Werk

Titel: Distribution of the Maximum of the Number of Impulses at Any Instant in a Type II...

Autor: Sankaranarayanan, G.; Suyambulingom, C.

Jahr: 1972

PURL: https://resolver.sub.uni-goettingen.de/purl?358794056_0018|log28

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Distribution of the Maximum of the Number of Impulses at Any Instant in a Type II Counter in a Given Interval of Time [1])

By G. SANKARANARAYANAN and C. SUYAMBULINGOM [2])

Summary: Here we have considered a type II counter in which the inter-arrival times of particles are independent identically distributed random variables following a given distribution. Each particle is assumed to give rise to an impulse and the state of the counter at an instant is defined by the number of impulses present at that instant. Two variations of the model are taken into account depending upon whether the system starts initially with an impulse or not. We have obtained integral equations for the distribution of the maximum of the number of impulses at any instant in a given interval of time for the two models considered. Special cases where the inter-arrival times and the duration of the impulses follow the exponential distribution have also been dealt with.

1. Introduction

Consider a process of events consisting in the arrival of particles to a counter. Let each particle arriving in the time interval $(0, \infty)$ independently of others gives rise to an impulse with probability one. We say that the counter at the instant "t" is in state E_k $(k = 0, 1, 2, \ldots)$ if k impulses are present at that instant. Hence the counter is in one or other of these mutually exclusive states at any instant of time.

Let $t_1, t_2, \ldots$ be the instants at which particles arrive at the counter. Let $(t_n - t_{n-1}), n = 1, 2, \ldots; t_0 = 0$ be identically distributed positive random variables independent of each other with a distribution function $F(x)$. Let $\chi_1, \chi_2, \ldots$ be the lengths of successive impulses. Let then be independent and identically distributed positive random variables with a distribution function $G(x)$. Further let them be lengths of successive impulses. Let them be independent and identically distributed

The model described here is called a type II counter. We now consider a type II counter with two different initial conditions.

---

[1]) Dedicated to Professor V. GANAPATHY IYER on the occasion of his sixty first birth day.

[2]) G. SANKARANARAYANAN and C. SUYAMBULINGOM (Student of the former), Department of Mathematics, Annamalai University, Annamalainagar (P. O.), South India.

In the first model that we consider, we assume that the system is set up at time $t=0$ in state E_0 and the first particle arrives at a later time t_1 followed by other particles at times $t_2, t_3, \ldots$. We call this model the type II (A) model. Let $\eta(t)$ denote the number of impulses present at time t in this model. Let $P_k(t)$ denote the probability that the system is in state E_k at time t so that $P_k(t) = P[\eta(t)=k]$. Let $\alpha(t) = \max_{0 \le u \le t} \eta(u)$ and let $Q_k(t) = P\{\alpha(t)=k\}$.

In the second model that we consider, other things being same as before, we assume that the instant $t=0$ when the system starts coincides with the arrival of a particle so that the system starts with an interval of state E_1. Let us denote this model by the type II (B) model. Let $\eta'(t)$ denote the number of impulses present at time t in this model. Let $P'_k(t) = P\{\eta'(t)=k\}$. Let $\alpha'(t) = \max_{0 \le u \le t} \eta'(t)$ and let $Q'_k(t) = P\{\alpha'(t)=k\}$. In this paper we are concerned with the study of the distributions of $\alpha'(t)$ and $\alpha(t)$.

TAKACS [1956] has determined $P_0(t)$ explicitly when $F(x)$ is exponential and here he has imposed no restriction on $G(x)$. He has considered [1959] the case when $F(x)$ is any general distribution with finite mean and standard deviation and $G(x)$ is exponential. He has determined [1956] an integral equation for $P'_0(t)$ in the general case when $F(x)$ and $G(x)$ are perfectly general, neither of them being exponential. SANKARANARAYANAN has determined $P_k(t)$ in the general case when $F(x)$ and $G(x)$ follow any general distribution. Here $P_k(t)$ and $P'_k(t)$ have been connected by means of an integral equation and using this a Volterra integral equation of the second kind has been obtained for the determination of $P_k(t)$. NEUTS has determined the distribution of the maximum length of a poisson Queue during a busy period when they are served by a single server whose service times are mutually independent identically distributed random variables with an exponential distribution. As far as the authors know no attempt has been made so far to determine $Q_k(t)$ and $Q'_k(t)$ for the models that we have described earlier. In this paper we have obtained integral equations to determine $Q_k(t)$ and $Q'_k(t)$, $k=0, 1, 2, \ldots$. We have also worked out a special case when $F(x)$ and $G(x)$ are both exponential distributions with parameters λ and μ respectively and has obtained and solved a difference equation for the determination of the LAPLACE transform of the distribution of $\alpha'(t)$.

