_2=u_1(g_1+f_1) \quad \text{on } \beta_2.$$

We will discretize the problem by considering the values of the function only at a finite set of points. Let ∂R_i^* denote a finite set of points on ∂R_i parti-

tioning ∂R_i. In general the notation $(\cdot)$ when applied to a set denote the partition points of that set. We have then the finite sets $\beta_1^\cdot, \beta_2^\cdot, \alpha_1^\cdot, \alpha_2^\cdot$. Let n_1, n_2, m_1, m_2 be the respective number of points in these sets. The notation $(\cdot)$ when applied to a function denotes its restriction to the partition points where it is defined; that is, the vector (discrete function) whose components are the values of the function at the partition points where it is defined. We have then the vectors, $g_1^\cdot, g_2^\cdot, f_1^\cdot, f_2^\cdot$. Vectors, other than those derived by $(\cdot)$ from functions, will be denoted by capital letters. In all that follows G's will denote vectors defined on the interior $\beta^\cdot$ arcs and identically zero on the exterior $\alpha^\cdot$ arcs.

Let the Dirichlet solution operator u_1, considered as a map from functions on ∂R_1 to functions on β_2, be approximated by a finite dimensional operator U_1, as explained in detail below. Likewise, let U_2 denote a finite dimensional operator approximating u_2, the Dirichlet solution operator on R_2.

Let $\|\ \|$ denote the supremum norm. That is, for a function or vector h defined on a set α, let $\|h\| = \sup_\alpha |h|$.

We now state four formal hypotheses:

(I) The functions g_1, g_2 satisfy (2.1).

(II) U_1, U_2 are known operators, mapping each n_1+m_1, n_2+m_2 dimensional vector defined on $\partial R_1^\cdot$, $\partial R_2^\cdot$ into a computable n_2, n_1 dimensional vector defined on $\beta_2^\cdot, \beta_1^\cdot$.

(III) U_1, U_2 are convergent approximations to u_1, u_2 for the boundary data at hand. That is,

$$\begin{aligned} u_1(g_1+f_1) - U_1(g_1^\cdot+f_1^\cdot) &= E_2 \quad \text{on } \beta_2^\cdot, \\ u_2(g_2+f_2) - U_2(g_2^\cdot+f_2^\cdot) &= E_1 \quad \text{on } \beta_1^\cdot, \end{aligned} \tag{2.2}$$

where a priori bounds are known for the errors $\|E_1\|$ and $\|E_2\|$, and these errors can be made arbitrarily small by choosing the partitions $\partial R_1^\cdot$, $\partial R_2^\cdot$ of ∂R_1, ∂R_2 sufficiently fine.

(IV) The $q<1$ property holds for U_1, U_2, and the arcs $\beta_1^\cdot, \beta_2^\cdot$. That is, for each partition there exist numbers $Q_1 \leqq 1$, $Q_2 \leqq 1$ such that $Q_1 Q_2 < 1$ and

$$\begin{aligned} \|U_1(G_1+f_1^\cdot) - U_1(G_1'+f_1^\cdot)\| &\leqq Q_1 \|G_1 - G_1'\|, \\ \|U_2(G_2+f_2^\cdot) - U_2(G_2'+f_2^\cdot)\| &\leqq Q_2 \|G_2 - G_2'\|, \end{aligned} \tag{2.3}$$

for arbitrary vectors G_1, G_1', G_2, G_2' on $\beta_1^\cdot$ and $\beta_2^\cdot$. Moreover, there exist numbers q_1, q_2, independent of the particular partition, such that $Q_1 \leqq q_1 \leqq 1$, $Q_2 \leqq q_2 \leqq 1$, and $q_1 q_2 < 1$.

We wish to approximate $g_1^\cdot, g_2^\cdot$ by the solution to the numerical problem

$$\begin{aligned} G_1 &= U_2(G_2+f_2^\cdot) \quad \text{on } \beta_1^\cdot, \\ G_2 &= U_1(G_1+f_1^\cdot) \quad \text{on } \beta_2^\cdot. \end{aligned} \tag{2.4}$$

Uniqueness for this solution follows easily from IV. Existence follows from the alternating procedure, which will be mentioned shortly. We now derive an error bound.

Lemma 2.1. *If G_1, G_2 is the solution to* (2.4), *then*

$$\|G_i - g_i^{\cdot}\| \leqq \frac{\|E_1\|+\|E_2\|}{(1-Q_1 Q_2)}, \qquad i=1,2. \tag{2.5}$$

This result follows by subtraction of (2.1) from (2.4) and application of III and IV. Notice also that Q_i above may be replaced by q_i, which are independent of the partition. Therefore, G_i converge to $g_i^{\cdot}$ as $\|E_i\|$ approach zero.

