# On the Uniqueness of Scales Derived from Canonical Representations

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In Pfanzagl [1968] a general theory of deriving scales from canonical representations was developed. In this paper we shall solve some mathematical questions connected with the uniqueness of these scales. To illustrate the scaling problem in question, let us consider the example of mental tests. Let $A$ be a set of dichotomous items, $B$ a set of subjects. As usual, we assume that item $a \in A$ will be solved by subject $b \in B$ with probability $P(a,b)$. It seems an admissible idealisation to assume that for all pairs $a,b$ we have $0 < P(a,b) < 1$.

If $P(a',b) = P(a'',b)$ for all $b \in B$, we may consider $a'$ and $a''$ as equivalent. Similarly, if $P(a,b') = P(a,b'')$ for all $a \in A$, we may consider $b'$ and $b''$ as equivalent. This divides the sets $A$ and $B$ into equivalence classes. Subjects in the same equivalence class are of equal intelligence, items in the same equivalence class of equal difficulty. We can suppose the function $P$ being defined on these equivalence classes or restrict the sets $A$ and $B$ to sets $A'$ and $B'$, respectively, which contain just one element from each equivalence class.

The interpretation of $b$ as the intelligence of the subject suggests to assume that for any pair $b',b'' \in B'$, $b' \neq b''$ we either have

$$P(a,b') < P(a,b'') \quad \text{for all} \quad a \in A'$$
or
$$P(a,b') > P(a,b'') \quad \text{for all} \quad a \in A'. \tag{1}$$

If condition (1) is fulfilled we may define a natural ordering in $B'$ by

$$b' \leqslant b'' \quad \text{iff} \quad P(a,b') \leqslant P(a,b'') \quad \text{for all} \quad a \in A'.$$

With this ordering, the function $P$ is strictly increasing in its second variable.

We remark that the dual condition, that for any pair $a'$, $a'' \in A'$, $a' \neq a''$, we either have

$$P(a',b) < P(a'',b) \quad \text{for all} \quad b \in B'$$
or
$$P(a',b) > P(a'',b) \quad \text{for all} \quad b \in B', \tag{2}$$

will be a reasonable idealization of reality only under special circumstances.

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We shall assume that to each intellegence level there corresponds a unique level of difficulty, which is characterized by the fact that a task of this level of difficulty is solved and not solved with equal probability. This correspondence may be formally defined by:

$$\text{To each } b \in B' \text{ there exists a uniquely determined } a \in A' \text{ such that } P(a,b) = 1/2. \tag{3}$$

(Despite the intuitive interpretation which the value 1/2 gives to this correspondence, any other constant between 0 and 1 could be used as well.)

If in addition:

$$\text{For every } a \in A' \text{ there exists exactly one } b \in B' \text{ such that } P(a,b) = 1/2, \tag{4}$$

then the order in $B'$ induces an order in $A'$ in a natural way:

$$a' \leqslant a'' \quad \text{iff} \quad P(a',b') = P(a'',b'') = 1/2 \quad \text{and} \quad b' \leqslant b''. \tag{5}$$

If condition (2) is fulfilled, then the order in $A'$ given by (5) is the same as the order defined by:

$$a' \leqslant a'' \quad \text{iff} \quad P(a',b) \geqslant P(a'',b) \quad \text{for all} \quad b \in B'.$$

In the simplest canonical representation it is assumed that each difficulty level $a \in A'$ can be expressed by a real number $m(a)$, each intelligence level $b \in B'$ by a real number $n(b)$ such that $P(a,b)$ is uniquely determined by $m(a)$ and $n(b)$ and $m(a)$, $n(b)$ are strictly increasing. One of the common assumptions is that

$$P(a,b) = F(n(b) - m(a)). \tag{6}$$

It will be true only in special cases that the dependence of $P$ on $a$ is expressible by a single scale $m$. In general, each item has its own "dispersion" and $P(a,b)$ is not uniquely determined by $m(a)$ and $n(b)$. In the following we shall assume that the dispersion of each item is expressible by a positive number, say $k(a)$, and that $P(a,b)$ is uniquely determined by $m(a)$, $k(a)$ and $n(b)$. One of the current psychometric models assumes that $m$, $k$ and $n$ can be chosen such that

$$P(a,b) = F\left(\frac{n(b) - m(a)}{k(a)}\right). \tag{7}$$

The function $F$ in (6) and (7) is strictly increasing since $n$ is strictly increasing with respect to the order (1).