2. Determination of $Q_0(t)$ and $Q'_0(t)$

Let us first consider the type II (B) counter. Since a particle arrives at time $t=0$, $\alpha'(t)$ will not be equal to zero and will surely be ≥ 1. Hence $Q'_0(t)=0$. For the type II (A) model $\alpha(t)=0$ implies that no particle arrives in the time interval $(0, t)$. Hence t_1, the time of arrival of the first particle should be greater than t. Thus we have

$$Q_0(t) = P(t_1 > t) = 1 - F(t). \tag{1}$$

3. Determination of $Q_1'(t)$

In the type II (B) model, there are two mutually exclusive possibilities under which $\alpha'(t)=1$. They are (i) Except the particle arriving at time $t=0$, no other particle arrives in the interval $(0, t)$. This implies that $t_1 > t$ and the probability of this possibility is

$$P_1(t) = P(t_1 > t) = 1 - F(t). \tag{2}$$

(ii) Apart from the particle arriving at time $t=0$, next particle arriving at $t_1 = u$, $0 < u \leq t$ is such that the impulse caused by the initial particle must end on or before the time u and that $\alpha'(t-u)=1$. The probability of this possibility is

Hence
$$P_2(t) = \int_0^t G(u)\, Q_1'(t-u) f(u)\, du. \tag{3}$$

$$Q_1'(t) = P_1(t) + P_2(t) = [1 - F(t)] + \int_0^t G(u)\, Q_1'(t-u) f(u)\, du. \tag{4}$$

Thus $Q_1'(t)$ satisfies the Volterra integral equation of the second kind.

4. Determination of $Q_k'(t)$, $(k > 1)$

In the type II (B) model, there are two mutually exclusive possibilities under which $\alpha'(t)=k$. They are,

(i) The duration χ_0 of the impulse caused by the initial particle arriving at time $t=0$ is greater than t. If the next particle arrives at $t_1 = u$, $0 < u < t, \alpha'(t) = k$ implies $\alpha'(t-u) = k-1$.

The probability $T_1(t)$ of this possibility is

$$T_1(t) = P\{\alpha'(t) = k/\chi_0 > t\} = [1 - G(t)] \int_0^t Q_{k-1}'(t-u) f(u)\, du. \tag{5}$$

(ii) The duration χ_0 of the impulse caused by the initial particle arriving at time $t=0$ is less than t. Let $T_2(t)$ be the probability that $\alpha'(t)=k$ under this condition, Then

$$T_2(t) = P\{\alpha'(t) = k/\chi_0 < t\}. \tag{6}$$

So
$$T_2(t) = Q_k'(t) - [1 - G(t)] \int_0^t Q_{k-1}'(t-u) f(u)\, du. \tag{7}$$

Let us now take it into account the instants of the first and the last arrivals in $(0, t)$.

Let u be the instant of first arrival and let v be the instant of last arrival in $(0, t)$. When $\alpha'(t)=k$ it is seen that $\alpha'(v)=k$. $\alpha'(t)=k$ can happen in two mutually exclusive ways.

(i) $\alpha'(v)=k$, $\chi_0 > v$ and (ii) $\alpha'(v)=k$, $\chi_0 < v$.

Let $T_1'(t)$ and $T_2'(t)$ be the corresponding probabilities.
$T_1'(t)$ is the probability that the first arrival is at u, $0<u<v$, the last arrival is at v, $u<v<t$, that the initial impulse is $\geq v$ and $\alpha'(v)=k$ so that $\alpha'(v-u)=k-1$.

$$T_1'(t)=\int_0^t du \int_u^t [1-G(v)]\,[1-F(t-v)]h(u)h(v)\,Q_{k-1}'(v-u)\,dv. \tag{8}$$

$T_2'(t)$ is the probability that the first arrival is at u, $0<u<v$, the last arrival is at v, $u<v<t$, that the initial impulse is $<v$ and $\alpha'(v)=k$.

$$T_2'(t)=\int_0^t G(v)\,[1-F(t-v)]h(v)\,dv\,P\{\alpha'(v)=k/\chi_0<v\} \tag{9}$$

$$=\int_0^t G(v)\,[1-F(t-v)]h(v)\,dv\,\{Q_k'(v)-[1-G(v)]\int_0^v Q_{k-1}'(v-u)h(u)\,du\}. \tag{10}$$

(obtained by using (7) with $t=v$)

$$Q_k'(t)=T_1'(t)+T_2'(t).$$

$$Q_k'(t)=\int_0^t du \int_u^t [1-G(v)]\,[1-F(t-v)]f(u)f(v)\,Q_{k-1}'(v-u)\,dv$$

$$+\int_0^t G(v)\,[1-F(t-v)]f(v)\,\{Q_k'(v)-[1-G(v)]\int_0^v Q_{k-1}'(v-u)f(u)\,du\}\,dv \tag{11}$$

