Werk

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Wrapped Distributions and Measurement Errors

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Summary: For a random variable X and $\alpha > 0$ let $U_\alpha := n_\alpha(X) - X$, where $n_\alpha(x) = n \in \mathbf{Z}$ iff $x \in (n\alpha - \alpha/2, n\alpha + \alpha/2]$. Random variables of this type are important in the theory of measurement errors. We derive formulas for the distribution of U_α and apply them to the case $X \sim N(\theta, \sigma^2)$. General conditions for the unimodality of U_α are given. The correlation of the measurement errors $X - E(X)$ and $U_\alpha(X)$ is seen to be $O(\alpha^j)$ with j depending on the smoothness and asymptotic behavior of the density of X. This gives a precise sense to the assertion that scale errors upwards and downwards are averagely well-balanced. In the normal case the density of U_α is shown to be constant up to $o(e^{-C\alpha^{-2}})$, as $\alpha \to 0$.

1. Introduction

Let X be an arbitrary random variable and $\alpha > 0$. We want to measure x by a scale with steps of length α. For $x \in \mathbf{R}$ we define $n_\alpha(x) := n \in \mathbf{Z}$, if and only if $x \in (n\alpha - \alpha/2, n\alpha + \alpha/2]$. Then the scale value nearest to X is $n_\alpha(X)$. We shall consider the deviation

$$U_\alpha := U_\alpha(X) := n_\alpha(X)\,\alpha - X \tag{1}$$

between the measured and the exact value. U_α takes values in $[-\alpha/2, \alpha/2)$, and for $u \in [-\alpha/2, \alpha/2)$ we have

$$P(U_\alpha \leqslant u) = \sum_{n=-\infty}^{\infty} P\left(X \in \left[n\alpha - u, \alpha\left(n + \frac{1}{2}\right)\right]\right). \tag{2}$$

If X has a density function p, an easy computation shows that U_α has the density

$$p_\alpha(u) := \sum_{n=-\infty}^{\infty} p(\alpha n - u), \quad u \in \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right). \tag{3}$$

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In (2) differentiation with respect to u and summation are thus Lebesgue-almost surely interchangeable. This clearly even holds for all u, if p is continuous and the sum in (3) converges uniformly – which is e.g. the case, if p is increasing on $(-\infty, a)$ and decreasing on (b, ∞) for some $a, b \in \mathbf{R}$.

Densities of the form (3) appear in the literature under the name of "wrapped distributions". They play an important role in the statistics of directional data [see e.g. *Mardia*, 1972, p. 53ff., where examples and many references are given, or *Johnson/Kotz*]. In this context the classical notions of mean, variance, correlation etc. are not appropriate and replaced by their "circular" counterparts [*Mardia,* 1972, 19–23]. However in the theory of measurement errors caused by discrete scaling these notions are clearly interpretable in the usual way. In practice it is often assumed that a measurement X of some parameter θ can be written as $X = \theta + Y$, where $E(Y) = 0$. In most situations, however, an additional error enters by using some discrete scale so that the total error is representable as $\alpha n_\alpha(X) - \theta$ or $U_\alpha(X) + Y$.

In this paper we study the distribution of U_α, give criteria for its unimodality and show that "usually" the two types of error U_α and Y have very small correlation. Finally we consider the case of normal distribution.

2. General Results on Wrapped Distributions

We start with the following lemma which is important for the concrete calculation of the distribution of U_α. It is a direct consequence of the Poisson summation formula [*Feller*, p. 630ff.]. By $\hat{p}$ ($\hat{p}_\alpha$) we denote the Fourier transform of p (p_α).

Lemma:

1. Let $\hat{p} \in L_1(\mathbf{R})$. Then we have for all $u \in [-\alpha/2, \alpha/2)$

$$p_\alpha(u) = \alpha^{-1} \sum_{n=-\infty}^{\infty} \hat{p}\left(\frac{2n\pi}{\alpha}\right) \exp\left\{\frac{2\pi i n u}{\alpha}\right\}, \tag{4}$$

if the right side is a continuous function of u.

2. If p is continuous and $\sum_{n=-\infty}^{\infty} p(\alpha n - u)$ is uniformly convergent, (4) holds for Lebesgue-almost all $u \in [-\alpha/2, \alpha/2)$. If p additionally has an integrable second derivative, we have

$$\hat{p}_\alpha(\lambda) = \frac{2}{\alpha} \sin\left(\frac{\lambda\alpha}{2}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n \hat{p}(2n\pi/\alpha)}{\lambda + (2n\pi/\alpha)} \tag{5}$$

for all $\lambda \in \mathbf{R}$ satisfying $\lambda\alpha/2\pi \notin \mathbf{Z}$.