Consider now the solution of (2.4) by a numerical analog to the Schwarz alternating procedure. We define a sequence of approximate solutions G_i^n as follows. Let

$$\begin{aligned} &G_1^0 \text{ be arbitrary on } \beta_1^{\cdot}, \\ &G_1^n = U_2(G_2^{n-1} + f_2^{\cdot}) \quad \text{on } \beta_1^{\cdot}, \qquad n=1,2,\ldots, \\ &G_2^n = U_1(G_1^n + f_1^{\cdot}) \quad \text{on } \beta_2^{\cdot}, \qquad n=0,1,2,\ldots. \end{aligned} \tag{2.6}$$

This shall be called the *numerical alternating procedure.*

One easily shows that G_1^n, G_2^n converge to a solution of (2.4), thus establishing existence. The following lemma establishes the rate of convergence.

Lemma 2.2. *If G_1, G_2 is the solution to* (2.4), *while G_1^n, G_2^n are defined by* (2.6), *then*

$$\|G_i - G_i^n\| \leqq (Q_1 Q_2)^n \|G_1 - G_1^0\|, \qquad i=1,2. \tag{2.7}$$

Proof. Subtracting (2.6) from (2.4) and applying IV, we obtain

$$\begin{aligned} \|G_1 - G_1^n\| &\leqq Q_1 \|G_2 - G_2^{n-1}\|, \\ \|G_2 - G_2^n\| &\leqq Q_2 \|G_1 - G_1^n\|. \end{aligned}$$

From this (2.7) immediately follows.

Now, $U_i(G_i^n + f_i^{\cdot})$ will usually be evaluated only approximately at each step of the procedure, either because of computer round-off error, or because the operator U_i is not known in explicit form, but instead must itself be solved by some approximate iterative method at each step. Instead of G_1^n, G_2^n of procedure (2.4) we would actually calculate J_1^n, J_2^n satisfying

$$\begin{aligned} &J_1^0 = G_1^0 \quad \text{on } \beta_1^{\cdot}, \\ &J_1^n = U_2(J_2^{n-1} + f_2^{\cdot}) + E_1^n \quad \text{on } \beta_1^{\cdot}, \\ &J_2^n = U_1(J_1^n + f_i^{\cdot}) + E_2^n \qquad \text{on } \beta_2^{\cdot}. \end{aligned} \tag{2.8}$$

The following lemma establishes stability with respect to such errors.

Lemma 2.3. *If G_1^n, G_2^n are defined by* (2.6), *while J_1^n, J_2^n are defined by* (2.8), *then*

$$\|J_i^n - G_i^n\| < \left(\frac{2}{1-Q_1 Q_2}\right)\varepsilon, \qquad i=1,2, \tag{2.9}$$

where ε is the maximum of the errors $\|\mathscr{E}_i^k\|$ for $k \leqq n$.

Proof. Subtracting (2.6) from (2.8) and applying IV, we obtain

$$\begin{aligned} &\|J_1^0 - G_1^0\| = 0, \\ &\|J_1^n - G_1^n\| \leqq Q_1 \|J_2^{n-1} - G_2^{n-1}\| + \varepsilon, \qquad n=1,2,\ldots, \\ &\|J_2^n - G_2^n\| \leqq Q_2 \|J_1^n - G_1^n\| + \varepsilon, \qquad n=0,1,2,\ldots. \end{aligned}$$

One first shows that it is sufficient to replace the inequalities above by equalities; that is, $\|J_i^n - G_i^n\|$ are dominated by the Z_i^n which satisfy

$$\begin{aligned} Z_1^0 &= 0, \\ Z_1^n &= Q_1 Z_2^{n-1} + \varepsilon, \qquad n = 1, 2, \ldots, \\ Z_2^n &= Q_2 Z_1^n + \varepsilon, \qquad n = 0, 1, 2, \ldots. \end{aligned}$$

Then (2.9) easily follows and the proof is complete.