In both representations the scales $m$ and $n$ can be subject to arbitrary shifts. On account of assumption (3) it is natural to assume (and we may do this without loss of generality) that the scales for intelligence and difficulty are tied together by the requirement that $m(a) = n(b)$ iff $P(a,b) = 1/2$ (i.e., that intelligence level and corresponding difficulty level have the same scale value). This standardization implies that for both representations (6) and (7)

$$F(0) = 1/2.$$

The concern of this paper is the question to what extent the scales $m$ and $n$ are uniquely determined, supposing that representations (6) and/or (7) exist (The reader interested in sufficient conditions for the existence of a representation (6) is referred to PFANZAGL [1968]). The uniqueness-problem will be solved here under the assumption that the ranges of $m$ and $n$ are intervals, which are then identical on account of (3) and (4). In the sequel we denote this interval by $\langle C, D\rangle$. It may be closed, open, half-open, finite or infinite. The problem of uniqueness assertions under less restrictive assumptions (e.g. that the range of $m$ is an interval, and that $B$ and hence the range of $n$ is a discrete set) which is of great importance for possible applications will not be considered here. We mention however that the method of the paper DARÓCZY, GYÖRY [1966] could be applied to this problem.

We shall consider the following three cases:

(i) There exist two representations (6),

(ii) There exists a representation (6) and a representation (7).

(iii) There exist two representations (7).

In the cases (i) and (ii) we shall suppose only strict monotony of $m$ and of $n$, an assumption which is quite natural. It remains an open problem to give uniqueness assertions under these conditions also in the case (iii). We succeeded, however, in solving this problem under additional weak differentiability conditions.

(i) We have to solve the functional equation

$$F(n(b)-m(a)) = F^*(n^*(b)-m^*(a)), \quad F(0) = F^*(0) = 1/2. \tag{8}$$

As the ranges of $m$ and $n$ are both $\langle C, D\rangle$, strict monotony of $m$ and $n$ implies that to every pair of real numbers $y,z \in \langle C, D\rangle$ there exists exactly one pair $a \in A'$, $b \in B'$ such that $y = m(a)$, $z = n(b)$, i.e., $b = n^{-1}(z)$, $a = m^{-1}(y)$.

We introduce new functions $u, v, \Phi$ by

$$u = m^* m^{-1}, \quad v = n^* n^{-1}, \quad \Phi = F^{*-1} F. \tag{9}$$

The above conditions on $m$ and $n$ imply that $u$ and $v$ are strictly increasing, continuous functions. Our equation (8) becomes

$$\Phi(z-y) = v(z) - u(y), \quad \Phi(0) = 0, \quad y,z \in \langle C, D\rangle. \tag{10}$$

If we put $y = z$ we get $u(z) = v(z)$ and (10) takes the form

$$\Phi(z-y) = v(z) - v(y), \quad y,z \in \langle C, D\rangle. \tag{11}$$

From (11) we deduce

$$v(z) - v\left(\frac{y+z}{2}\right) = \Phi\left(\frac{z-y}{2}\right) = v\left(\frac{y+z}{2}\right) - v(y),$$

i.e.,

$$v\left(\frac{y+z}{2}\right) = \frac{1}{2}\left[v(z) + v(y)\right], \quad y,z \in \langle C, D\rangle. \tag{12}$$

Since $v$ is continuous and strictly increasing, (12) implies [ACZEL, pp. 44–45] that $v(z) = \alpha z + \beta = u(z)$ where $\alpha, \beta$ are constants and $\alpha > 0$.

Hence, we have proved the following:

*Theorem 1*

Let $m$ and $n$ be strictly increasing functions defined on $A'$ and $B'$, respectively, having the same real interval as range. Let $m^*$ and $n^*$ be another such pair of functions. If these functions satisfy (8) with $F$ and $F^*$ strictly increasing, then

$$\left.\begin{aligned} m^*(a) &= \alpha m(a) + \beta, \\ n^*(b) &= \alpha n(b) + \beta, \\ F^*(t) &= F\left(\frac{t}{\alpha}\right), \end{aligned}\right\} \tag{13}$$

where $\alpha$ and $\beta$ are appropriate constants.