Proof:

1. follows from the formulation (5.9) in *Feller* [1971, p. 632] of the Poisson summation formula, if one sets $s = -u$, $\alpha = \pi/\lambda$ and $\zeta = 0$.
2. p_α is continuous and therefore equals Lebesgue-almost surely its Fourier series [*Carleson*]:

$$p_\alpha(u) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n u/\alpha} \quad \text{a.s.} \tag{6}$$

where

$$a_n = \alpha^{-1} \int_{-\alpha/2}^{\alpha/2} p_\alpha(u)\, e^{-2\pi i n u/\alpha}\, du = \alpha^{-1} \int_{-\infty}^{\infty} p(-s)\, e^{-2\pi i n s/\alpha}\, ds$$
$$= \alpha^{-1}\, \hat{p}\left(\frac{2\pi n}{\alpha}\right). \tag{7}$$

The implicit interchange of summation and integration is justified by uniform convergence. This proves (4). (5) follows from (4), as

$$\hat{p}_\alpha(\lambda) = \alpha^{-1} \int_{-\alpha/2}^{\alpha/2} \left(\sum_{n=-\infty}^{\infty} \hat{p}\left(\frac{2\pi n}{\alpha}\right) e^{(2\pi i n t/\alpha) + it\lambda} \right) dt$$

$$= \alpha^{-1} \sum_{n=-\infty}^{\infty} \hat{p}\left(\frac{2\pi n}{\alpha}\right) \int_{-\alpha/2}^{\alpha/2} \exp\left\{ i \left(\lambda + \frac{2n\pi}{\alpha} \right) t \right\} dt \tag{8}$$

$$= \frac{2}{\alpha} \sin\left(\frac{\lambda\alpha}{2}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n\, \hat{p}(2n\pi/\alpha)}{\lambda + (2n\pi/\alpha)},$$

where $\lambda\alpha/2\pi \notin \mathbf{Z}$ is used for the last equation. Again summation and integration could be interchanged, since $p'' \in L_1(\mathbf{R})$ implies $\hat{p}(\lambda) = o(|\lambda|^{-2})$, as $|\lambda| \to \infty$.

Remarks:

1. By (4), U_α is uniformly distributed on $[-\alpha/2, \alpha/2)$, if $\hat{p}(2\pi n/\alpha) = 0$ for all $n \neq 0$. This is for example true for all $\alpha \in (0, 2\pi M)$, i.e. for sufficiently small α, if p vanishes outside the interval $(-M^{-1}, M^{-1})$.
2. If X is an arbitrary and V a uniformly (on $[-\alpha/2, \alpha/2)$) distributed random variable, the measurement deviation $U_\alpha(X + V)$ of $X + V$ is uniformly distributed and independent of X. For we have, if $x \in \mathbf{R}$, $n = n_\alpha(x)$ and $u \in [-\alpha/2, \alpha/2)$,

$$P(U_\alpha(X+V) \leqslant u \mid X=x) = P\left(x - \frac{\alpha}{2} \leqslant x + V < n\alpha + u\right) + P\left(n\alpha + \frac{\alpha}{2} \leqslant x + V < x + \frac{\alpha}{2}\right)$$

$$= \frac{u}{\alpha} + \frac{1}{2} \tag{9}$$

and thus for $t \in \mathbf{R}$

$$P(U_\alpha(X+V) \leqslant u, X \leqslant t) = \int_{-\infty}^{t} P(U_\alpha(X+V) \leqslant u \mid X=x) P(X \in dx)$$

$$= \left(\frac{u}{\alpha} + \frac{1}{2}\right) P(X \leqslant t). \tag{10}$$

Taking the convolution of X with V means that the origin of the underlying scale with span α is not fixed in advance, but is given as the value of a uniformly distributed random variable independent in X.

3. The bilateral exponential distribution (with density $p(x) = e^{-|x|}$) is one of the rare cases for which p_α can be given in closed form: Using identity 1.445,2, in *Gradshteyn/Rychik* [1980, p. 40] we get, e.g., for $\alpha = 1$, by (4), $p_\alpha(u) = \cos(u - 1/2)/\sin 1$.

We now consider some classes of densities p which lead to strictly unimodal densities p_α. This property has often been studied in the literature because of its importance for circular models [*Mardia*, 1972, 51–68]. In our context unimodality is desirable, as it indicates a high probability for small measurement errors. *Lévy* [1939] already proved that U_α is unimodal for $X \sim N(0, \sigma^2)$. This has been generalized by *Wintner* [1947] to all symmetric stable distributions using a different method. We shall give the first general theorem on unimodality of wrapped distributions. First we define for $s, t > 0$

$$P_t := \{p \text{ density on } \mathbf{R} \mid p \text{ symmetric around 0 and strictly convex on } [t, \infty)\} \tag{11}$$

$$\tilde{P}_s := \left\{p \text{ density on } \mathbf{R} \mid \hat{p} \text{ real and non-negative, } \hat{p}(\lambda) > \frac{\pi}{2} \sum_{n=2}^{\infty} n^2 \hat{p}(n\lambda) \text{ for all } \lambda > \frac{\pi}{2s}\right\}. \tag{12}$$

$\bigcup_{t>0} P_t$ contains nearly all important densities with non-compact support. For p to be in $\tilde{P}_s$, its Fourier transform must fall off sufficiently fast. For example all convex

(in $\mathbf{R}_+$), non-negative and even functions f with $f(0) = 1$ are characteristic functions [*Feller*, p. 509]. If f decreases in the right way, its Fourier inverse is in $\tilde{P}_s$. However if $\hat{p}$ has compact support, p is in no $\tilde{P}_s$; for these p see remark 1 above.