5. Relationship connecting $Q_k(t)$ and $Q_k'(t)$

For the type II (A) model, $\alpha(t)=k$ implies that the first particle arrives at time $t=u$, $0<u<t$ and that $\alpha'(t-u)=k$. Hence

$$Q_k(t)=\int_0^t Q_k'(t-u)f(u)\,du. \tag{12}$$

Equation (12) is true for all $k=1, 2, 3, \ldots$

Let $$\theta_k'(s)=\int_0^\infty e^{-st}\,Q_k'(t)\,dt$$

and $$\theta_k(s)=\int_0^\infty e^{-st}\,Q_k(t)\,dt.$$

In particular when $F(x)=1-e^{-\lambda x}$, $\lambda>0$, $x\geq 0$ it follows from (12) that

$$\theta_k(s)=[\lambda/(\lambda+s)]\,\theta_k'(s). \tag{13}$$

6. An Example

Here we take

$$F(x)=1-e^{-\lambda x},\ \lambda>0,\ x\geq 0$$

$$G(x)=1-e^{-\mu x},\ \mu>0,\ x\geq 0.$$

In this case

$$Q_0(t) = 1 - F(t) = e^{-\lambda t}. \text{ So } \theta_0(s) = [1/(\lambda + s)] \tag{14}$$

$$Q_1'(t) = e^{-\lambda t} + \lambda \int_0^t e^{-\lambda u}(1 - e^{-\mu u})\, Q_1'(t-u)\, du. \tag{15}$$

$$\theta_1'(s) = \{1/(\lambda+s)\} + \lambda \int_0^\infty e^{-st}\, dt \int_0^t e^{-\lambda u}(1-e^{-\mu u})\, Q_1'(t-u)\, du$$

$$= \{1/(\lambda+s)\} + \lambda\, \theta_1'(s) \int_0^\infty e^{-(\lambda+s)u}(1-e^{-\mu u})\, du.$$

i.e. $$\theta_1'(s)\{1 - \lambda \int_0^\infty e^{-(\lambda+s)u}(1-e^{-\mu u})\, du\} = \{1/(\lambda+s)\}.$$

$$\theta_1'(s)[1 - \{\lambda/(\lambda+s)\} + \{\lambda/(\lambda+\mu+s)\}] = [1/(\lambda+s)].$$

$$\theta_1'(s) = \{(\lambda+\mu+s)/[\lambda^2 + s(2\lambda+\mu) + s^2]\}. \tag{16}$$

But $\theta_1(s) = [\lambda/(\lambda+s)]\, \theta_1'(s)$.

So

$$\theta_1(s) = \{\lambda(\lambda+\mu+s)/(\lambda+s)[\lambda^2 + s(2\lambda+\mu) + s^2]\}. \tag{17}$$

In this case

$$Q_k'(t) = \int_0^t du \int_u^t e^{-\mu v} e^{-\lambda(t-v)} \lambda^2 e^{-\lambda u} e^{-\lambda v} Q_{k-1}'(v-u)\, dv$$

$$+ \int_0^t (1-e^{-\mu v})\, e^{-\lambda v} \lambda\, e^{-\lambda(t-v)} Q_k'(v)\, dv$$

$$- \int_0^t (1-e^{-\mu v})\, e^{-\mu v} e^{-\lambda(t-v)} \lambda\, e^{-\lambda v}\, dv \int_0^v Q_{k-1}'(v-u)\, \lambda\, e^{-\lambda u}\, du \tag{18}$$

$$= \lambda^2 e^{-\lambda t} \int_0^t du \int_u^t e^{-\lambda u - \mu v} Q_{k-1}'(v-u)\, dv$$

$$+ \lambda e^{-\lambda t} \int_0^t (1-e^{-\mu v})\, Q_k'(v)\, dv$$

$$- \lambda^2 e^{-\lambda t} \int_0^t (1-e^{-\mu v})\, e^{-\mu v}\, dv \int_0^v Q_{k-1}'(v-u)\, e^{-\lambda u}\, du \tag{19}$$

$$= \lambda e^{-\lambda t} \int_0^t [1-e^{-\mu v}]\, e^{\lambda v} Q_k'(v)\, dv + \lambda^2 e^{-\lambda t} \int_0^t e^{-2(2\mu-\lambda)} [Q_{k-1}'(v) * e^{-\lambda v}]\, dv$$

where $$A(v) * B(v) = \int_0^v B(v-n)\, A(n)\, du \tag{21}$$

i.e. $$Q'_k(t) = \lambda^2 e^{-\lambda t} \int_0^t e^{-\mu v} \{Q'_{k-1}(v) * e^{-\lambda v}\} dv$$

$$+ \lambda e^{-\lambda t} \int_0^t (1 - e^{-\mu v}) Q'_k(v) dv$$

$$- \lambda^2 e^{-\lambda t} \int_0^t (1 - e^{-\mu v}) e^{-\mu v} \{Q'_{k-1}(v) * e^{-\lambda v}\} dv$$

$$= \lambda e^{-\lambda t} \int_0^t (1 - e^{-\mu v}) Q'_k(v) dv$$

$$+ \lambda^2 e^{-\lambda t} \int_0^t e^{-2\mu v} \{Q'_{k-1}(v) * e^{-\lambda v}\} dv. \tag{22}$$

