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KOEBE Arcs and FATOU Points of Normal Functions

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Let C be the unit circle and D be the open unit disk in the complex z -plane, and C_w, D_w be the corresponding entities in the complex w -plane. The closure of a point set S will be denoted by $\overline{S}$, and the LEBESGUE measure of a measurable set E by $m(E)$.

We begin by setting down some definitions.

Definition 1. Let A be an open arc of C , possibly C itself. A **KOEBE sequence of arcs (relative to A)** is a sequence of JORDAN arcs $\{J_n\}$ in D such that (a) for some sequence $\{\varepsilon_n\}$ satisfying the conditions $0 < \varepsilon_n < 1$ ($n = 1, 2, 3, \dots$) and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, J_n lies in the ε_n -neighborhood of A ($n = 1, 2, 3, \dots$), and (b) every open sector Δ of D subtending an arc of C that lies strictly interior to A has the property that, for all values of n except at most a finite number, the arc J_n contains at least one JORDAN subarc lying wholly in Δ except for its two end points which lie on distinct sides of Δ .

The terminology in Definition 1 is suggested by the appearance of such arcs in KOEBE's lemma [2, p. 19].

Definition 2. A **strong KOEBE sequence of arcs** is a KOEBE sequence of arcs $\{J_n\}$ with the property that, to every $\zeta \in C$, there corresponds a rectilinear segment extending from ζ to a point of D , which is intersected by infinitely many of the arcs J_n ($n = 1, 2, 3, \dots$).

It is easily verified that a strong KOEBE sequence of arcs is a KOEBE sequence of arcs relative either to C itself or to C minus a single point of C .

Definition 3. If $f(z)$ is a meromorphic function in D and c is a constant, finite or ∞ , we say that $f(z) \rightarrow c$ *along a KOEBE sequence of arcs* $\{J_n\}$, provided that, for some sequence of positive numbers $\{\eta_n\}$, where $\eta_n \rightarrow 0$ as $n \rightarrow \infty$, we have, for every $z \in J_n$ ($n = 1, 2, 3, \dots$), $|f(z) - c| < \eta_n$ or $|f(z)| > 1/\eta_n$, according as c is finite or infinite.

Definition 4. If $f(z)$ is a meromorphic function in D , we say that $f(z)$ *is bounded by M on a KOEBE sequence of arcs* $\{J_n\}$, provided that there

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exists a finite positive constant M such that $|f(z)| < M$ for every $z \in J_n$ ($n = 1, 2, 3, \dots$).

Definition 5. Let $z' = S(z)$ denote an arbitrary one-to-one conformal mapping of D onto itself. A function $f(z)$, meromorphic in D , is said to be *normal* in D [5, p. 53], if the family of functions $\{f(S(z))\}$ is normal in D in the sense of MONTEL, where convergence is defined in terms of the spherical metric.

Definition 6. A *FATOU point* of a meromorphic function in D is a point $\zeta \in C$ such that, for some complex number c (possibly ∞), as $z \rightarrow \zeta$ in any STOLZ angle at ζ , $f(z) \rightarrow c$; c is then called a *FATOU value* of $f(z)$.

We show first (Theorem 1) that a normal meromorphic function that tends to a constant along a KOEBE sequence of arcs is identically constant. This generalizes a result due to GROSS [4, pp. 35–36] as well as a result due to the present authors [1, Corollary 1, p. 266]. Next we prove (Theorem 2) that a normal holomorphic function that is bounded on a strong KOEBE sequence of arcs must be a bounded function. This generalizes [1, Corollary 2, p. 266]. (The two results in [1] alluded to involve “boundary paths” instead of KOEBE sequences of arcs.)

Theorem 3 asserts that if the set of FATOU points of a normal holomorphic function in D is of measure zero on an arc of C , then that arc contains an everywhere dense set of FATOU points of the function at each of which the corresponding FATOU value is ∞ . This generalizes [1, Theorem 5, p. 267]. It follows immediately that the set of FATOU points of a normal holomorphic function in D is everywhere dense on C , which sharpens [1, Theorem 4, p. 267]. This result is to be contrasted with one given in [5, p. 58], according to which there exist normal *meromorphic* functions in D possessing no FATOU points. (Cf. also [1, Remark 4, p. 267].) Theorem 4 shows that a normal holomorphic function in D can have its set of FATOU points of arbitrarily small positive measure without having ∞ as a FATOU value. This leads us to pose the following problem, which we have not solved.

