A Note on Horvitz's and Thompson's T_3 Class of Linear Estimators[1]

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1. Introduction

Whenever a random sample of size n is drawn without replacement from a finite population of size N, with varying probabilities of selection at each draw, Horvitz and Thompson [2] have formulated certain classes of linear estimators, depending on the weights associated with sample observations, to furnish a sample appraisal of the total of a population characteristic. It is demonstrated by the author [6,7] that for a given sampling procedure a class of linear estimators might be empty or non-empty with the definitions given as follows: (1) Empty Class: Whenever there does not exist even a single linear unbiased estimator independent of the population values for a class, then that class is called an empty class; (2) Non-empty Class: For a non-empty class there always exists a linear unbiased estimator with weights independent of the population values. Further the usual criterion, employed to choose a unique estimator for a class of linear estimators, is that of least variance. The present paper is concerned with the T_3-class of linear estimators, formulated by Horvitz and Thompson [2], and examines it in the light of the above discussion. It is noted that the T_3-class is non-empty. However, the best estimator for the T_3-class depends on the population values even when the elements are selected with equal probabilities. Incidentally it is observed that the best estimator for the T_3-class belongs to a sub-class of T_3, where the order of the elements constituting the sample, is not taken into account.

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2. The T_3-Class of Linear Estimators

As has already been described that a random sample of size n is drawn from a finite population of size N, let $Y_{i_1}, Y_{i_2}, \ldots, Y_{i_n}$ be the population elements entered into the sample at the first, second, ..., nth draws respectively and the probability of such ordered sample be $p_{i_1, i_2, \ldots, i_n}$. The T_3-class of linear estimators formulated by HORVITZ and THOMPSON [2] for the estimation of the population total

$$T = \sum_{i=1}^{N} Y_i, \tag{1}$$

is defined symbolically as

$$T_3 = w_{i_1, i_2, \ldots, i_n} \left\{ \sum_{r=1}^{n} Y_{i_r} \right\} \tag{2}$$

where $w_{i_1, i_2, \ldots, i_n}$ is a constant used as the weight associated with the sample in which the population elements $Y_{i_1}, Y_{i_2}, \ldots, Y_{i_n}$ appear at the first, second, ..., nth draws respectively.

It is easy to note that the present sampling procedure incorporates $M = \binom{N}{n} n!$ different samples when the order of the elements constituting the sample is taken into account. T_3 should be an unbiased estimator of the population total, therefore we consider

$$E(T_3) = \sum_{s_n}^{\binom{N}{n} n!} w_{i_1, i_2, \ldots, i_n} \left\{ \sum_{r=1}^{n} Y_{i_r} \right\} p_{i_1, i_2, \ldots, i_n} =$$

$$= Y_1 \Big[\sum_{s_n(1, i_2, \ldots, i_n)} w_{1, i_2, \ldots, i_n} p_{1, i_2, \ldots, i_n} + \sum_{s_n(i_1, 1, i_3, \ldots, i_n)} w_{i_1, 1, i_3, \ldots, i_n} p_{i_1, 1, i_3, \ldots, i_n} +$$

$$+ \cdots + \sum_{s_n(i_1, i_2, \ldots, i_{n-1}, 1)} w_{i_1, i_2, \ldots, i_{n-1}, 1} p_{i_1, i_2, \ldots, i_{n-1}, 1} \Big]$$

$$+ Y_2 [\quad]$$

$$\vdots$$

$$+ Y_n \Big[\sum_{s_n(N, i_2, \ldots, i_n)} w_{N, i_2, \ldots, i_n} p_{N, i_2, \ldots, i_n} + \cdots +$$

$$+ \sum_{s_n(i_1, \ldots, i_{n-1}, N)} w_{i_1, \ldots, i_{n-1}, N} p_{i_1, i_2, \ldots, i_{n-1}, N} \Big] \tag{3}$$

where s_n $(i_1, i_2, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_n)$ stands for all possible samples in which the jth population element enters into the sample on the rth draw. For each Y_j, the summation '$\sum$' extends over all possible $(n-1)! \binom{N-1}{n-1}$ samples containing the jth population element. It is easily noted that T_3 is an unbiased estimator of the population total, whatever may be the Y_i's, if the coefficient of Y_j $(j = 1, 2, \ldots, N)$ of the above equation is unity. Thus we obtain,

$$\sum_{r=1}^{n} \sum_{s_n(i_1, i_2, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_n)}^{(n-1)!\binom{N-1}{n-1}} w_{i_1, i_2, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_n} p_{i_1, i_2, \ldots, i_{r-1}, j, i_{r+1} \ldots, i_n} = 1 \tag{4}$$

for $j = 1, 2, \ldots, N$. Consequently there are N conditions on these $M = n! \binom{N}{n}$ number of w's.

The T_3-class of linear estimators is non-empty if the above system of linear equations arrived at in (3), is consistent. However, if an estimator, independent of the population values, exists for the class, the class is a non-empty one. Accordingly, when

$$w_{i_1, i_2, \ldots, i_n} = \frac{K}{p_{i_1, i_2, \ldots, i_n}} \tag{5}$$

for a suitable value of K, it is observed that the coefficients w's do not depend upon the population values, and subsequently the T_3-class is non-empty, whatever may be the sampling procedure.