Certain simplifications occur if U_1 and U_2 are linear. The numerical problem (2.4) becomes equivalent to a diagonally dominant system of linear equations, and the numerical alternating procedure becomes equivalent to the usual Gauss-Seidel interative solution of such systems. Also, the convergence factors Q_i may often be easily evaluated. Let us develop the linear equations. Let $1, \ldots, n_1$; $n_1+1, \ldots, n_1+m_1$; $1, \ldots, n_2$; $n_2+1, \ldots, n_2+m_2$ index the points of $\beta_1^*, \alpha_1^*, \beta_2^*, \alpha_2^*$ respectively. Let G_{1l} denote the lth component of G_1, and likewise for the other vectors. Let (a_{1lk}) and (a_{2lk}) denote the matrices for U_1 and U_2. Then (2.4) may be written in component form as the following system of linear equations.

$$\begin{aligned} G_{1l} - \sum_{k=1}^{n_2} a_{2lk} G_{1k} &= \sum_{k=n_2+1}^{n_2+m_2} a_{2lk} (f_2^*)_k, \qquad l = 1, \ldots, n_1; \\ G_{2l} - \sum_{k=1}^{n_1} a_{1lk} G_{1n} &= \sum_{k=n_1+1}^{n_1+m_1} a_{1lk} (f_1^*)_k, \qquad l = 1, \ldots, n_2. \end{aligned} \tag{2.11}$$

This system has the matrix

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} -a_{2lk} \\ l = 1, \ldots, n_1 \\ k = 1, \ldots, n_2 \end{matrix} \\ \hline \begin{matrix} -a_{1lk} \\ l = 1, \ldots, n_2 \\ k = 1, \ldots, n_1 \end{matrix} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right) \tag{2.12}$$

Notice that the smallest possible value for the factor Q_1 of (IV) is given by

$$Q_1 = \max_{l=1, \ldots, n_2} \left[\sum_{k=1}^{n_1} |a_{1lk}| \right]. \tag{2.13}$$

The condition $Q_1 Q_2 < 1$ from IV is thus just the condition that the matrix (2.12) be diagonally dominant.

If the coefficients a_{1lk} are non-negative, then we have in analogy with the harmonic case that the factor Q_1 is the maximum on β_2^* of the discrete analog of the harmonic measure of β_1^*. That is,

$$Q_1 = \|U_1(\chi(\beta_1^*))\| \tag{2.14}$$

where $\chi(\beta_1^*)$ is the vector which is 1 on β_1^* and 0 on α_1^*.

Extension of the numerical alternating procedure to the case of three or more overlapping regions is possible of course. However, new decisions have to be made in each situation. For example, a portion of ∂R_1 may lie in both R_2^0 and R_3^0. It must be decided whether to treat the values on this portion as

coming from the Dirichlet solution on R_2 or the Dirichlet solution on R_3. What we have derived with Lemmas 2.1, 2.2, and 2.3, at least in the linear case, is a block analysis for a diagonally dominant system of equations. The important fact established by this block analysis is that the rate of convergence and stability of the alternating procedure depend only on the geometry of the overlapping regions, and not on the fineness of the partitions. This result carries over to the case of more than two regions.

3. The linear interpolation approximation

Let R_1, R_2 be closed plane regions with intersecting interiors, each bounded by a piecewise smooth simple closed Jordan curve. Let R, β_i, α_i be defined as in Section 2, $i=1, 2$. Assume that $\partial R=\alpha_1\cup\alpha_2$, that β_i consists of a finite number of non-adjacent proper subarcs of ∂R_i, and that the closure of β_j meets ∂R_i only at end points of the arcs of β_i, making non-zero interior angles of intersection with these arcs, $i\neq j$. Thus the supremum on β_j of the harmonic measure of β_i is a number $q_i<1$.

Consider the Dirichlet problem for Laplace's equation on R for a fixed boundary function f on ∂R; let u denote the solution. Suppose that u is sufficiently smooth on R that

$$|u_{ss}|\leqq M \tag{3.0}$$

on the smooth arcs of ∂R_i, where u_{ss} denotes the second derivative of u with respect to arc length s. Let g_i and f_i be defined as in Section 2.

Let ∂R_i^* denote a finite set of points $\{s_k\}$ on ∂R_i, partitioning ∂R_i with maximum arc length interval δ, with the corners between the smooth arcs of ∂R_i included as partition points, and with all end points of the arcs of β_i included as partition points. Let $\beta_i^*, \alpha_i^*, n_i, m_i, g_i^*, f_i^*$ be defined as in Section 2.

Let $u_i(h; x, y)$ denote the solution of the Dirichlet problem, evaluated at the point (x, y) in R_i, for a given boundary function $h(s)$ on ∂R_i. Corresponding to each function $h(s)$ on ∂R_i there is the vector (discrete function) h^* on ∂R_i^* with components $(h^*)_k=h(s_k)$. We wish a numerical approximation to $u(h; x, y)$ in terms of h^* only. Our first reaction is to attempt to approximate h itself, and in such a fashion as to preserve the $q_i<1$ property of the operator u_i. Let $\hat{h}(s)$ denote the polygonal function equal h at the partition points and defined by linear interpolation with respect to arc length between. Then let U_i be the finite dimensional operator defined for every vector H on ∂R_i^* and each (x, y) in R_i by