Problem. Let $f(z)$ be a normal holomorphic function in D . Suppose that an arc A of C exists such that the measure of the set of FATOU points of $f(z)$ on every subarc of A is less than the length of that subarc. Does A contain a FATOU point of $f(z)$ at which the corresponding FATOU value is ∞ ?

We proceed now to the proofs of our theorems.

Theorem 1. Let $f(z)$ be a normal meromorphic function in D . If $f(z) \rightarrow c$ along a KOEBE sequence of arcs $\{J_n\}$, then $f(z) \equiv c$.

Proof. We may assume that $c = 0$, for otherwise we can replace the normal meromorphic function $f(z)$ by the normal meromorphic function $f(z) - c$ if c is finite, or $1/f(z)$ if $c = \infty$.

Let the given sequence $\{J_n\}$ be a KOEBE sequence relative to the arc A (see Definition 1), and consider an arc $B = \{z : |z| = 1, q_1 < \arg z < q_2\}$ strictly interior to A . Denote by Δ the open sector of D with vertex at the origin and vertex angle β , subtending the arc B . The sides of Δ will be called s_1, s_2 , where these segments terminate in e^{iq_1}, e^{iq_2} , respectively. In view of (b) in Definition 1, there is no loss of generality in asserting now that for every n the arc J_n contains a JORDAN subarc Γ_n lying wholly in Δ except for its endpoints $P_n^{(1)}, P_n^{(2)}$ which lie on s_1, s_2 , respectively. It is obvious that $\{\Gamma_n\}$ is a KOEBE sequence of arcs relative to B .

Set

$$r_n = \min_{z \in \Gamma_n} |z|, \quad R_n = \max_{z \in \Gamma_n} |z| \quad (n = 1, 2, 3, \dots).$$

It follows from (a) in Definition 1 that

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} R_n = 1. \quad (1)$$

For $n = 1, 2, 3, \dots$, we now define a JORDAN curve K_n . Let the circle $|z| = R_n$ intersect s_1 and s_2 in the respective points $Q_n^{(1)}, Q_n^{(2)}$, and denote the radial segments $P_n^{(1)} Q_n^{(1)}, P_n^{(2)} Q_n^{(2)}$ by $t_n^{(1)}, t_n^{(2)}$, respectively (these segments may reduce to single points). Then, if B_n is the open arc of the circle $|z| = R_n$ which lies in Δ and B_n^* is the complementary arc, we put

$$K_n = t_n^{(1)} \cup B_n^* \cup t_n^{(2)} \cup \Gamma_n.$$

The interior of K_n will be called Ω_n , and we set $G_n = \{z : |z| < R_n\}$.

CARLEMAN'S Extension Principle for harmonic measure implies [7, p. 70] that

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) \geq \omega(0, B_n, G_n) = \frac{\beta}{2\pi}.$$

We have [7, p. 26]

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) = \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) + \omega(0, \Gamma_n, \Omega_n).$$

An inequality due to OSTROWSKI [3, p. 42] shows that

$$\omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) \leq \frac{4}{\pi} \arcsin \frac{2 \sqrt{\frac{R_n - r_n}{2} \cdot \frac{R_n + r_n}{2}}}{\frac{R_n - r_n}{2} + \frac{R_n + r_n}{2}} = \frac{4}{\pi} \arcsin \frac{\sqrt{R_n^2 - r_n^2}}{R_n},$$

and (1) implies that $\lim_{n \rightarrow \infty} \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) = 0$. Hence

$$\lim_{n \rightarrow \infty} \inf \omega(0, \Gamma_n, \Omega_n) \geq \frac{\beta}{2\pi}.$$

Consequently, if D_w is mapped conformally onto Ω_n by means of the function $z = \psi_n(w)$, where $\psi_n(0) = 0$ and the point $w = e^{t_{q_1}}$ corresponds to the point $z = P_n^{(1)}$, then each arc Γ_n , for n sufficiently large, is the image of an arc of C_w of length at least $\beta/2$ with its end point of smaller argument at $e^{t_{q_1}}$.