(ii) In this case we have to solve the functional equation

$$F(n(b) - m(a)) = F^*\left(\frac{n^*(b) - m^*(a)}{k^*(a)}\right), \quad F(0) = F^*(0) = 1/2. \tag{14}$$

We introduce again $u, v, \Phi$ by (9) and $g$ by

$$g = k^* m^{-1}. \tag{15}$$

The equation (14) is transformed into

$$\Phi(z-y) = \frac{v(z) - u(y)}{g(y)}, \quad \Phi(0) = 0, \quad y, z \in \langle C, D \rangle. \tag{16}$$

This equation implies that $\Phi$ and $g$ are continuous. Of course, we consider $\Phi$ as being defined only in the interval $\{x \mid x = z - y,\ y, z \in \langle C, D\rangle\}$. Similar remarks apply to $F$ and $F^*$. Putting $y = z$ we get again $u(z) = v(z)$ and

$$v(z) - v(y) = \Phi(z-y) g(y), \quad y, z \in \langle C, D \rangle.$$

By interchange of $y$ and $z$ we get

$$v(y) - v(z) = \Phi(y-z) g(z), \quad y, z \in \langle C, D \rangle.$$

Adding the last two equations we get

$$\Phi(z-y) g(y) + \Phi(y-z) g(z) = 0, \quad y, z \in \langle C, D \rangle.$$

Defining

$$h(z-y) = \frac{-\Phi(z-y)}{\Phi(y-z)}, \quad y \neq z, \quad y, z \in \langle C, D \rangle$$

we obtain

$$g(z) = g(y) h(z-y). \tag{17}$$

From (17) we deduce

$$g(y) h(u-y) = g(u) = g(z) h(u-z) = g(y) h(z-y) h(u-z).$$

Since $g(y) \neq 0$ we infer that

$$h(u-y) = h(u-z)h(z-y), \quad u \neq z \neq y \neq u, \quad u,y,z \in \langle C,D \rangle .$$

The only continuous solution of this functional equation is (cf. ACZEL, pp. 37–39)

$$h(z-y) = e^{\beta(z-y)}$$

where $\beta$ is a constant. Now (17) implies that $g(z) = \delta e^{\beta z}$ where $\delta$ is a positive constant. The equation (16) leads to

$$v(z) - v(y) = \delta \Phi(z-y) e^{\beta y}, \quad y,z \in \langle C,D \rangle .$$

This equation implies

$$\frac{1}{\delta} e^{-\beta y}\left[v\left(\frac{z+y}{2}\right) - v(y)\right] = \Phi\left(\frac{z-y}{2}\right) = \frac{1}{\delta} e^{-\beta\frac{z+y}{2}}\left[v(z) - v\left(\frac{z+y}{2}\right)\right],$$

i.e.,

$$v\left(\frac{z+y}{2}\right) = \frac{v(z)e^{\frac{-\beta z}{2}} + v(y)e^{\frac{-\beta y}{2}}}{e^{\frac{-\beta z}{2}} + e^{\frac{-\beta y}{2}}}, \quad y,z \in \langle C,D \rangle . \tag{18}$$

If $\beta = 0$ then $g$ = constant and we have essentially case (i). Let $\beta \neq 0$. One can easily verify that $v(z) = \alpha e^{\beta z} + \gamma$ is a solution of (18) for arbitrary constants $\alpha$ and $\gamma$. By ACZEL [1964] there are no other continuous solutions. From (16) we get $\Phi(x) = \frac{\alpha}{\delta}(e^{\beta x} - 1)$.

Hence, we obtain:

*Theorem 2*

Let $m$ and $n$ be strictly increasing functions defined on $A'$ and $B'$, respectively, having the same real interval as range. Let $m^*$ and $n^*$ be another such pair of functions. Let $k^*$ be a positive function defined on $A'$. Assume that these functions satisfy (14) where $F$ and $F^*$ are real-valued strictly increasing functions. If $k^*$ reduces to a constant then the solution of (14) can be obtained from Theorem 1. If $k^*$ does not reduce to a constant then we have

$$\begin{aligned} m^*(a) &= \alpha e^{\beta m(a)} + \gamma, \\ n^*(b) &= \alpha e^{\beta n(b)} + \gamma, \\ k^*(a) &= \delta e^{\beta m(a)}, \\ F^*(t) &= F\left(\frac{1}{\beta}\log\left(1 + \frac{\delta t}{\alpha}\right)\right), \end{aligned} \tag{19}$$

with appropriate constants $\alpha$, $\beta$, $\gamma$, $\delta$.

