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Lefschetz formulae for arithmetic varieties

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0 Introduction

Let G be the identity component of the set of real points of a semisimple algebraic group defined over $\mathbb{Q}$, K a maximal compact subgroup, and Γ a neat arithmetic subgroup. We assume that the symmetric space $X \equiv G/K$ is Hermitian and $\mathbb{Q}$-irreducible. Denote by X_Γ the locally symmetric space $\Gamma \backslash X$. Let (ρ, E) be a finite dimensional, complex irreducible representation of G. Let $\mathscr{E}$ denote the corresponding flat vector bundle over X_Γ, and $L_2^\cdot(X_\Gamma, \mathscr{E})$ the space of square integrable $\mathscr{E}$-valued forms. Define the associated signature operator

$$D_\rho = d + d^* ,$$

where d is the exterior derivative on $\mathscr{E}$-valued forms and d^* is its formal adjoint.

Let $T_G(\alpha)$ denote the Hecke operator associated to an element α of the commensurator of Γ (defined in §2). Let τ be an involution on $L_2^\cdot(X_\Gamma, \mathscr{E})$ which anticommutes with D_ρ and commutes with $T_G(\alpha)$. A natural extension of the Lefschetz fixed point formula in this context would be a formula computing the number $L(\alpha, \tau, E)$ defined by

$$L(\alpha, \tau, E) = \operatorname{Trace} T_G(\alpha)_{|H_+} - \operatorname{Trace} T_G(\alpha)_{|H_-} ,$$

where $H_\pm \equiv \operatorname{Ker}(D_\rho) \cap \pm 1$ eigenspace of τ.

When X_Γ is compact, such a formula has been obtained by Moscovici [17b]. Using the trace formula, Arthur [3, 4] has given a solution to this problem for X_Γ noncompact when τ is the involution corresponding to the Gauss–Bonnet complex. Recently, Goresky and MacPherson have also announced (see [13b]), for the Gauss–Bonnet complex, a solution to the problem using intersection cohomology techniques. The purpose of this paper is to compute $L(\alpha, \tau, E)$ for X_Γ a $\mathbb{Q}$-irreducible arithmetic variety (in general noncompact) and τ the involution corresponding to the signature complex. We shall also sketch the extension of these results to the case where τ^e is the involution corresponding to the Gauss–Bonnet complex. For this complex, we no longer need assume that X_Γ is $\mathbb{Q}$-irreducible. Such formulae may be of some use in understanding the Hasse-Weil zeta functions of Shimura varieties.

Our main results (Theorems 7.9, 7.10, 8.7 and 8.10) state that the Lefschetz numbers are given in most cases by the sum of a local contribution associated to the coincidence locus exactly as in the compact case and an eta invariant term associated to the boundary components. The computation of the eta term is then reduced to the computation of Lefschetz numbers on the boundary components. Inductively, this gives the Lefschetz number as a sum of local coincidence indices on the boundary components. In some degenerate cases (for the signature complex only) there also arises a zeta function term. Following a suggestion of John Millson, we have briefly indicated (in §6) the relationship between this zeta function and the zeta functions associated to prehomogeneous vector spaces studied by Sato and Shintani [24]. We remark that Muller (see [18]) has previously studied the link between zeta functions arising in his work on index theory for $\mathbb{Q}$-rank one spaces and the Sato-Shintani zeta functions.

Suppose now that G is the set of real points of $R_{k|\mathbb{Q}}G$ for some totally real extension $k \neq \mathbb{Q}$, where $R_{k|\mathbb{Q}}$ denotes the restriction of scalars functor. Then the eta terms associated to the boundary components of X_Γ vanish. If we further assume that for all maximal rational parabolic subgroups P and all $\beta \in P \cap (\Gamma\alpha\Gamma)$, ad β has no fixed vector in the center of the Lie algebra of the unipotent radical of P, then the zeta function terms vanish. Combining the assumptions on α and G, one obtains a Lefschetz theorem in which there is no contribution from the boundary components, and the Lefschetz numbers are given by the same formula as in the compact case.