$$U_i(H; x, y)=u_i(\hat{H}; x, y). \tag{3.1}$$

Notice that if a vector G is identically zero on α_i^* then its linear interpolation $\hat{G}(s)$ is identically zero on α_i since all end points of the arcs of β_i belong to α_i^*, not to β_i^*. Also, $\|\hat{G}\|_{\beta_i}=\|G\|_{(\beta_i^*)}$ since the linear interpolation assumes its extremums at the partition points. The subscript here indicates the set over which the supremum norm is taken of course. Thus,

$$\|U_i(G)\|_{\beta_j}=\|u(\hat{G})\|_{\beta_j}\leqq q_i\|\hat{G}\|_{\beta_i}=q_i\|G\|_{(\beta_i^*)}; \tag{3.2}$$

in other words, U_i preserves the $q<1$ property of u_i.

Consider an error bound for the approximation U_i. Suppose that the second derivative of $h(s)$ satisfies the bound $|h_{ss}| \leqq M$ on ∂R_i. It is easily shown by Taylor's expansion with remainder that $h(s)$ differs from its linear interpolation $\hat{h}(s)$ by at most $\frac{1}{4}M(\delta)^2$. Also, $|u_i(h; x, y) - u_i(\hat{h}; x, y)| \leqq \|h - \hat{h}\|$ by the maximum principle. Therefore,

$$|u_i(h; x, y) - U_i(h^*; x, y)| \leqq \tfrac{1}{4}M(\delta)^2 \tag{3.3}$$

at each point (x, y) in R_i.

We now put the operator U_1 in matrix form. We are actually interested in the values of $U_1(H; x, y)$ only at the points of β_2^*. Let these points have coordinates (x_l, y_l) in R_1, $l = 1, \ldots, n_2$. Let $t_k(s)$ be the linear interpolation of the vector which is 1 at the kth point s_k on ∂R_1^* and 0 at s_j for $j \neq k$; that is, $t_k(s)$ is the triangular peak between s_{k-1} and s_{k+1} with height 1 at s_k. Then $\hat{H}(s)$ on ∂R_1 may be written

$$\hat{H}(s) = \sum_{k=1}^{n_1+m_1} H_k t_k(s).$$

The Dirichlet solution for $\hat{H}(s)$ may therefore be written as the linear combination

$$U_1(H; x, y) = \sum_{k=1}^{n_1+m_1} H_k u(t_k; x, y).$$

Thus, the operator U_1 maps each vector $\{H_k;\ k = 1, \ldots, n_1 + m_1\}$ on ∂R_1^* into the vector $\{[U_1(H)]_l:\ l = 1, \ldots, n_2\}$ on β_2^* by the rule

$$[U_1(H)]_l = \sum_{k=1}^{n_1+m_1} a_{1lk} H_k, \tag{3.4}$$

where the matrix (a_{1lk}) is given by

$$a_{1lk} = u_1(t_k; x_l, y_l). \tag{3.5}$$

Similarly define the operator U_2 with matrix a_{2lk} mapping vectors on ∂R_2^* to vectors on β_1^*.

We make the final assumption that R_1, R_2 are regions for which the coefficients a_{1lk} and a_{2lk}, given by (3.5), can be easily evaluated. Several examples will be mentioned shortly.

Lemma 3.1. *The hypotheses I—IV are satisfied in the present examples. The errors* $\|E_1\|$, $\|E_2\|$ *here are bounded by* $\frac{1}{4}M(\delta)^2$.

Proof. I and II are clear by the definitions of u_i, f_i, g_i, and U_i. III and the error bound $\frac{1}{4}M(\delta)^2$ follow from (3.0) and (3.3). IV and the convergence factors q_1, q_2 follow from (3.2). This completes the proof.

After calculation of the coefficients, the numerical problem (2.4) may be written as the diagonally dominant system of linear equations (2.11), and the numerical alternating procedure may be carried out as the usual Gauss-Seidel iteration for such systems. At the nth iteration we have the following error bound, which follows immediately from Lemmas 3.1, 2.1, and 2.2.

Lemma 3.2.

$$\|G_i^n - g_i^*\| = O(\delta^2) + O\left((q_1 q_2)^n\right). \tag{3.6}$$

The coefficients a_{1lk} and a_{2lk} can be evaluated easily by classical methods for certain simple regions. For example, for circular regions one has the Poisson integral formula. These coefficients can then be evaluated by high order correct numerical integration techniques; perhaps the indefinite integrals can even be found in finite form. Also, for circular and rectangular regions one easily writes down the Fourier series formula for these coefficients and obtains good estimates on rates of convergence.