Thus

$$\theta'_k(s) = \lambda \int_0^\infty e^{-(s+\lambda)t} \{\int_0^t (1 - e^{-\mu v}) Q'_k(v) dv\} dt$$

$$+ \lambda^2 \int_0^\infty e^{-(\lambda+s)t} [\int_0^t e^{-2\mu v} \{Q'_{k-1}(v) * e^{-\lambda v}\} dv] dt \tag{23}$$

$$= \lambda \{[e^{-(s+\lambda)t}/-(\lambda+s)] \int_0^t (1 - e^{-\mu v}) Q'_k(v) dv\}_0^\infty$$

$$+ [\lambda/(\lambda+s)] \int_0^\infty e^{-(s+\lambda)t} (1 - e^{-\mu t}) Q'_k(t) dt$$

$$+ \lambda^2 \{[e^{-(s+\lambda)t}/-(s+\lambda)] \int_0^t e^{-2\mu v} [Q'_{k-1}(v) * e^{-\lambda v}] dv\}_0^\infty$$

$$+ [\lambda^2/(\lambda+s)] \int_0^\infty e^{-(\lambda+s+2\mu)t} [Q'_{k-1}(t) * e^{-\lambda t}] dt.$$

So

$$\theta'_k(s) = [\lambda/(\lambda+s)] [\theta'_k(s+\lambda) - \theta'_k(s+\lambda+\mu)] + [\lambda^2/(\lambda+s)(2\lambda+s+2\mu)] \times \times [\theta'_{k-1}(s+\lambda+2\mu)]. \tag{24}$$

Let

$$E_1 \theta'_k(s) = \theta'_k(s+\lambda)$$
$$E_2 \theta'_k(s) = \theta'_k(s+\mu). \tag{25}$$

Equation (24) now becomes

$$\theta'_k(s) = -\frac{\lambda}{s} E_2 \theta'_k(s) + \Phi'_{k-1}(s), \text{ where } \Phi'_{k-1}(s) = E_2^2 \left\{\frac{\theta'_{k-1}(s)}{s+\lambda}\right\} \tag{26}$$

where

$$\Phi_{k-1}(s) = [\lambda^2/(\lambda+s)(s+2\lambda+2\mu)] \theta'_{k-1}(s+\lambda+2\mu). \tag{27}$$

Successively operating on each side by $[\lambda/(\lambda+s)]\, E_1(1-E_2)$ we get

$$\theta'_k(s) = [\lambda/(\lambda+s)]\, E_1(1-E_2)\, \theta'_k(s) + \Phi_{k-1}(s)$$

$$[\lambda/(\lambda+s)]\, E_1(1-E_2)\, \theta'_k(s) = \lambda/(\lambda+s)\, E_1(1-E_2)\, \frac{\lambda}{\lambda+s}\, E_1(1-E_2)\, \theta'_k(s)$$

$$[\lambda/(\lambda+s)]\, E_1(1-E_2)\, \Phi_{k-1}(s)$$

. . . . . .

Adding all these equations and noting the fact that the LAPLACE transform $\theta'_k(s+n\lambda+r\mu) \to 0$ as n and $r \to \infty$, we have

$$\theta'_k(s) = \sum_1^\infty t_n, \text{ where } t_{n+1} = -\frac{\lambda}{s}\, E_2\, t_n \tag{28}$$

and $t_1 = \Phi_{k-1}(s)$.

References

NEUTS, MARCEL, F.: The distribution of the maximum length of a Poisson Queue during a busy period. Operations Res., **12**, 281–285, 1964.

SANKARANARAYANAN, G.: On some probability problems in a type II counter. J. Indian Math. Soc., **30**, 79–86, 1966.

TAKACS, L.: On some probability problem arising in the theory of counters. Proc. Camb. Phil. Soc., **53**, 488–498, 1956.

– On the limiting distribution of the number of coincidences concerning telephone traffic. Ann. Math. Statist., **30**, 134–142, 1959.

– On the sequence of events selected by a counter from a recurrent process of events. Theory of probability and its applications, Vol. I, 81–91, 1956.

WIDDER, D. V.: The Laplace transform. Princeton University Press, Princeton 1946.