As a special case of the computation of the Lefschetz number for the Gauss–Bonnet complex, we show (Theorem 8.9) how to extend our L^2-signature theorem [26] to compute the L^2-Euler characteristic. Goresky and MacPherson [13a] have previously obtained similar formulae using intersection cohomology techniques. The zeta terms of the signature theorem do not arise here and the eta term is substantially different from that of the signature theorem. Thus, one is led to expect that an index theorem for general Dirac operators would have one more boundary term than arises in the signature theorem. We include (see (8.14)) a sample computation for Sp(4).

1 Algebraic preliminaries

Let V be a real oriented $2m$-dimensional vector space with inner product $(\cdot,\cdot)$. Let $\Lambda^\cdot V^*$ denote the full exterior algebra of V^*. Given $X \in V$, let $X^* \in V^*$ denote the covector dual to X determined by the inner product. Let $\varepsilon(X)$ denote exterior multiplication by X^* on the left and $\varepsilon^*(X)$ its adjoint. We let

$$C(X) = \varepsilon(X) - \varepsilon^*(X)$$

denote left Clifford multiplication by X. We also define

$$\hat{C}(X) = \varepsilon(X) + \varepsilon^*(X)\,.$$

Given distinct orthonormal vectors $X_1, \ldots, X_r \in V$, we call $C(X_1)\ldots C(X_r)$ (respectively $\hat{C}(X_1)\ldots\hat{C}(X_r)$) real (respectively imaginary) Clifford multiplication by $X_1^* \cdots X_r^*$. We say that an endomorphism of this form has real (respectively imaginary) Clifford degree r. Let $\{Z_i\}_{i=1}^{2m}$ be an oriented orthonormal basis of V.

We define the involutions τ_V and $\hat{\tau}_V$ by:

$$\tau_V = i^m C(Z_1) \ldots C(Z_{2m})$$

and

$$\hat{\tau}_V = i^{-m} \hat{C}(Z_1) \ldots \hat{C}(Z_{2m}) .$$

We will need the following standard trace identities (see, for example, [26, (4.4.2)]).

(1.1) Proposition. *Let Y_1 an endomorphism of $\Lambda^\cdot V^*$ of real Clifford degree r_1 and Y_2 an endomorphism of imaginary Clifford degree r_2. Then*

$$\operatorname{tr} Y_1 = 0 \text{ unless } r_1 = 0, \text{ and}$$

$$\operatorname{tr} Y_1 Y_2 = 0, \text{ unless } r_1 = 0 \text{ and } r_2 = 0 .$$

Let $Y(x)$ be a $\Lambda^\cdot V^*$ endomorphism-valued formal power series of the form $Y(x) = \sum_{|I| \geq 0} x^I Y_I(0)$. Suppose further that $Y_I(0) = \sum_{i,j} L_{i,I} R_{j,I}$, where $L_{i,I}$ (respectively $R_{j,I}$) is of real Clifford degree i (respectively imaginary Clifford degree j).

(1.2) Definition. The mass of $Y(x)$ is defined to be

$$\operatorname{mass} Y(x) = \max\{i - |I| : L_{i,I} R_{j,I} \neq 0\} .$$

We shall also require a refined mass, mass_e, defined by

$$\operatorname{mass}_e Y(x) = \max\{i + j - |I| : L_{i,I} R_{j,I} \neq 0\} .$$

From (1.1) we may immediately deduce the following corollary.

(1.3) Corollary. (i) *If* $\operatorname{mass} Y(x) < 2m$, *then* $\operatorname{tr} \tau_V Y(0) = 0$.
(ii) *If* $\operatorname{mass}_e Y(x) < 4m$, *then* $\operatorname{tr} \tau_V \hat{\tau}_V Y(0) = 0$.

(1.4) Remark. (1.3.i) is among the essential tools for obtaining vanishing results and estimates in [26]. We have included (1.3.ii) so that we may later discuss the relation of the results of [26] to the computation of L^2-Euler characteristics.

2 Hecke operators

We shall employ the convention of denoting the Lie algebra of a group by the corresponding lower case bold face letter. If H is a group, L a subgroup, and $x \in H$, we will denote by L_x the centralizer of x in L.