As a specific example, let us consider the case of two overlapping discs. For simplicity, assume that both are unit discs and that we may choose partitions with equal subdivisions of length δ. Let ϑ be the interior angle of intersection of the two boundary circles. Now, it is well-known that the level curves for the harmonic measure of a boundary arc β of a disc are just the circles through its end points. Along a circle of this family which meets β with interior angle ϑ the harmonic measure has the value $1-(\vartheta/\pi)$. Therefore, the convergence factors in the present case are given by $q_1=q_2=1-(\vartheta/\pi)$.

Let us investigate the work requirements to insure approximation of g_1 and g_2 with an error no larger than $O(\delta^2)$. By (3.6) it is sufficient to choose $n=O(|\log\delta|)$. The matrix contains $O(\delta^{-2})$ coefficients and this is the number of arithmetical operations per iteration. Therefore, the method requires initial evaluation of $O(\delta^{-2})$ matrix coefficients, followed by $O(|\log\delta|)$ iterations or $O(\delta^{-2}|\log\delta|)$ arithmetical operations.

Finally, we investigate the work requirements for evaluation of the coefficients by the Fourier series method. Let ε_{lk} denote an error in evaluation of the coefficient a_{1lk}. Now, analysis shows that if $\max_l \sum_k |\varepsilon_{lk}|<\varepsilon$ then the coefficient error introduces an error in the final answer which is $O(\varepsilon)$. We want $\varepsilon=\delta^2$ and there are $O(\delta^{-1})$ coefficients on each row of the matrix. Thus, it suffices to make $\varepsilon_{lk}=O(\delta^3)$. The L_2 norm of the triangular peak $t_k(s)$ is $O(\delta)$. Thus, by the Schwarz inequality applied to the Fourier series expansion of the harmonic function with these boundary values, the uniform error of the Nth partial Fourier expansion evaluated at the radius $r<1$ is less than $O\left(\frac{\delta r^N}{1-r}\right)$. Let the points (x_l, y_l) of β_2^* have polar coordinates (r_l, ϑ_l), ordered in order of decreasing radius, $r_{l+1}\leqq r_l$. Since the two circles intersect at a non-zero angle ϑ, there exists a geometrical constant K, depending only on ϑ, such that $r_l\leqq 1-lK\delta$. Thus, the number N_{lk} of Fourier terms required to insure $\varepsilon_{lk}=O(\delta^3)$ is given by

$$O[(1-lK\delta)^N l\, k]=O(\delta^3),$$

or

$$N_{lk}=O\left(\frac{1}{l\,\delta}|\log\delta|\right).$$

Adding with respect to l and k, we have that the total number of trigonometric terms which must be evaluated is

$$\sum_{l,k} N_{lk}=O(\delta^{-2}|\log\delta|^2).$$

We point out for the sake of emphasis that this method, as opposed to finite difference methods, requires only bounded second derivatives, rather than bounded fourth derivatives, to insure $O(\delta^2)$ accuracy.

It is clear that the linear interpolation method developed above for Laplace's equation applies equally well in theory to uniformly elliptic equations, since the $q<1$ property and the maximum principle used in the arguments are satisfied also for these equations. However, the method is of value only when the coefficients a_{1lk} and a_{2lk} are easily evaluated.

4. The Laplace difference equation on two overlapping rectangles

We first state some results for the Laplace difference equation on a somewhat general region. Let $R=R^0\cup\partial R$ where R^0 is a bounded domain in the (x, y) plane with boundary ∂R. In the usual manner we cover the plane with a square grid of grid size h, with grid points (mh, nh) for all integers m, n. Let R^*, R^{0*}, and ∂R^* denote the grid points of R, R^0, and ∂R. The grid interior of R^* consists of those points of R^* each of whose four grid neighbors also belongs to R^*. The grid boundary of R^* consists of the rest of the points of R^*. We assume that R is a "grid region" for which R^{0*} and ∂R^* coincide with the grid interior and grid boundary of R^*. A function defined at the grid points of R^* will be called a grid function.

Let $u(x, y)$ be the solution on R of the Dirichlet problem for Laplace's equation for a given boundary function f on ∂R. Let $w(x, y)$ be the grid function solution of the corresponding Dirichlet problem for the Laplace difference equation

$$\begin{aligned} &w(x,y)=\tfrac{1}{4}\{w(x+h,y)+w(x-h,y)+\\ &\qquad +w(x,y+h)+w(x,y-h)\}, && (x,y)\in R^{0*},\\ &w(x,y)=f(x,y), && (x,y)\in\partial R^*. \end{aligned} \tag{4.1}$$

Uniqueness, existence, etc. are well-known for this problem. In particular we will use the maximum principle: A grid function satisfying the Laplace difference equation on R^{0*} assumes its maximum and minimum on ∂R^*. It is well-known [4] that under quite general conditions w converges to u as h tends to 0. In particular, if f is such that u has bounded fourth order derivatives on R, then

$$|u(x,y)-w(x,y)|<M(h^2), \qquad (x,y)\in R^{0*}, \tag{4.2}$$

where M is proportional to this bound.

