On a Formulation of Discrete, N-Person Non-Cooperative Games

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Abstract: We present a method of formulating n-person non-cooperative games and a means of finding their equilibrium points.

Introduction

An n-person game is defined by n pure strategy index sets $T_k = \{i \mid i = 1, \cdots, m_k\}$, $k = 1, \cdots, n$, together with n real-valued payoff functions $M_k(x^1, x^2, \cdots, x^n)$ defined on $\prod_{i=1}^{n} X^i$, where X^i is the set of all m_i-dimensional probability vectors $x^i \cdot x^i_j$ gives the probability with which player P_i plays his pure strategy j. A set of vectors $(\bar{x}^1, \ldots, \bar{x}^k, \ldots, \bar{x}^n)$ is called an equilibrium if $M_k(\bar{x}^1, \ldots, \bar{x}^k, \ldots \bar{x}^n) \geq M_k(\bar{x}^1, \ldots, x^k, \cdots, \bar{x}^n)$ for all k.

Let $D^k = \prod_{i \neq k} T_i$ be the cartesian product of the pure strategy index sets T_i, $i \neq k$, so that the cardinality of D^k is $\prod_{i \neq k} m_i = r_k$. Let $S_k = \{i \mid i = 1 \cdots r_k\}$ index the points in D^k, and, for $k = 1, \cdots n$, let A^k be the $m_k \times r_k$ matrix (a_{ij}), where $a_{ij} = M_k(i_k, d^k_j) = M_k(i_1, \cdots i_{k-1}, i_k, i_{k+1}, \cdots i_n)$, $i_k \in T_k$. Thus the rows of A^k are indexed by T_k and the columns by S_k.

Let Y^k be that r_k-dimensional vector whose jth element Y^k_j is given by $\prod_{i \neq k} x^i_{p_i}$, where p_i is the index from T_i in d^k_j. (Alternately, one can define, for 2 vectors, the product $x^i \ast x^j = (x^i_1\, x^j_1,\, x^i_2\, x^j_1, \cdots, x^i_{n_i}\, x^j_1,\, x^i_1\, x^j_2, \cdots, x^i_{n_i}\, x^j_2, \cdots, x^i_1\, x^j_{m_j}, \cdots, x^i_{m_i}\, x^j_{m_j})$, then define $Y^k = (x^1 \ast x^2 \ast \cdots x^{k-1} \ast x^{k+1} \cdots \ast x^n)$. It is easy to see that $M_k(x^1, \cdots x^n) = x^k A^k Y^k$ and that Y^k is a probability vector. The game is thus formulated in a fashion analogous to the usual formulation of two person games, and each player „views" the game as being played against the aggregate of all other players.

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Let $T'_k, k = 1, \cdots n$, be subsets of T_k, and let $X^{k*} = \{x \in X^k \mid x_i = 0 \quad i \in T_k - T'_k = \bar{T}'_k\}$. Given $x^{i*} \in X^{i*}, i = 1, \cdots n$, let Y^k be the vector thus defined and let $S'_k = \{p \in s_k \mid Y^k_p > 0\}$.

If $T'_k = \{i \mid x^k_i > 0\}$, define the deletion, $x^{k\prime}$, of x^k under T'_k as follows: Let $x^k = (x^k_1, \cdots x^k_{m_k})$ and $I(x^k) = \{i \in T_k \mid x^k_i > 0\}$. Let $m'_k = \text{card } I(x^k)$ and index $I(x^k)$ by $j, j = 1, \cdots m'_k$. Then $x^{k\prime} = (x^k_{i_1}, x^k_{i_2}, \cdots x^k_{i_{m'k}})$, where $i_j < i_{j+1}, j = 1, \cdots m'_{k-1}$. Let x^k be the extension of $x^{k\prime}$, and define the deletion and extension of Y^k and $Y^{k'}$ in the same manner.

Let $A^k_{i\cdot}$ and $A^k_{\cdot j}$ represent, respectively, the ith row and jth column of A^k, and let A^{kt} be the $m_k \times r_k$ matrix, each of whose rows is $A^k_{t\cdot}$, for any $t \in T_k$. Finally, let U represent the set of all equilibrium points of the game.

Theorem I: $(\bar{x}^1, \cdots \bar{x}^n) \in U$ if and only if $\bar{x}^k A^k \bar{Y}^k \geq A^k_{i\cdot} \bar{Y}^k$ for all k.

Proof: This follows directly from the formulation above and from the fact that $\sum_i x^k_i = 1$.