The normal density $p^0(x) = (2\pi)^{-1/2} e^{-x^2/2}$ satisfies $p^0 \in P_1 \cap \tilde{P}_1$: to see $p^0 \in \tilde{P}_1$, compute for $\lambda > \pi/2$

$$\frac{(n+1)^2 \hat{p}^0((n+1)\lambda)}{n^2 \hat{p}^0(n\lambda)} = \left(1 + \frac{1}{n}\right)^2 \exp\{-(2n+1)\lambda^2/2\} \tag{13}$$

$$< 4 e^{-3\pi^2/8} < 1/4.$$

(13) implies $n^2 \hat{p}^0(n\lambda) < 4^{-n+1} \hat{p}^0(\lambda)$, $n = 2, 3, \ldots$. Thus

$$\frac{\pi}{2} \sum_{n=2}^{\infty} n^2 \hat{p}^0(n\lambda) < \frac{\pi}{2} \hat{p}^0(\lambda) \sum_{n=2}^{\infty} 4^{-n+1} < \hat{p}^0(\lambda). \tag{14}$$

We shall now show that p_α is strictly unimodal for $\alpha < 4t$, if $p \in \tilde{P}_t$ (i.e. for small α). Moreover, if additionally $p \in P_t$, then p_α is also strictly unimodal for $\alpha \geqslant 4t$. For the other conditions see remark 1 below.

Theorem 1: Let (4) be term by-term differentiable and (for (ii)) (3) twice term-by-term differentiable.

(i) If $p \in \tilde{P}_t$ and $\alpha < 4t$, p_α is strictly unimodal.

(ii) If $p \in P_t \cap \tilde{P}_t$, p_α is strictly unimodal for all $\alpha > 0$.

Proof:

(i) An easy induction argument shows that

$$f_n(t) := |\sin n\pi t| + n\pi t \leqslant n\pi,\ t \in [1/2, 1],\ n \in \mathbf{N}. \tag{15}$$

For $f_1(t) = \sin \pi t + \pi t$ has no extremum in $(1/2, 1)$, and $f_1(1) = \pi$, $f_1(1/2) = 1 + \pi/2 < \pi$. But if (15) holds for some n, then

$$|\sin(n+1)\pi t| + (n+1)\pi t = |\cos \pi t \sin n\pi t + \cos n\pi t \sin \pi t|$$

$$+ n\pi t + \pi t \leqslant |\sin n\pi t| + n\pi t + |\sin \pi t| + \pi t \leqslant n\pi + \pi.$$

Further it is easily seen that

$$\sin \pi t + 2t \geqslant 2,\quad t \in [1/2, 1]. \tag{16}$$

Let us suppose that $u \in [\alpha/4, \alpha/2)$. Then by (4), using (16) for $n = 1$ and (15) for $n \geqslant 2$ with $t = 2u/\alpha \in [1/2, 1)$ and the defining property of $\tilde{P}_t$ (note that for $\alpha < 4t$ we have $\lambda = 2\pi/\alpha > \pi/2t$ and that the sin-series in (4) vanishes) we obtain

$$p'_\alpha(u) = -\frac{2}{\alpha} \sum_{n=1}^{\infty} \hat{p}(2\pi n/\alpha) \frac{2\pi n}{\alpha} \sin(2\pi n u/\alpha)$$

$$\leqslant -\frac{4\pi}{\alpha^2}\left(\hat{p}\left(\frac{2\pi}{\alpha}\right)\left(2 - \frac{4u}{\alpha}\right) - \sum_{n=2}^{\infty} n\,\hat{p}(2\pi n/\alpha)\left(n\pi - \frac{2\pi n u}{\alpha}\right)\right)$$

$$= -\frac{4\pi}{\alpha^3}\left(\frac{\alpha}{2} - u\right)(4\hat{p}(2\pi/\alpha) - 2\pi \sum_{n=2}^{\infty} n^2\,\hat{p}(2\pi n/\alpha)) < 0. \tag{17}$$

Thus p_α is strictly decreasing in $[\alpha/4, \alpha/2)$. Now let $u \in (0, \alpha/4)$. We need the functional equation

$$p_{\alpha/2}(u) = p_\alpha(u) + p_\alpha\left(\frac{\alpha}{2} - u\right), \quad u \in \left(0, \frac{\alpha}{4}\right), \tag{18}$$

which is valid for arbitrary p symmetric around 0 (this is easily seen, if one inserts (3) in (18)). Differentiating (18)) one gets

$$p'_{\alpha/2}(u) = p'_\alpha(u) + \frac{d}{du} p_\alpha\left(\frac{\alpha}{2} - u\right). \tag{19}$$