(iii) Here we have to solve the functional equation

$$F\left(\frac{n(b) - m(a)}{k(a)}\right) = F^*\left(\frac{n^*(b) - m^*(a)}{k^*(a)}\right), \quad F(0) = F^*(0) = 1/2 . \tag{20}$$

We introduce again $u$, $v$, $\Phi$, $g$ by (9) and (15) and also $f$ by

$$f = km^{-1}. \tag{21}$$

The equation (20) is transformed into

$$\Phi\left(\frac{z-y}{f(y)}\right) = \frac{v(z)-u(y)}{g(y)}, \quad \Phi(0) = 0, \quad y,z \in \langle C,D\rangle.$$

Once more, $u(z) = v(z)$ follows by putting $y = z$. So we have

$$\Phi\left(\frac{z-y}{f(y)}\right) = \frac{v(z)-v(y)}{g(y)}, \quad y,z \in \langle C,D\rangle. \tag{22}$$

Since $v$ is strictly increasing it has a finite derivative almost everywhere. We shall prove that $v$ and $\Phi$ are differentiable everywhere in corresponding intervals. Let $\xi$ be an interior point of $\langle C,D\rangle$ such that $v$ is differentiable at $\xi$. Put $y = \xi$ in (22):

$$\Phi\left(\frac{z-\xi}{f(\xi)}\right) = \frac{v(z)-v(\xi)}{g(\xi)}. \tag{23}$$

The right hand side in (23) is differentiable as a function of $z$ at $z = \xi$. Hence, the left hand side is also differentiable at $z = \xi$. This means that $\Phi'(0)$ exists and is finite. Let now $\eta \in \langle C,D\rangle$ be arbitrary. Replace $y$ by $\eta$ in (22):

$$\Phi\left(\frac{z-\eta}{f(\eta)}\right) = \frac{v(z)-v(\eta)}{g(\eta)}. \tag{24}$$

Now, the left hand side in (24) is differentiable at $z = \eta$. We deduce that the right hand side, i.e. $v$, is differentiable at $z = \eta$. Hence, $v$ is differentiable in $\langle C,D\rangle$. The equation (22) implies that also $\Phi$ is differentiable in the interval where it is defined.

Concerning $f$ and $g$ we suppose that they are positive and differentiable in $\langle C,D\rangle$.

Let us differentiate (22) with respect to $z$ and $y$:

$$\Phi'\left(\frac{z-y}{f(y)}\right)\frac{1}{f(y)} = \frac{v'(z)}{g(y)}, \tag{25}$$

$$\Phi'\left(\frac{z-y}{f(y)}\right)\frac{f(y) + (z-y)f'(y)}{f(y)^2} = \frac{v'(y)g(y) + (v(z)-v(y))g'(y)}{g(y)^2}.$$

By eliminating $\Phi'$ we get

$$v'(z)[f(y) + (z-y)f'(y)] - \frac{f(y)}{g(y)}[v'(y)g(y) + (v(z)-v(y))g'(y)] = 0.$$

We can write this as follows [ACZEL, 1961]

$$v'(z)\left[1 - y\frac{f'(y)}{f(y)}\right] + zv'(z)\frac{f'(y)}{f(y)} - v(z)\frac{g'(y)}{g(y)} + \left[v(y)\frac{g'(y)}{g(y)} - v'(y)\right] = 0. \tag{26}$$

Consider the vectors

$$\{v'(z), z v'(z), -v(z), 1\}, \quad z \in \langle C, D \rangle, \tag{27}$$

$$\left\{1 - y\frac{f'(y)}{f(y)}, \frac{f'(y)}{f(y)}, \frac{g'(y)}{g(y)}, v(y)\frac{g'(y)}{g(y)} - v'(y)\right\}, \quad y \in \langle C, D \rangle. \tag{28}$$

The vectors (27) span a subspace $V_z$ in the four dimensional Euclidean space, and the vectors (28) span a subspace $V_y$. By (26) these subspaces are orthogonal to each other. We have $\dim V_z \geqslant 2$ since $v(z)$ is not a constant. On the other hand $\dim V_y \geqslant 1$ since the first two components of (28) cannot vanish simultaneously. Hence, we have only two possibilities: $\dim V_z = 2$ or 3.