We consider the Laplace difference equation on a rectangle and prove the $q<1$ property for lines intersecting its boundary at right angles. Let R be the rectangle $\{(x, y)\colon 0\leqq x\leqq a,\ 0\leqq y\leqq b\}$ whose corners fall on grid points for a grid size h. Let γ be the interior vertical segment $\{(x, y)\colon 0<x=c<a;\ 0<y<b\}$ where c is also a multiple of h. Let β be the open boundary portion to the right $\{(x, y)\in\partial R\colon x>c\}$ and α closed boundary portion to the left $\{(x, y)\in\partial R\colon x\leqq c\}$.

Lemma 4.1. *Let $w(x, y)$ be the discrete harmonic measure of β^* on R^*, that is, the solution of the Laplace difference equation with boundary values 1 on β^* and 0 on α^*. Then*

$$w(x,y)<\max\left(\frac{1}{2},\frac{c}{a}\right) \tag{4.3}$$

on and to the left of γ^*.

Proof. The following argument employs only symmetry, the maximum principle, and the fact that any linear function satisfies the Laplace difference equation. Consider first the case that $c/a \leqq \frac{1}{2}$. Consider the rectangle $R' < R$, $R' = \{(x, y): 0 \leqq x \leqq 2c,\ 0 \leqq y \leqq b\}$. Let $v(x, y)$ be the solution of the Laplace difference equation on $R'^{\cdot}$ with boundary values $=0$ for $x < c$, $=\frac{1}{2}$ for $x = c$, and $=1$ for $x > c$. By symmetry $v(x, y) = \frac{1}{2}$ on $\gamma^{\cdot}$. But also $v(x, y) \geqq w(x, y)$ on the boundary of $R'^{\cdot}$. Hence $w(x, y) \leqq v(x, y) = \frac{1}{2}$ on $\gamma^{\cdot}$ by the maximum principle.

Consider next the case that $c/a > \frac{1}{2}$. Subtract the linear solution x/a. Then it is sufficient to show that the solution $z(x, y) = w(x, y) - x/a$ is $\leqq 0$ on $\gamma^{\cdot}$. Now, z has boundary values $= -x/a$ for $x \leqq c$, $\leqq 1 - x/a$ for $x > c$. Consider the rectangle $R' > R$, $R' = \{(x, y): 0 \leqq x \leqq 2c,\ 0 \leqq y \leqq b\}$. Let $v(x, y)$ be the solution of the Laplace difference equation on $R'^{\cdot}$ with boundary values $= -x/a$ for $x < c$, $=0$ for $x = a$, $= \frac{2c}{a} - \frac{x}{a}$ for $x > c$. By symmetry $v(x, y) = 0$ on $\gamma^{\cdot}$. Thus, by the maximum principle applied to the right half of $R'^{\cdot}$, $v(x, y) \geqq 0$ on $x = a$. Thus $v(x, y) \geqq z(x, y)$ on the boundary of $R^{\cdot}$. Hence $z(x, y) \leqq v(x, y) = 0$ on $\gamma^{\cdot}$. This completes the proof.

We are now prepared to formulate the numerical alternating procedure for two overlapping rectangles. Let R_1, R_2 be two closed rectangular regions, neither wholly containing the other, with interiors intersecting, with sides parallel to the $x-y$ axes, and with all corners falling on grid points for some square grid of grid size h. Let $R = R_1 \cup R_2$ and let f be a boundary function on ∂R whose Dirichlet solution u on R has bounded fourth order derivatives. Let α_i, β_i, f_i, g_i, u_i, $(i = 1, 2)$ be defined as in Section 2.

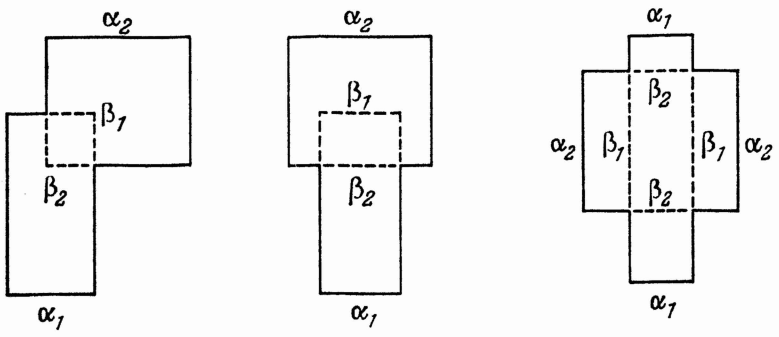

Fig. 2

For a given vector H on $\partial R_1^{\cdot}$ let $U_1(H)$ be defined as the restriction to $\beta_2^{\cdot}$ of $w(x, y)$, where $w(x, y)$ is the grid function solution of the Laplace difference equation on $R_1^{\cdot}$ for boundary values H. Similarly define the map U_2 from vectors on $\partial R_2^{\cdot}$ to vectors on $\beta_1^{\cdot}$.