Because of $\alpha/2 - u \in [\alpha/4, \alpha/2)$, it follows that $(d/du)\, p_\alpha(\alpha/2 - u) = -$ $-p'_\alpha(\alpha/2 - u) > 0$. Now suppose $p'_\alpha(u) \geqslant 0$. Then (19) yields

$$p'_{\alpha/2}(u) > 0.$$

Let us choose $N \in \mathbf{N}$ such that $\alpha/2^{N+1} \leqslant u < \alpha/2^N$. If for some $n \in \{1, \ldots, N-2\}$ it is proved that $p'_{\alpha/2^n}(u) \geqslant 0$, we can conclude as above that

$$p'_{\alpha/2^{n+1}}(u) = p'_{\alpha/2^n}(u) - p'_{\alpha/2^n}(\alpha 2^{-n-1} - u), \tag{20}$$

and therefore $p'_{\alpha/2^{n+1}}(u) > 0$. Especially we get $p'_{\alpha/2^{N-1}}(u) \geqslant 0$. But this is impossible, for we have already seen that $p'_\beta(u) < 0$, if $\beta/4 \leqslant u < \beta/2 < 2t$. Hence $p'_\alpha(u) < 0$ also for $u \in (0, \alpha/4)$. As p_α is symmetric around 0, this proves the strict unimodality.

(ii) By the strict convexity of p on $[t, \infty)$ and on $(-\infty, t]$ we clearly have from (3) $p''_\alpha(u) > 0$, if $u \in [t, \alpha/2)$. By (4), $\lim_{u \to \alpha/2} p'_\alpha(u) = 0$. Therefore p_α is strictly convex and decreasing on $[t, \alpha/2)$. If $u \in (0, t)$, as in (i) we choose $N \in \mathbf{N}$ such that $\alpha/2^{N+1} \leqslant u < \alpha/2^N$. Suppose that $p'_{\alpha/2^n}(u) \geqslant 0$. Then if $\alpha/2^n \geqslant 4t$, we have by the above reasoning $p'_{\alpha/2^n}(\alpha 2^{-n-1} - u) < 0$; if $\alpha/2^n < 4t$, the same inequality follows from (i). This holds for $n \in \{0, \ldots, N-2\}$. By (20) we can conclude that $p'_{\alpha/2^{n+1}}(u) > 0$. Thus by induction $p'_{\alpha/2^{N-1}}(u) > 0$. But $\alpha/2^{N-1} \leqslant 4u < 4t$, and this contradicts the result of (i). Therefore $p'_\alpha(u) < 0$, and the proof is complete.

Remarks:

1. The interchange of differentiation and summation in (4) needed in the proof of theorem 1 is, e.g., legitimate, if p'_α has only a finite number of discontintuities and p_α, p'_α are of bounded variation, for example if p'_α has a bounded derivative. This is usually easy to verify starting from (3). For (ii), we need $p''_\alpha(u) = \sum_n p''(\alpha n - u)$. This is ensured e.g., if p is twice differentiable, there is a u_0 such that $\sum_n p'(\alpha n - u_0)$ converges and $\sum_n p''(\alpha n - u)$ converges uniformly [*Rudin*, p. 140; these conditions are sufficient, as $\sum_n p(\alpha n - u)$ converges almost everywhere. For differentation of (4) see *Whittaker/Watson*, p. 168].
2. It is clear that p_α exists, if X has a density. The converse is also true. In fact, X is discrete resp. singular resp. absolutely continuous, if, and only if U_α has the respective property. The discrete case is trivial. If X is singular, there is a Lebesgue null set A such that $P(X \in A) = 1$, and X has no atoms. Then $A_\alpha := \{u \in [-\alpha/2, \alpha/2) \mid \exists\, n\colon \alpha n - u \in A\}$ also has Lebesgue measure 0 and $P(U_\alpha \in A_\alpha) = 1$. Obviously U_α has no atoms and thus is singular. If conversely $P(U_\alpha \in B) = 1$ for some Lebesgue null set B, X is concentrated on $\bigcup_{n=-\infty}^{\infty} (\alpha n - B)$. Finally if U_α has a density, from the above it is clear that the distribution of X cannot have a discrete or singular component. Hence X has a density (related to p_α by (3)).

In the "central limit problem" for wrapped distributions the place of the normal law is taken by the uniform one in the following sense: If $X_1, X_2, \ldots$ is a sequence of independent, identically distributed random variables for which the common distribution is not concentrated on a lattice of the type $\{b + \alpha q k \mid k \in \mathbf{Z}\}$ for some