If we set

$$g_n(w) = f(\psi_n(w)) \quad (n = 1, 2, 3, \dots), \quad (2)$$

then [5, p. 57] $g_n(w)$ is a normal meromorphic function in D_w . Since $f(z)$ is normal in D , there exists [5, p. 56] a finite positive constant γ such that for every $z \in D$,

$$\frac{|f'(z)|}{1 + |f(z)|^2} (1 - |z|^2) \leq \gamma. \quad (3)$$

Now from (2) we obtain

$$\frac{|g'_n(w)|}{1 + |g_n(w)|^2} (1 - |w|^2) = \frac{|f'(\psi_n(w))| \cdot |\psi'_n(w)|}{1 + |f(\psi_n(w))|^2} (1 - |w|^2). \quad (4)$$

According to [9, p. 133], if $D_1(z)$ denotes the radius of univalence at the point $z = \psi_n(w)$ of the region Ω_n , we have

$$(1 - |w|^2) |\psi'_n(w)| \leq 4D_1(z), \quad (5)$$

and since Ω_n lies in D ,

$$D_1(z) \leq 1 - |z| \leq 1 - |z|^2. \quad (6)$$

Combining (3) to (6), we find that

$$\frac{|g'_n(w)|}{1 + |g_n(w)|^2} (1 - |w|^2) \leq \frac{4|f'(z)|}{1 + |f(z)|^2} \leq 4\gamma. \quad (7)$$

Let S denote the subarc of C_w whose end point of smaller argument is $e^{t_{q_1}}$ and whose length is $\beta/2$. The hypothesis that $f(z) \rightarrow 0$ along the KOEBE sequence $\{J_n\}$ implies that $\lim_{n \rightarrow \infty} g_n(w) = 0$ uniformly on S . This together with (7) shows, in view of [5, p. 64], that the sequence $\{g_n(w)\}$ tends uniformly to zero on every compact subset of D_w .

We shall now show that $f(z) \equiv 0$. Suppose that, on the contrary, $f(z_0) \neq 0$ for some $z_0 \in D$. By (a) in Definition 1, $z_0 \in \Omega_n$ for all sufficiently large values

of n . Let $w = \varphi_n(z)$ be the inverse of the function $z = \psi_n(w)$. Then, according to (2),

$$g_n(\varphi_n(z_0)) = f(z_0)$$

for all sufficiently large values of n . Since $\{g_n(w)\}$ tends uniformly to zero on every compact subset of D_w , but $f(z_0) \neq 0$, we must have $\lim_{n \rightarrow \infty} |\varphi_n(z_0)| = 1$.

But this is impossible; for if we fix ϱ so that $|z_0| < \varrho < 1$, then SCHWARZ'S lemma yields

$$|\varphi_n(z_0)| \leq \frac{|z_0|}{\varrho} < 1$$

for all sufficiently large values of n . Our supposition has thus led to a contradiction, and the theorem is proved.

Theorem 2. *Let $f(z)$ be a normal holomorphic function in D . If $f(z)$ is bounded by M on a strong KOEBE sequence of arcs $\{J_n\}$, then $f(z)$ is bounded by M throughout D .*

Proof. If $f(z)$ is bounded in D , then Definition 2 implies that none of its radial limits, except perhaps one, is greater than M in modulus, and the representation of $f(z)$ by its POISSON integral shows immediately that $|f(z)| < M$ throughout D .

We shall now suppose that $f(z)$ is unbounded in D , and show that this leads to a contradiction of the hypothesis that $\{J_n\}$ is a strong KOEBE sequence. The set of all points $z \in D$ at which $|f(z)| > M + 1$ is open and not empty; let R_1 be some component of this set. At all boundary points of R_1 that lie in D , we have $|f(z)| = M + 1$, and the maximum principle implies that R_1 cannot lie wholly in some disk $|z| < \varrho < 1$. Hence, the boundary of R_1 contains at least one point of C . The region R_1 cannot have more than one accessible boundary point on C , for if it had two such points ζ_1 and ζ_2 , they could be connected by a JORDAN arc Γ lying, except for its end points ζ_1 and ζ_2 , in R_1 , and Γ would decompose D into two subregions. But R_1 , and hence Γ , meets none of the arcs J_n ($n = 1, 2, 3, \dots$), and therefore infinitely many of these arcs would have to lie in one of the two subregions of D , contradicting the remark following Definition 2 and (b) in Definition 1.