Lemma 4.2. *The hypotheses I—IV are satisfied in the present example. The errors* $\|E_1\|$, $\|E_2\|$ *here are* $O(h^2)$.

Proof. I and II are clear by the definitions of u_i, f_i, g_i and U_i, $i = 1, 2$. III and the error bound $O(h^2)$ follow from (4.2).

We show next that IV follows from Lemma 4.1. Several of the different manners in which R_1 and R_2 can overlap are shown in Fig. 2. We consider only the first configuration; the proofs for the others are similar. Suppose $G_1 \leqq 1$ on $\beta_1^{\cdot}$ and $G_1 = 0$ on $\alpha_1^{\cdot}$. Extending the vertical portion of β_2 downward to divide R_1 we see that $G_1 \leqq 1$ to the right and $G_1 = 0$ to the left; thus, by Lemma 4.1 and the maximum principle, the solution on $R_1^{\cdot}$ with these boundary values is $\leqq \max(\frac{1}{2}, 1 - r_H)$ on and to the left of the vertical portion of $\beta_2^{\cdot}$, where r_H

is the ratio of the width of β_1 to the width of R_1. Similarly, this solution is $\leqq \max(\frac{1}{2}, 1-r_V)$ on and below the horizontal portion of β_2^*, where r_V is the ratio of the height of β_1 to the height of R_1. Hence, we may take $q_1=\max(\frac{1}{2}, 1-r_V, 1-r_H)$, which factor depends only on the geometry and not on h. This completes the proof.

The numerical alternating procedure (2.6) may be written as follows:

(4.4)
Let $w_1^0(x, y)$ be the solution of the Laplace difference equation on R_1^* for boundary values f_1^* on α_1^* and arbitrary values G_1^0 on β_1^*;

let $w_1^n(x, y)$ be the solution of the Laplace difference equation on R_1^* for boundary values f_2^* on α_2^* and $w_2^{n-1}(x, y)$ on β_1^*, $n=1, 2, \ldots$;

let $w_2^n(x, y)$ be the solution of the Laplace difference equation on R_1^* for boundary values f_2^* on α_2^* and $w_1^n(x, y)$ on β_2^*, $n=0, 1, 2, \ldots$.

Clearly w_1^n converges on R_1^* (and w_2^n converges on R_2^*) to the solution w of the Laplace difference equation on R^* for boundary values f^* on ∂R^*. Therefore, use of Lemma 2.1 is superfluous in the present example, for we already know by (4.2) that the discretization error $\|u-w\|$ is $O(h^2)$.

WALSH and YOUNG [8] prove results similar to Lemma 2.1 and Lemma 4.1. They use these results to extend to more general regions certain precise a priori discretization error bounds developed on a rectangle.

It is the rate of convergence of the alternating procedure which is of interest presently. Since $w(x, y)$ and $w_1^n(x, y)$ have boundary values $G_1+f_1^*$ and $G_1^n+f_1^*$ on ∂R_1^*, $\|w-w_1^n\|=\|G_1-G_1^n\|$ by the maximum principle. Thus, (2.7) states

$$\|w-w_1^n\| \leqq (q_1 q_2)^n \|w-w_1^0\|.$$

At each step in (4.4) we will not solve for w_1^n and w_2^n exactly, but only to within an error of ε in the uniform norm. By Lemma 2.3 this introduces an additional error in $w-w_1^n$ no larger than $\left(\frac{2}{1-q_1 q_2}\right)\varepsilon$. We may for example employ the alternating direction procedure to solve the Laplace difference equation on R_1^* and R_2^*. On a rectangle very good estimates are known for the rate of convergence and work requirements of this method; see [3] for details. The number of arithmetical operations required to reduce the l_2 norm of the error by a factor of ε is $O\left(h^{-2}|\log h|\,|\log \varepsilon|\right)$. Now, the uniform norm on R_1^* is bounded below by the l_2 norm and above by $N^{\frac{1}{2}}$ times the l_2 norm, where $N=O(h^{-2})$ is the number of points in R_1^*. Thus, to reduce the uniform norm of the error by a factor of ε it is sufficient to reduce the l_2 norm by a factor of approximately $\varepsilon\, h$. Therefore, we see that $O\left(h^{-2}|\log h|^3\right)$ arithmetical operations are required to insure that $\|u-w_1^n\|=O(h^2)$.