$b \in \mathbf{R}$ and some $q \in \mathbf{Q}$, it follows for $S_k := X_1 + \ldots + X_k$

$$U_\alpha(S_k) \to M_\alpha \text{ weakly } (k \to \infty) \quad \text{for each } \alpha > 0, \tag{21}$$

where M_α is the uniform distribution on $[-\alpha/2, \alpha/2)$ [*Mardia*, 88–89]. (21) can, e.g., be applied to show a result converse to remark 2 following the introductory lemma: Let X and V be independent and let the support of $P(X \in dx)$ not be contained in a lattice of the above type. Then if $V \overset{D}{=} U_\alpha(X + V)$, V is uniformly distributed on $[-\alpha/2, \alpha/2)$. For let $X, X_1, X_2, \ldots$ be i.i.d. and independent of V. Then

$$V \overset{D}{=} U_\alpha(X_1 + V) \overset{D}{=} U_\alpha(U_\alpha(X_1 + V) + X_2) = U_\alpha(U_\alpha(X_1 + X_2) + V)$$

$$\overset{D}{=} \ldots \overset{D}{=} U_\alpha(U_\alpha(X_1 + \ldots + X_n) + V)$$

for all $n \in \mathbf{N}$. As $U_\alpha(X_1 + \ldots + X_n) \overset{D}{\to} M_\alpha$, we get $V \overset{D}{=} U_\alpha(Y + V)$, where $Y \sim M_\alpha$ and Y is independent of V. But $U_\alpha(Y + V) \sim M_\alpha$, so $V \sim M_\alpha$, as claimed.

We prove now that a result similar to (21) holds for $(tX)_{t>0}$, if $t \to \infty$; If the variance of X is increased by multiplication with t, the measurement deviation will be approximately uniformly distributed for large t.

Theorem 2: Let X have an almost everywhere continuous density p such that $h_\epsilon(x) := \sup\{p(y) \mid y \in (x - \epsilon, x + \epsilon)\}$ is integrable for some $\epsilon > 0$. If for $\alpha > 0$ $p_\alpha^{(t)}$ denotes the density of $U_\alpha(tX)$, we have

$$\lim_{t \to \infty} p_\alpha^{(t)}(u) = 1/\alpha \text{ for Lebesgue-almost all } u \in \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right). \tag{22}$$

Especially

$$U_\alpha(tX) \to M_\alpha \ (t \to \infty) \text{ in total variation.} \tag{23}$$

Proof: Without restriction of generality let $\alpha = 1$. By $[v]$ we denote the largest integer not exceeding v. Then Lebesgue-almost everywhere we get

$$|p_1^{(t)}(u) - 1| = \left| \sum_{n=-\infty}^{\infty} \frac{p((n-u)/t)}{t} - \frac{1}{t} \int_{-\infty}^{\infty} p\left(\frac{v}{t}\right) dv \right|$$

$$\leqslant \frac{1}{t} \int_{-\infty}^{\infty} \left| p\left(\frac{[v]-u}{t}\right) - p\left(\frac{v}{t}\right) \right| dv$$

$$= \int_{-\infty}^{\infty} |p(y + \epsilon_t(y)) - p(y)| \, dy, \tag{24}$$

where $\epsilon_t(y) := ([yt] - yt - u)/t \to 0\ (t \to \infty)$. By assumption on p, $p(y + \epsilon_t(y)) \to$ $\to p(y)$ almost everywhere, and the integrand in (24) is bounded by $h_\epsilon(y) + p(y)$, if $t > (|u| + 1)/\epsilon$. Thus the integral tends to 0, as claimed.

Remark: h_ϵ is e.g. integrable, if p is bounded by some integrable and monotonously decreasing function. Theorem 2 is another version of example (b) in *Feller* [1971, 62–63], where it is proved that $\sup_u |p_1^{(t)}(u) - 1| \leqslant t^{-1} \sup_u p(u)$ holds, if p satisfies some monotonicity conditions. Weak convergence in (23) is also shown in *Mardia* [1972, 91–92]. The problem traces back to *Poincaré* [1912].

Clearly X and $U_\alpha(X)$ are (extremely) dependent random variables, since the second one is a function of the first one. However, our next theorem shows that "usually" X and its measurement deviation have a small correlation coefficient $r_\alpha = \mathrm{Cor}(X, U_\alpha(X))$, if α is small. Examples of uncorrelated, but functionally dependent random variables Z_1 and $Z_2 = f(Z_1)$ are constructed by choosing Z_2 in a "well-balanced" manner such that Z_2 is "as often" larger as smaller than Z_1. Our theorem shows that the measurement error due to a small-step scale and the measured variable are connected in this way. Thus in our context this dependence structure does not only occur in artificial examples, but quite general. We shall show $r_\alpha = O(\alpha^{2k-1})$ with k depending on the smoothness and the asymptotic behavior of p. This gives a quantitative version of the statement that the measurement deviations upwards and downwards compensate each other on an average. Further the theorem implies that the variance of the total error $\alpha n_\alpha(X) - \theta$ is almost equal to the sum of the variance of the "random error" and that of the "scale error". Another result of this type is proved in III, where we show that the density of U_α is exponentially near to the uniform one, if α is small.