We now map D_w conformally onto the universal covering surface R_1^* of R_1 by means of the single-valued function $z = \varphi(w)$, and set

$$g(w) = f(\varphi(w))$$

in D_w . We have $|\varphi(w)| < 1$ in D_w . The FATOU values of $\varphi(w)$ are of

modulus 1 on at most a subset of measure zero of C_w ; this follows from the RIESZ uniqueness theorem [7, p. 209] and the fact that R_1 has at most one accessible boundary point on C . Since R_1^* is unbranched over R_1 , almost all the FATOU values of $\varphi(w)$ are points in D that lie on the boundary of R_1 . Hence, $g(w)$ possesses limits of modulus $M + 1$ along almost all radii of C_w . It follows that $f(z)$ is unbounded in R_1 , because otherwise we should have $|g(w)| < M + 1$ throughout D_w , contradicting the definition of R_1 .

The set of all points $z \in R_1$, at which $|f(z)| > M + 2$ is open and not empty; let R_2 be some component of this set. Then $R_2 \subset R_1$, and if we apply to R_2 the foregoing argument for R_1 , we arrive at the conclusion that $f(z)$ is unbounded in R_2 . Proceeding in this manner, we obtain a sequence of nested regions

$$R_1 \supset R_2 \supset R_3 \supset \dots$$

such that, for $n = 1, 2, 3, \dots$,

$$|f(z)| > M + n \quad (z \in R_n). \quad (8)$$

Now take

$$z_1 \in R_1, z_2 \in R_2 - \{z_1\}, z_3 \in R_3 - \{z_1, z_2\}, \dots, z_n \in R_n - \{z_1, z_2, \dots, z_{n-1}\}, \dots,$$

and join z_1 to z_2 by means of a JORDAN arc K_1 lying in R_1 , join z_2 to z_3 by means of a JORDAN arc K_2 lying in R_2 and having no point except z_2 in common with $K_1, \dots$, join z_n to z_{n+1} by means of a JORDAN arc K_n lying in R_n and having no point except z_n in common with $K_1 \cup K_2 \cup \dots \cup K_{n-1}, \dots$. We thus obtain a path

$$P = \bigcup_{n=1}^{\infty} K_n$$

in D . Its initial point is z_1 , and its "end" lies on C because, due to (8) and the fact that $K_n \subset R_n$ ($n = 1, 2, 3, \dots$),

$$\lim_{n \rightarrow \infty} \min_{z \in K_n} |f(z)| = \infty,$$

and $f(z)$, by hypothesis, is holomorphic in D . The path P then is a "boundary path" in D along which $f(z) \rightarrow \infty$. According to [1, Corollary 1, p. 266], the end of P is a single point $\zeta \in C$. Since $f(z)$ is normal in D , ζ is a FATOU point of $f(z)$ with ∞ as the corresponding FATOU value [5, p. 53]. But, in view of Definition 2, this contradicts the hypothesis that $\{J_n\}$ is a strong KOEBE sequence, because $f(z)$ is bounded on $\{J_n\}$; and the theorem is proved.

Theorem 3. *Let $f(z)$ be a normal holomorphic function in D and A be an open subarc of C . If the set of FATOU points of $f(z)$ on A is of measure zero, then A contains a FATOU point of $f(z)$ at which the corresponding FATOU value is ∞ .*