The method of the present example, including the use of the alternating direction procedure for the inner iteration, has been developed independently by D'JAKONOV [2]. Also, the present example is just a simple case of one of the block methods mentioned in [1]. That article, however, establishes convergence by monotonicity arguments, with no consideration of rates of convergence, stability against error, etc.

5. An example with an overlapping disc and rectangle

We now give an example which shows that classical and finite difference techniques can be combined at will. Let $R=R_1\cup R_2$ where R_1 is a closed disc and R_2 is a closed rectangular region with sides parallel to the $x-y$ axes and with corners falling on grid points for some square grid of grid size h. Let R_1^0 and R_2^0 overlap. Let α_i and β_i be defined as in Section 2, $i=1, 2$.

Consider the Dirichlet problem for Laplace's equation on R for a fixed boundary function f on ∂R; let u denote the solution. Suppose u has bounded fourth and second order derivatives on R. Let g_i and f_i be defined as in Section 2.

Again, let ∂R_1^* denote a finite set of points on ∂R_1, partitioning ∂R_1 with maximum arc length interval δ and with all end points of the arcs of β_1 included as partition points. Let ∂R_2^* denote the grid points of ∂R_2; it is not necessary that the end points of the arcs of β_2 be included.

On the rectangle we employ the Laplace difference equation approximation of Section 4, extended between the grid points by bilinear interpolation. That is, for each vector H on ∂R_2^* we let $U_2(H; x, y)$ on R_2^* be the grid function solution of the Laplace difference equation for boundary values H on ∂R_2^*. We then extend this function to points (x, y) in R_2 between the grid points by bilinear interpolation.

On the disc we employ the linear interpolation approximation of Section 3. That is, for each vector H on ∂R_1^* we let $U_1(H; x, y)$ be the solution of the Dirichlet problem for Laplace's equation on R_1 for boundary values $\hat{H}(s)$ on ∂R_1, where $\hat{H}(s)$ is the linear interpolation of H.

The numerical alternating procedure (2.6) can then be written as follows:

(5.1) Let G_1^0 be an arbitrary vector on β_1^*;

(5.2) let G_1^n be the restriction of $U_2(G_2^{n-1}+f_2^*; x, y)$ to β_1^*, $n=1, 2, \ldots$;
let G_2^n be the restriction of $U_1(G_1^n+f_1^*; x, y)$ to β_2^*, $n=0, 1, 2, \ldots$.

Lemma 5.1. *The hypotheses I—IV are satisfied in the present example. The errors* $\|E_2\|$ *and* $\|E_1\|$ *are* $O(\delta^2)$ *and* $O(h^2)$ *respectively.*

Proof. I and II are clear by the definitions of u_i, f_i, g_i, and U_i. III and the error bounds for $\|E_2\|$ and $\|E_1\|$ follow from Sections 3 and 4. It was shown in Section 3 that U_1 preserves the $q<1$ property of u_1. Therefore, we may let $q_1<1$ be the supremum on β_2 of the harmonic measure of β_1. For q_2 we have no decent analysis, except in certain situations where Lemma 4.1 can be applied. However, we may certainly take $q_2=1$, since bilinear interpolation preserves the maximum principle of the Laplace difference equation. That is, $U_2(H; x, y)$ assumes its maximum and minimum values on ∂R_2^*. This completes the proof.

Consider the work requirements to insure a final accuracy of $O(h^2)$. We choose δ approximately equal h for maximum efficiency. The matrix coefficients a_{1lk} for U_1 should first be calculated to $O(h^3)$ accuracy. According to the analysis of Section 3 this requires evaluation of $O(h^{-2}|\log h|^2)$ trigonometric terms when the Fourier series method is used. Then evaluation of $U_1(G_1^{n-1}+f_1^*; x, y)$ on β_2^* at each iteration involves only a matrix multiplication, or $O(h^{-2})$ arithmetical operations. $U_2(G_2^n+f_2^*; x, y)$ should be evaluated by the alternating direction

procedure to $O(h^2)$ uniform accuracy at each iteration; this requires $O(h^{-2}|\log h|^2)$ arithmetical operations. Finally, $O(|\log h|)$ iterations are required, giving a total requirement of $O(h^{-2}|\log h|^3)$ arithmetical operations in addition to the preliminary evaluation of the matrix coefficients.

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(Received July 21, 1964)