Theorem 3: Let $E(X) = 0$ and let X have a $2k$-times continuously differentiable density p which satisfies $p^{(i)}(u) = O(1/|u|^{3+\epsilon})$ for $|u| \to \infty$, $i = 0, 1, \ldots, 2k$, and some $\epsilon > 0$. If r_α is the correlation coefficient of X and $U_\alpha(X)$, then

$$r_\alpha = O(\alpha^{2k-1}) \quad \text{for } \alpha \to 0. \tag{25}$$

Proof: We need the following consequence of the Euler summation formula [*Knopp*, p. 542]: If $f: \mathbf{R} \to \mathbf{R}$ is $(2k+1)$-times continuously differentiable, f and $f^{(2k+1)}$ are integrable and $\lim_{|u| \to \infty} f^{(i)}(u) = 0$ for $i = 0, 1, 2, \ldots, 2k-1$, the following identity holds

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(u)\,du + \int_{-\infty}^{\infty} H_{2k+1}(u) f^{(2k+1)}(u)\,du \tag{26}$$

where we have set [*Knopp*, p. 541]

$$H_{2k+1}(u) := (-1)^{k+1} \sum_{n=1}^{\infty} \frac{2 \sin(2n\pi u)}{(2n\pi)^{2k+1}}. \tag{27}$$

(26) immediately leads to

$$\sum_{n=-\infty}^{\infty} f(\alpha n) = \int_{-\infty}^{\infty} f(\alpha u)\, du + \alpha^{2k+1} \int_{-\infty}^{\infty} H_{2k+1}(u) f^{(2k+1)}(\alpha u)\, du. \tag{28}$$

Next we define $f(u) := \int_{u-\alpha/2}^{u+\alpha/2} t\, p(t)\, dt$. Then we get

$$E(X U_\alpha(X)) = E(X(\alpha n_\alpha(X) - X)) = \sum_{n=-\infty}^{\infty} \alpha n \int_{\alpha n-\alpha/2}^{\alpha n+\alpha/2} u\, p(u)\, du - E(X^2)$$

$$= \int_{-\infty}^{\infty} \alpha x \left(\int_{\alpha x-\alpha/2}^{\alpha x+\alpha/2} u\, p(u)\, du \right) dx - E(X^2)$$

$$+ \alpha^{2k+1} \int_{-\infty}^{\infty} H_{2k+1}(u) f^{(2k+1)}(\alpha u)\, du. \tag{29}$$

We want to obtain an upper bound for $\int_{-\infty}^{\infty} H_{2k+1}(u) f^{(2k+1)}(\alpha u)\, du$. First we note that $f^{(2k+1)}(u)$ is a linear combination of functions of the form $p^{(j)}(u \pm \alpha/2)$, $u\, p^{(j)}(u \pm \alpha/2)$, $0 \leqslant j \leqslant 2k$, which is easily seen by induction. Therefore it can be concluded from the assumptions about p

$$f^{(2k+1)}(u) = O(|u|^{-2-\epsilon}) \quad (|u| \to \infty). \tag{30}$$

(30) yields the existence of a constant M satisfying

$$|f^{(2k+1)}(u)| \leqslant \frac{M}{1 + |u|^{2+\epsilon}} \quad \text{for all } u \in \mathbf{R}. \tag{31}$$

By its definition (27), $|H_{2k+1}(u)|$ is bounded by a constant M': $|H_{2k+1}(u)| \leqslant 2^{2-k} \pi^{-k}$. Thus there is a constant M'' for which

$$\left| \int_{-\infty}^{\infty} H_{2k+1}(u) f^{(2k+1)}(\alpha u)\, du \right| \leqslant M M' \int_{-\infty}^{\infty} (1 + |\alpha u|^{2+\epsilon})^{-1}\, du \leqslant M'' \alpha^{-1}. \tag{32}$$

Using $p(u) = 0(|u|^{-3-\epsilon})$ for $|u| \to \infty$ it is easily verified that for certain constants C, C', K

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xu|\, p(u)\, 1_{[x-\alpha/2, x+\alpha/2)}(u)\, du\, dx \leqslant C + C' \int_{[-K,K]^c} |u|^{-1-\epsilon}\, du < \infty$$

holds true. Hence the Fubini theorem can be applied to the first integral on the right-hand side of (29), and one obtains

$$\int_{-\infty}^{\infty} x \left(\int_{\alpha x-\alpha/2}^{\alpha x+\alpha/2} u\, p(u)\, du \right) dx = \int_{-\infty}^{\infty} \left(\int_{u/\alpha-1/2}^{u/\alpha+1/2} x\, dx \right) u\, p(u)\, du$$

$$= \alpha^{-1} \int_{-\infty}^{\infty} u^2\, p(u)\, du = \alpha^{-1} E(X^2). \tag{33}$$

(29), (32) and (33) together give

$$|E(X U_\alpha(X))| \leqslant M \alpha^{2k}. \tag{34}$$

Setting $f(t) := t \int_{t-\alpha/2}^{t+\alpha/2} p(u)\, du$ resp. $f(t) := t^2 \int_{t-\alpha/2}^{t+\alpha/2} p(u)\, du$ we have $f^{(2k+1)}(t) = O(|t|^{-2-\epsilon})$ resp. $f(t) = O(|t|^{-1-\epsilon})$ so that by the same method as above

$$E(\alpha n_\alpha(X)) = \sum_{n=-\infty}^{\infty} \alpha n \int_{n\alpha-\alpha/2}^{n\alpha+\alpha/2} p(u)\, du$$