For the selection of a unique estimator for the T_3-class, which is non-empty, we employ the usual procedure of the least variance. Thus

$$\text{Var}(T_3) = E(T_3^2) - (T)^2$$

where

$$E(T_3^2) = \sum w_{i_1, i_2, \ldots, i_n}^2 \left\{ \sum_{r=1}^{n} Y_{i_r} \right\}^2 p_{i_1, i_2, \ldots, i_n}. \tag{6}$$

Further, for the minimization of $E(T_3^2)$ for variation of w's with respect to the conditions of unbiasedness arrived at in (4), we employ the Lagrangian method of undetermined multipliers.

Let

$$\varphi = \text{Var}(T_3) - 2 \sum_{j=1}^{N} \lambda_j \sum\sum w_{i_1, i_2, \ldots, i_n} p_{i_1, i_2, \ldots, i_n} \tag{7}$$

where λ_j $(j = 1, 2, \ldots, N)$ is LAGRANGE's multiplier corresponding to the jth equation in (3). Equating $\partial\varphi/\partial w_{i_1, i_2, \ldots, i_n}$ to zero, we obtain after some simplification

$$W_{i_1, i_2, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_n} =$$

$$= \frac{\lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_{r-1}} + \lambda_j + \lambda_{i_{r+1}} + \ldots + \lambda_{i_n}}{(Y_{i_1} + Y_{i_2} + \ldots + Y_{i_{r-1}} + Y_j + Y_{i_{r+1}} + \ldots + Y_{i_n})^2} \tag{8}$$

for $i_1 \neq i_2 \neq \ldots \neq j \neq i_{r+1} \neq \ldots \neq i_n = 1, 2, \ldots, N$. Moreover, the second differentiation of φ discloses that for the above specification of w's $E(T_3^2)$ is minimum.

From the above specification of w's, arrived at in (8), it is noted that the weights attached to the sample observations, for the best estimator in the T_3-class do not depend on the order in which the elements enter into the sample. Accordingly, the same weight is associated, with the $n!$ samples obtained by the different permutations of n sampling units. It is interesting to compare this result to that derived by MURTHY [5] in his first theorem, wherein it is stated that if Y_{s_n} be an unbiased estimator of a population parameter θ, based on one of the samples out $n! \binom{N}{n}$ possible samples, there exists an estimator

$$\theta_0 = \sum Y_{s_n} p_{i_1, i_2, \ldots, i_n} / \sum p_{i_1, i_2, \ldots, i_n},$$

the summation being over all permutations of the given n units in the sample s_n, whose variance is less than or equal to the variance of the estimator Y_{s_n}. It is also pertinent here to note one of KOOP's [3] results where a special estimator of the present T_3-class is proved to have greater variance than another special estimator of a sub-class of T_3, formulated by ignoring the order in which elements enter into the sample.

Apparently, the weights w's for the best estimator, are determined from the equations in (4) and (8). However, that these weights w's for the best estimator depend on the population values, even when the population elements are selected with equal probabilities, is shown in the following section.

Let the weights w's of the best estimator be independent of the population values. From equations in (8), we derive the following relations:

$$w_{i_1, i_2, \ldots, i_n}(Y_{i_1} + Y_{i_2} + \ldots + Y_{i_n})^2 = \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_n},$$

$$w_{j_1, i_2, \ldots, i_n}(Y_{j_1} + Y_{i_2} + \ldots + Y_{i_n})^2 = \lambda_{j_1} + \lambda_{i_2} + \ldots + \lambda_{i_n},$$

$$w_{j_1, j_2, \ldots, j_n}(Y_{j_1} + Y_{j_2} + \ldots + Y_{j_n})^2 = \lambda_{j_1} + \lambda_{j_2} + \ldots + \lambda_{j_n},$$

$$w_{i_1, j_2, \ldots, j_n}(Y_{i_1} + Y_{j_2} + \ldots + Y_{j_n})^2 = \lambda_{i_1} + \lambda_{j_2} + \ldots + \lambda_{j_n}.$$

From these four relations, after suitably eliminating λ's, we arrive at the following equation,

$$\sum A_{rs} Y_r Y_s = 0 \tag{9}$$

where

$$r, s = i_1, i_2, \ldots, i_n; j_1, j_2, \ldots, j_n$$

and A_{rs}'s are functions of the w's which are assumed to be independent of the Y_i's. The above equation holds good whatever may be the Y_i's, in which case every coefficient of $Y_r Y_s$ must vanish. Accordingly, we obtain

$$w_{i_1, i_2, \ldots, i_n} = w_{j_1, i_2, \ldots, i_n} = w_{j_1, j_2, \ldots, j_n} = w_{i_1, j_2, \ldots, i_n} = 0. \tag{10}$$

Considering the other similar equations and following the same procedure we arrive at the result that the coefficients w's of the best estimator are equal to zero. However, this conclusion contradicts the result deduced in (3). Hence the coefficients w's, associated with the best estimator depend on the population values.

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