Theorem II: Given any collection of the subsets $T'_1 \cdots T'_n$ of the T_k, $k = 1, \cdots n$, and the resulting subsets $S'_1 \cdots S'_n$, let $B^1 \cdots B^n$ be the submatrices of $A^1, \cdots A^n$ indexed by the $T'_1, \cdots T'_n$ and $S'_1 \cdots S'_n$. Let $t^k \in T'_k$, $k \in \{i \mid i = 1 \cdots n\}$, and consider the system of equalities and inequalities given by:

1) $(B^{kt} - B^k) \bar{Y}^{k'} = 0$
2) $(A^{kt} - A^k) Y^k \geqq 0$

where $Y^{k'}$ is the deletion of Y^k, Y^k a probability vector. If $(\bar{x}^1, \cdots \bar{x}^n)$ solves 1) and 2) for all k, then $(\bar{x}^1, \cdots \bar{x}^n) \in U$.

Proof: Let $\bar{Y}^{k'}$ be a solution of $(B^{kt} - B^k) Y^{k'} = 0$.
Then $B^k_{j\cdot} \bar{Y}^{k'} = B^k_{i\cdot} \bar{Y}^{k'}$ for all $i, j \in T'_k$, so that

$$A^k_{i\cdot} \bar{Y}^k = A^k_j \bar{Y}^k \text{ for all } i, j \in T'_k. \tag{1}$$

Also, if $\bar{Y}^k$ is a solution of $(A^{kt} - A^k) Y^k \geqq 0$, then

$$A^k_{j\cdot} \bar{Y}^k \geqq A^k_{i\cdot} \bar{Y}^k \text{ for all } i \in T'_k. \tag{2}$$

Since $\bar{x}^k$ is the extension of $\bar{x}^{k'}$ under T_k, we have

$$\bar{x}_k = 0 \text{ for all } i \in \bar{T}'_k;$$

now

$$\begin{aligned} \bar{x}^k A^k \bar{Y}^k &= \sum_{i \in T'_k} \bar{x}^k_i A^k_{i\cdot} Y^k \\ &+ \sum_{i \in \bar{T}'_k} \bar{x}^k_i A^k_{i\cdot} \bar{Y}^k \\ &= \sum_{i \in T'_k} \bar{x}^k_i A^k_i \bar{Y}^k \end{aligned}$$

but since $\sum_{i \in T_k} \bar{x}^k_i = 1$ and from (1) and (2),

$$\bar{x}^k A^k \bar{Y}^k = A^k_{i\cdot} \bar{Y}^k \geqq A^k_{j\cdot} \bar{Y}, \quad i \in T'_k, \quad j \in T_k,$$

or $\bar{x} A^k \bar{Y}^k \geqq A^k_{i\cdot} \bar{Y}^k$ for all $i \in T_k$ and for all k. The conclusion follows from Theorem I.

Lemma 1: If $(\bar{x}^1_, \cdots \bar{x}^n) \in U$ then $\bar{x}^k_i = 0$ for all i such that

$$\bar{x}^k A^k \bar{Y}^k > A^k_{i\cdot} \bar{Y}^k.$$

Proof: Assume $\bar{x}^k_i \neq 0$ and $\bar{x}^k A^k \bar{Y}^k > A^k_{i\cdot} \bar{Y}^k$ for $i \in T'_k$. As before, let $\bar{T}'_k$ be the complement of T'_k in T_k. Then

$$\bar{x}^k A^k \bar{Y}^k = \sum_{i \in T'_k} \bar{x}_i A^k_i \bar{Y}^k + \sum_{i \in \bar{T}'_k} \bar{x}_i A^k_i \bar{Y}^k.$$

However, if

$i \notin T'_k$ (i. e. $i \in \bar{T}'_k$) then either $\bar{x}^k_i = 0$ or $\bar{x}^k A^k \bar{Y}^k = A^k_{i\cdot} \bar{Y}^k$, so

$$\bar{x}^k A^k \bar{Y}^k = \sum_{i \in T'_k} \bar{x}_i A^k_i \bar{Y}^k + \sum_{i \in \bar{T}'_k} \bar{x}^k_i (\bar{x}^k A^k \bar{Y}^k)$$

$$= \sum_{i \in T_k} \bar{x}^k_i A^k_{i\cdot} \bar{Y}^k + [1 - \sum_{i \in T_k} \bar{x}^k_i] \bar{x}^k A^k \bar{Y}^k$$

so

$$\sum_{i \in T'_k} \bar{x}^k_i A^k_{i\cdot} \bar{Y}^k = \sum_{i \in T'_k} \bar{x}^k_i (\bar{x}^k A^k \bar{Y}^k).$$

But from the definition of T'_k, this cannot hold unless T'_k is empty; thus proving the lemma for all k.