$$= \int_{-\infty}^{\infty} x\alpha \left(\int_{\alpha x-\alpha/2}^{\alpha x+\alpha/2} p(u)\, du \right) dx + O(\alpha^{2k})$$

$$= \int_{-\infty}^{\infty} u\, p(u)\, du + O(\alpha^{2k}) = O(\alpha^{2k}) \tag{35}$$

$$E(\alpha^2 n_\alpha(X)^2) = \sum_{n=-\infty}^{\infty} (\alpha n)^2 \int_{\alpha n-\alpha/2}^{\alpha n+\alpha/2} p(u)\, du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha x)^2\, p(u)\, 1_{[\alpha x-\alpha/2, \alpha x+\alpha/2)}(u)\, du\, dx + O(\alpha^{2k})$$

$$= \frac{\alpha^2}{3} \int_{-\infty}^{\infty} \left(3\left(\frac{u}{\alpha}\right)^2 + \frac{1}{4} \right) p(u)\, du + O(\alpha^{2k})$$

$$= \int_{-\infty}^{\infty} u^2\, p(u)\, du + \frac{\alpha^2}{12} \int_{-\infty}^{\infty} p(u)\, du + O(\alpha^{2k})$$

$$= E(X^2) + \frac{\alpha^2}{12} + O(\alpha^{2k}). \tag{36}$$

As $E(X) = 0$, we get using (34), (35) and (36)

$$r_\alpha = \frac{E(U_\alpha(X)X)}{(\mathrm{Var}(X)[E((\alpha n_\alpha(X)-X)^2)-(E(\alpha n_\alpha(X)-X))^2])^{1/2}}$$

$$\leqslant \frac{M\alpha^{2k}}{(\mathrm{Var}(X)(E(\alpha^2 n_\alpha(X)^2)-2E(\alpha n_\alpha(X)X)+E(X^2)-(E(\alpha n_\alpha(X)))^2))^{1/2}}$$

$$= O(\alpha^{2k-1}). \tag{37}$$

Remark: If $E(X) \neq 0$, but the other conditions of theorem 3 hold, one uses the same reasoning as in the above proof to show that the correlation coefficient r_α of $U_\alpha(X)$ and $X - E(X)$ is of order $O(\alpha^{2k-1})$.

3. The Wrapped Normal Distribution

For $X \sim N(\theta, \sigma^2)$, p_α is given by

$$p_\alpha(u) = \alpha^{-1} + \frac{2}{\alpha}\sum_{n=1}^{\infty} \exp\left\{-\frac{2\pi^2\sigma^2 n^2}{\alpha^2}\right\}\cos(2\pi n(u+\theta)/\alpha).$$

Our final theorem indicates how good the derivatives $p_\alpha^{(j)}$, $j = 0, 1, 2, \ldots$, of p_α (including p_α itself) can be approximated by the respective partial sums. Especially it is seen the maximal difference between p_α and the uniform density approaches 0 with exponential speed, as $\alpha \to 0$. In this sense the measurement error is nearly uniform.

Let

$$f_j(u) := \begin{cases} \sin\dfrac{2\pi j}{\alpha}(u+\theta), & j \text{ odd} \\ \cos\dfrac{2\pi j}{\alpha}(u+\theta), & j \text{ even.} \end{cases}$$

Theorem 4: For all $j = 1, 2, \ldots$ there exists a constant K_j depending only on j such that for all $N \in \mathbf{N}$

$$\sup_{u\in(-\alpha/2,\alpha/2]} |p_\alpha^{(j)}(u) - (-1)^{[(j+1)/2]}\, 2^{j+1}\pi^j\alpha^{-j-1}\sum_{n=1}^{N} n^j e^{-2\pi^2\sigma^2n^2/\alpha^2} f_j(u)|$$

$$\leqslant K_j e^{-2\pi^2\sigma^2N^2/\alpha^2}\left((\sqrt{2}\pi\sigma)^{j-1} + \left(\frac{N}{\alpha}\right)^{j+1}\right). \tag{39}$$

(39) remains true for $j = 0$, if $p_\alpha^{(0)} = p_\alpha$ is replaced by $p_\alpha - 1/\alpha$.

Proof: Since the term with the sum in (39) is the sum of the first N terms of the respective derivative of (38), the proof easily follows from the lemma below which is given in somewhat more generality than we need it here, because it has some interest in this own.

Lemma: For all $a > 0$ and $N \in \mathbf{N}$ we have

$$F(a, b, N) := \sum_{n=N}^{\infty} n^b \, e^{-an^2} \leqslant \begin{cases} K_b \, e^{-aN^2} \, (a^{-(b+1)/2} + N^{b+1}), b > -1 \\ K_b \, e^{-aN^2} \, N^{b+1}, b < -1, \end{cases} \tag{40}$$

where K_b depends only on b.

Proof: We distinguish four cases:

(1) $-1 < b \leqslant 0$

(2) $b < -1$

(3) $b > 0, N < \left(\frac{b}{2a}\right)^{1/2}$

(4) $b > 0, N \geqslant \left(\frac{b}{2a}\right)^{1/2}$.