Proof. Take a point $\zeta \in A$. The function $f(z)$ cannot be bounded in any neighborhood of ζ , because otherwise, by a simple extension of FATOU's theorem, the set of FATOU points of $f(z)$ on A would be of positive measure, contrary to hypothesis. Hence, there exists a number $\delta > 0$ such that the region $H = D \cap \{z : |z - \zeta| < \delta\}$ satisfies the conditions that $\overline{H} \cap C \subset A$ and $f(z)$ is unbounded in H . Consequently there exists a sequence of points $\{z_n\}$ in D such that $z_n \rightarrow \zeta$ and $M_n = |f(z_n)| \rightarrow \infty$ as $n \rightarrow \infty$, where $1 < M_1 < M_2 < \dots < M_n < \dots$. For $n = 1, 2, 3, \dots$, let V_n be the open set of all points of D at which $|f(z)| > M_n - 1$, and denote by R_n that component of V_n which contains the point z_n . Evidently $|f(z)| = M_n - 1$ at all boundary points of R_n that lie in D . The maximum principle implies that $\overline{R_n} \cap C$ is not empty. As $n \rightarrow \infty$, the diameter of R_n tends to zero. For if $r_n = \min_{z \in \overline{R_n}} |z|$, the hypothesis that $f(z)$ is holomorphic in D implies that $\lim_{n \rightarrow \infty} r_n = 1$, so that if the diameter of R_n did not tend to zero as $n \rightarrow \infty$, one could obtain a KOEBE sequence of arcs along which $f(z) \rightarrow \infty$, which is impossible in view of Theorem 1. Thus there exists a natural number N such that $R_N \subset H$, and we set $G_1 = R_N$.

We shall show that $f(z)$ is unbounded in G_1 . Let G_1^* be the smallest simply connected region containing G_1 , and $z = \varphi(w)$ be a function that maps D_w conformally onto G_1^* . The set $B^* = \overline{G_1^*} \cap C$ is not empty; we denote by B_1^* the set of all points of B^* that are accessible from the region G_1^* . According to FATOU's theorem, $\varphi(w)$ has a radial limit at almost all points of C_w ; we put

$$\varphi^*(e^{i\mu}) = \lim_{r \rightarrow 1} \varphi(re^{i\mu})$$

for every μ for which the limit exists. The set

$$E_1 = \{e^{i\mu} : |\varphi^*(e^{i\mu})| = 1\}$$

is a BOREL set, and is therefore measurable, and we have

$$B_1^* = \{\varphi^*(e^{i\mu}) : e^{i\mu} \in E_1\}.$$

Consider the function

$$g(w) = f(\varphi(w))$$

in D_w . We are going to show that $g(w)$ is unbounded in D_w . Assume that $g(w)$ is bounded in D_w . We have either $m(E_1) > 0$ or $m(E_1) = 0$.

Suppose first that $m(E_1) > 0$. Let E_0 be the BOREL subset of positive measure of E_1 at each point of which $g(w)$ possesses a radial limit, and B_0^* be the image of E_0 under the mapping $z = \varphi(w)$. An application of an extension of LÖWNER's theorem [10, p. 322] shows that B_0^* is a measurable subset of B_1^* with $m(B_0^*) > 0$. Let $\zeta_0 \in B_0^*$. Then there is a path in G_1^* terminating in ζ_0 , and this path is the image, under the mapping $z = \varphi(w)$, of a path in D_w that terminates in a point $e^{i\mu_0} \in E_0$. Now $\varphi^*(e^{i\mu_0}) = \zeta_0$, and $g(w)$ has a radial limit at the point $e^{i\mu_0}$; therefore $f(z)$ tends to a limit along a path in G_1^* terminating in ζ_0 . By hypothesis, $f(z)$ is normal in D , and consequently [5, p. 53] ζ_0 is a FATOU point of $f(z)$. Since ζ_0 was an arbitrary point of B_0^* , and $m(B_0^*) > 0$, we have arrived at a contradiction of the hypothesis that the set of FATOU points of $f(z)$ on A is of measure zero.

Suppose next that $m(E_1) = 0$. Since every boundary point of G_1^* is a boundary point of G_1 , the italicized remark in the first paragraph of the proof implies that the FATOU values of $g(w)$ are equal to $M_N - 1$ in modulus almost everywhere on C_w . The representation of $g(w)$ by its POISSON integral shows that $|g(w)| \leq M_N - 1$ throughout D_w , which implies that $|f(z)| \leq M_N - 1 = L$ throughout $G_1 = R_N$, contrary to the definition of R_N .