Lemma 2: If $(\bar{x}^1 \cdots \bar{x}^n) \in U$ then for all $i, j \in T'_k = \{i \mid \bar{x}^k_i > 0\}$,

$$A^k_{i\cdot} \bar{Y}^k = A^k_{j\cdot} \bar{Y}^k \geq A^k_{r\cdot} \bar{Y}^k \text{ for all } r \in \bar{T}'_k.$$

Proof: $\bar{x}^k A^k \bar{Y}^k = \sum_{i \in T'_k} \bar{x}^k_i A^k_{i\cdot} \bar{Y}^k + \sum_{i \in \bar{T}'_k} \bar{x}^k_i A^k_{i\cdot} \bar{Y}^k,$

$$= \sum_{i \in T'_k} \bar{x}^k_i A^k_{i\cdot} \bar{Y}^k$$

so that Lemma 1 gives

$$\bar{x}^k A^k \bar{Y}^k = A^k_{i\cdot} \bar{Y}^k \quad i \in T'_k$$

and

$$A^k_{i\cdot} \bar{Y}^k = A^k_{j\cdot} \bar{Y}^k \quad i, j \in T'_k;$$

since $(\bar{x}^1_, \cdots \bar{x}^n) \in U$,

$$\bar{x}^k A^k \bar{Y}^k \geqq A^k_{r\cdot} \bar{Y}^k \quad r \in \bar{T}'_k,$$

thus proving the lemma.

Theorem III: If $(\bar{x}^1, \cdots \bar{x}^n) \in U$ there exist subsets $T'_1, \cdots T'_n$ and resulting submatrices $B^1, \cdots B^n$ for which $\bar{x}^1, \cdots \bar{x}^m$ yield the solution to the sets of equalities (1) and inequalities (2) defined in Theorem II.

Proof: Choose index sets $T'_1, \cdots T'_n$ such that $T'_k = \{i \mid \bar{x}^k_i > 0\}$, and derive $S'_1, \cdots S'_n$ and $B^1, \cdots B^n$. From Lemma 2:

$$A^k_{i\cdot}\ \bar{Y}^k = A^k_{j\cdot}\ \bar{Y}^k \geqq A^k_{r\cdot}\ \bar{Y}^k \text{ for all } i \in T'_k, j \in T'_k, r \in \bar{T}'_k.$$

Let $\bar{Y}^{k'}$ be the deletion of $\bar{Y}^k$. Choose any $t^k \in T'_k$. Then

$$B^k_{t\cdot}\ \bar{Y}^{k'} = B^k_{i\cdot}\ \bar{Y}^{k'} \qquad i \in T'_k$$

and

$$A^k_{t\cdot}\ \bar{Y}^k \geqq A^k_{i\cdot}\ \bar{Y}^k \qquad i \in T_k,$$

or

$$(B^{kt} - B^k)\ \bar{Y}^{k'} = 0 \text{ for all } k$$

and

$$(A^{kt} - A^k)\ \bar{Y}^k \geqq 0 \text{ for all } k.$$

Thus the $T'_1, \cdots T'_n$ that were chosen produce the equalities and inequalities for which $(\bar{x}^1, \cdots \bar{x}^n)$ yield solutions, and the theorem is proved.

Theorems II and III together show that we can find all equilibrium points of an n-person game by solving the appropriate equalities and inequalities for all possible subsets T'_k.

Given any m_k-probability vector x^k let, as before, $T'_k = \{i \mid x^k_i > 0\}$. Also let $Q(x^k)$ be the set of all m_k-probability vectors with the same index sets T'_k as x^k.

Theorem IV: If $(\bar{x}^1, \cdots \bar{x}^n) \in U$ and $x^k \in Q(\bar{x}^k)$ then

$$x^k\ A^k\ \bar{Y}^k = \bar{x}^k\ A^k\ \bar{Y}^k \text{ for all } k.$$

Proof: From Lemma 2, if $(\bar{x}^1, \cdots \bar{x}^n) \in U$ then $A^k_{i\cdot}\ \bar{Y}^k = A^k_{j\cdot}\ \bar{Y}^k \geqq A^k_{r\cdot}\ \bar{Y}^k$ for all $i \in T'_k, j \in T'_k, r \in \bar{T}'_k$. If $x^k \in Q(\bar{x}^k)$ then

$$x^k\ A^k\ \bar{Y}^k = \sum_{i \in T'_k} x^k_i\ A^k_{i\cdot}\ \bar{Y}^k + \sum_{r \in \bar{T}'_k} x^k_{r\cdot}\ A^k_{r\cdot}\ \bar{Y}^k$$

$$= \sum_{i \in T'_k} x^k_i\ A^k_{i\cdot}\ \bar{Y}^k.$$

But, from Lemma 1, $A^k_{i\cdot}\ \bar{Y}^k = \bar{x}^k\ A^k\ \bar{Y}^k$, $i \in T'_k$, therefore

$$x^k\ A^k\ \bar{Y}^k = \bar{x}^k\ A^k\ \bar{Y}^k \text{ for all } k$$

and the theorem has been proved.

References

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