In the cases 1 and 2 the function $t \to t^b \, e^{-at^2}$ is decreasing and therefore

$$F(a, b, n) \leqslant N^b \, e^{-aN^2} + \int_N^{\infty} t^b \, e^{-at^2} \, dt$$

$$= N^b \, e^{-aN^2} + \frac{1}{2} a^{-(b+1)/2} \int_{aN^2}^{\infty} y^{(b-1)/2} \, e^{-y} \, dy. \tag{41}$$

In (41) the substitution $y = at^2$ is used. In case 1 the last integral in (41) does not exceed $\Gamma((b + 1)/2)$. Thus, if additionally $aN^2 < 1$,

$$F(a, b, N) \leqslant N^b \, e^{-aN^2} + \frac{1}{2} a^{-(b+1)/2} \, \Gamma\left(\frac{b+1}{2}\right) e^{1-aN^2}$$

$$= K_{b,1} \, e^{-aN^2} \, (a^{-(b+1)/2} + N^b), \tag{42}$$

where $K_{b,1} = 1 + \frac{e}{2} \Gamma((b + 1)/2)$.

If $aN^2 \geqslant 1$, the substitution $y = aN^2\,(1+u)$ yields

$$\int_{aN^2}^{\infty} y^{(b-1)/2}\, e^{-y}\, dy = e^{-aN^2}\,(aN^2)^{(b+1)/2} \int_0^{\infty} (1+u)^{(b-1)/2} e^{-aN^2 u}\, du$$

$$\leqslant e^{-aN^2}\, a^{(b+1)/2}\, N^{b+1} \int_0^{\infty} (1+u)^{(b-1)/2}\, e^{-u}\, du \tag{43}$$

so that by (41)

$$F(a, b, N) \leqslant K_{b,2}\, N^{b+1}\, e^{-aN^2} \tag{44}$$

with

$$K_{b,2} = 1 + \frac{1}{2} \int_0^{\infty} (1+u)^{(b-1)/2}\, e^{-u}\, du.$$

In case 2 we also use transformation (43), for then

$$\int_0^{\infty} (1+u)^{(b-1)/2}\, e^{-aN^2 u}\, du \leqslant \int_0^{\infty} (1+u)^{(b-1)/2}\, du < \infty. \tag{45}$$

Combined with (41) and (43), we get from (45)

$$F(a, b, N) \leqslant K_{b,3}\, N^{b+1}\, e^{-aN^2} \tag{46}$$

with

$$K_{b,3} = 1 + \frac{1}{2} \int_0^{\infty} (1+u)^{(b-1)/2}\, du.$$

If $b > 0$, the function $t \to t^b\, e^{-at^2}$ is increasing in $(0, (b/2a)^{1/2})$, decreasing in $((b/2a)^{1/2}, \infty)$ and satisfies $t^b\, e^{-at^2} \leqslant (b/2a)^{b/2}\, e^{-b/2}$. In case 3 we can thus conclude that

$$F(a, b, N) \leqslant (b/2a)^{b/2}\, e^{-b/2} + \int_N^{\infty} t^b\, e^{-at^2}\, dt. \tag{47}$$

Since in that case $N < (b/2a)^{1/2}$, we have $b > 2a$ and $aN^2 < b/2$. Therefore

$$\left(\frac{b}{2a}\right)^{b/2} e^{-b/2} \leqslant \left(\frac{b}{2}\right)^{(b+1)/2} a^{-(b+1)/2}\, e^{-aN^2}. \tag{48}$$

For the integral in (47), we apply the method of (41) and (42), if $aN^2 < 1$, and the method of (43), if $aN^2 \geqslant 1$. Then we obtain

$$F(a, b, N) \leqslant K_{b,4}\, (a^{-(b+1)/2} + N^{b+1})\, e^{-aN^2} \tag{49}$$

with

$$K_{b,4} = \max\left[\left(\frac{b}{2}\right)^{(b+1)/2}, \frac{e}{2}\Gamma\left(\frac{b+1}{2}\right), \frac{1}{2}\int_0^\infty (1+u)^{(b-1)/2}\, e^{-u}\, du \quad .\right]$$

It remains to consider the case 4. Then because of the monotonicity of $t \to t^b\, e^{-at^2}$ in (N, ∞) (41) is again valid, the integral term in (42) is estimated from above as in case 3, and we get (49) with $K_{b,4}$ replaced by $K_{b,5} = \max[1, K_{b,4}]$. This completes the proof of the lemma.

Remark: For $j = 0$ and $N = 1$ (39) takes the form

$$\sup_u |p_\alpha(u) - \frac{1}{\alpha}| \leqslant \left(1 + \frac{e}{2}\sqrt{\pi}\right)\left(\frac{2}{\alpha} + \frac{\sqrt{2}}{\pi\sigma}\right) e^{-2\pi^2\sigma^2/\alpha^2}. \tag{50}$$

For example, if $\sigma = 1$, $\alpha = 1/10$, the maximal difference between p_α and the uniform density is smaller than $70\, e^{-200\pi^2}$!

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