Case 1. $\dim V_z = 2$. The projections of $V_z$ onto the coordinate hyperplanes have at most dimension 2. Therefore there exist relations of linear dependence

$$p_1 v'(z) + q_1 z v'(z) + r_1 = 0,$$
$$p_2 v'(z) + q_2 z v'(z) - r_2 v(z) = 0,$$
$$p_3 v'(z) - q_3 v(z) + r_3 = 0.$$

The first two equations imply that $v$ is of the form

$$v(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Taking this into account the third relation implies that $v(z) = \alpha z + \beta$. The equation (26) reduces to

$$f'(y) g(y) - f(y) g'(y) = 0$$

so that $g(y) = \gamma f(y)$ where $\gamma$ is some constant. We obtain the following solution:

$$\left.\begin{aligned} m^*(a) &= \alpha m(a) + \beta, \\ n^*(b) &= \alpha n(b) + \beta, \\ k^*(a) &= \gamma k(a), \\ F^*(t) &= F\left(\frac{\gamma t}{\alpha}\right). \end{aligned}\right\} \tag{29}$$

Case 2. $\dim V_z = 3$ and consequently $\dim V_y = 1$. The projection of $V_y$ on any coordinate plane is at most a one dimensional subspace. Therefore there exist constants $p$, $q$, not both zero, such that

$$p\left[1 - y\frac{f'(y)}{f(y)}\right] + q\frac{f'(y)}{f(y)} = 0.$$

If $p = 0$ then $f =$ constant and we are in the case (ii). If $p \neq 0$ let us write $\beta = -q/p$. Since $f(y)$ is positive we infer that $f(y) = \alpha(y + \beta)$. For the same reason

$$\frac{g'(y)}{g(y)} = \gamma\frac{f'(y)}{f(y)} = \frac{\gamma}{y + \beta},$$

which implies

$$g(y) = \delta|y + \beta|^{\gamma}.$$

Since $f(y) > 0$ for all $y \in \langle C, D\rangle$ we must have $\beta \notin \langle C, D\rangle$. The equation (26) reduces to

$$(z+\beta)v'(z) - \gamma v(z) = (y+\beta)v'(y) - \gamma v(y).$$

If $\gamma = 0$ then $g$ = constant and we are in the case (ii). If $\gamma \neq 0$ we infer that

$$(z+\beta)v'(z) - \gamma v(z) = -\gamma\varepsilon,$$

$$\frac{v'(z)}{v(z)-\varepsilon} = \frac{\gamma}{z+\beta},$$

$$v(z) = \lambda|z+\beta|^\gamma + \varepsilon.$$

Now from (22) we find that

$$\Phi(x) = \tfrac{\lambda}{\delta}(|1+\alpha x|^\gamma - 1).$$

Finally we obtain

$$\left.\begin{aligned} m^*(a) &= \lambda|m(a)+\beta|^\gamma + \varepsilon, \\ n^*(b) &= \lambda|n(b)+\beta|^\gamma + \varepsilon, \\ k^*(a) &= \delta|m(a)+\beta|^\gamma, \\ k(a) &= \alpha(m(a)+\beta), \\ F(x) &= F^*(\tfrac{\lambda}{\delta}(|1+\alpha x|^\gamma - 1)). \end{aligned}\right\} \tag{30}$$

Hence, we have proved the following:

*Theorem 3*

Let $m$ and $n$ be strictly increasing functions defined on $A'$ and $B'$, respectively, having the same real interval as range. Let $m^*$ and $n^*$ be another such pair of functions. Let $k$ and $k^*$ be positive functions defined on $A'$. Assume that the functions $f = km^{-1}$ and $g = k^*m^{-1}$ are differentiable. Finally, assume that these functions satisfy (20) with $F$ and $F^*$ real-valued strictly increasing functions. If $k$ or $k^*$ reduces to a constant the solution of (20) is supplied by Theorem 2. If $k$ and $k^*$ do not reduce to a constant then the solution of (20) is given by (29) or (30) with appropriate constants $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\lambda$.

## References


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