Thus $g(w)$ is unbounded in D_w , which implies that $f(z)$ is unbounded in G_1^* and hence in G_1 . It follows that the open set of all points of G_1 at which $|f(z)| > L + 1$ is not empty, and letting G_2 denote a component of this set, we conclude as above that $f(z)$ is unbounded in G_2 . Continuing in this manner, we obtain a sequence of nested subregions $G_1 \supset G_2 \supset G_3 \supset \dots$ of H , and now an argument employed in the proof of Theorem 2 enables us to infer the existence of a FATOU point of $f(z)$ on A at which the corresponding FATOU value is ∞ , thus completing the proof of the theorem.

Corollary 1. *The set of FATOU points of a normal holomorphic function in D is everywhere dense on C .*

Theorem 4. *Given $\varepsilon > 0$, there exists a normal holomorphic function $f(z)$ in D whose set of FATOU points is of measure less than ε but for which ∞ is not a radial limit.*

Proof. Consider first the function $\varphi(w) = g(w) + h(w)$ in D_w , where $g(w)$ is the elliptic modular function, holomorphic and normal in D_w , whose set of FATOU points E is enumerable and whose FATOU values are 0, 1, ∞ , and $h(w)$ is bounded and holomorphic in D_w and possesses a radial limit at every point of $C_w - E$ but no radial limit at any point of E [6, Theorem 6,

p. 14]. Now $\varphi(w)$ is holomorphic and normal in D_w [5, p. 53]; its set E_0 of FATOU points is enumerable, and ∞ is its only FATOU value.

Choose a positive number δ so small that, if $\varrho = \cos \frac{\delta}{2}$, then

$$\frac{L}{\left| \log \frac{\delta}{\varrho} \right| + 1} < \varepsilon, \quad (9)$$

where L is a certain positive absolute constant to be specified later. Let P be a perfect nowhere dense set on C_w that contains no point of E_0 and for which $m(P) > 2\pi - \delta$, and set $H = C_w - P$. Denote by R the simply connected subregion of D_w whose boundary consists of the points of P and the open chords of C_w that subtend the components of the open set H . The boundary of R is evidently a rectifiable JORDAN curve of length less than 2π . Since each component of H is of length less than δ , the region R contains the disk $|w| < \varrho$. Let the function $w = \lambda(z)$ map D conformally onto R so that $\lambda(0) = 0$, and let S be the set of all points on C that correspond under this mapping to points on the chords of C_w subtending components of H . Since the sum of the lengths of these chords is less than δ , we have, by a theorem of LAVRENTIEV [8, p. 125],

$$m(S) < \frac{L}{\left| \log \frac{\delta}{\varrho} \right| + 1}. \quad (10)$$

Now consider the function $f(z) = \varphi(\lambda(z))$ in D . It is holomorphic and normal in D [5, p. 57], does not have ∞ as a FATOU value, and its set of FATOU points is S . According to (9) and (10), $m(S) < \varepsilon$, and this completes the proof of the theorem.

REFERENCES

- [1] F. BAGEMHIL and W. SEIDEL, *Behavior of meromorphic functions on boundary paths, with applications to normal functions*. Arch. Math. 11 (1960), 263–269.
- [2] L. BIEBERBACH, *Lehrbuch der Funktionentheorie*. Bd. II, 2. Aufl., Leipzig und Berlin, 1931.
- [3] C. GATTEGNO et A. OSTROWSKI, *Représentation conforme à la frontière; domaines généraux*. Paris, 1949.
- [4] W. GROSS, *Über die Singularitäten analytischer Funktionen*. Monatsh. Math. Phys. 29 (1918), 3–47.
- [5] O. LEHTO and K. I. VIRTANEN, *Boundary behaviour and normal meromorphic functions*. Acta Math. 97 (1957), 47–65.
- [6] A. J. LOHWATER and G. PIRANIAN, *The boundary behavior of functions analytic in a disk*. Ann. Acad. Sci. Fenn. A I 239 (1957), 1–17.
- [7] R. NEVANLINNA, *Eindeutige analytische Funktionen*. 2. Aufl., Berlin, Göttingen und Heidelberg, 1953.

- [8] I. I. PRIWALOW, *Randeigenschaften analytischer Funktionen*. Berlin, 1956.
- [9] W. SEIDEL and J. L. WALSH, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p -valence*. Trans. Amer. Math. Soc. 52 (1942), 128–216.
- [10] M. TSUJI, *Potential theory in modern function theory*. Tokyo, 1959.

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