Let G be the identity component of the set of real points of a semisimple algebraic group defined over $\mathbb{Q}$, K a maximal compact subgroup, and Γ a neat arithmetic subgroup. We assume that $X \equiv G/K$ is a $\mathbb{Q}$-irreducible Hermitian symmetric space of dimension n. Denote by X_Γ the locally symmetric space $\Gamma \backslash X$ endowed with the metric induced by the Killing form. Let (ρ, E) be a finite dimensional, complex irreducible representation of G endowed with an admissible inner product (see [16]). Let $\mathscr{E}$ denote the corresponding flat vector bundle over X_Γ, and $L_2^\cdot(X_\Gamma, \mathscr{E})$ the space of square integrable $\mathscr{E}$-valued forms. Here the L_2-inner product is computed with respect to the admissible inner product. Define the associated signature operator

$$D_\rho = d + d^* ,$$

where d is the exterior derivative on $\mathscr{E}$-valued forms and d^* is its formal adjoint. We let

$$\mathscr{H}^\cdot(X_\Gamma, \mathscr{E}) \equiv L_2^\cdot(X_\Gamma, \mathscr{E}) \cap \text{kernel of } D_\rho$$

denote the space of harmonic representatives of the L_2-cohomology of X_Γ with coefficients in $\mathscr{E}$. We shall also denote this space by $\mathscr{H}^\cdot$ when no confusion shall result.

Let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the Cartan decomposition of $\mathbf{g}$. We may identify $L_2^\cdot(X_\Gamma, \mathscr{E})$ with $(L^2(\Gamma\backslash G) \otimes \Lambda^\cdot \mathbf{p}^* \otimes E)^K$ – the elements of $(L^2(\Gamma\backslash G) \otimes \Lambda^\cdot \mathbf{p}^* \otimes E$ invariant under the representation $R \otimes \sigma \otimes \rho$ of K. Here σ denotes the coadjoint representation, and R denotes the right regular representation. Let τ_p denote the involution defined in §1 for $V = \mathbf{p}$. In [17a], Moscovici defines an involution τ_E of E which satisfies

$$\tau_E \rho(X) = -\rho(X)\tau_E, \text{ for every } X \in \mathbf{p}, \quad \text{and} \tag{2.1}$$

$$\tau_E \rho(Y) = \rho(Y)\tau_E, \text{ for every } Y \in \mathbf{k}.$$

This involution induces a decomposition of E of the form $E = E^+ \oplus E^-$, where $E^\pm$ denotes the ± 1 eigenspace of E. Let $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ be the corresponding decomposition of $\mathscr{E}$. Set

$$\tau = 1 \otimes \tau_{\mathbf{p}} \otimes \tau_E\,, \tag{2.2}$$

to obtain an involution of $L_2^\cdot(X_\Gamma, \mathscr{E})$ which anticommutes with D_ρ. When no confusion will result we will also denote $1 \otimes \tau_{\mathbf{p}} \otimes 1$ and $1 \otimes 1 \otimes \tau_E$ by τ_p and τ_E respectively.

Let $\Omega_\pm$ denote the ± 1 eigenspaces of τ acting on $L_2^\cdot(X_\Gamma, \mathscr{E})$ and $\mathscr{H}^\cdot_\pm(X_\Gamma, \mathscr{E})$ (or merely $\mathscr{H}^\cdot_\pm$) the ± 1 eigenspaces of τ acting on $\mathscr{H}^\cdot(X_\Gamma, \mathscr{E})$. We let $D_\pm$ denote the restriction of D_ρ to $\Omega_\pm$. The index of D_+ (given by $\dim \mathscr{H}^\cdot_+ - \dim \mathscr{H}^\cdot_-$) computes the signature of the L^2-cohomology of X_Γ with coefficients in $\mathscr{E}$ (see [17a]).

The commensurator $C(\Gamma)$ of Γ is defined to be the set of $\alpha \in G$ such that $\Gamma \cap \alpha\Gamma\alpha^{-1}$ is of finite index in both Γ and $\alpha\Gamma\alpha^{-1}$. Fix $\alpha \in C(\Gamma)$. Let $C(\alpha) = \{\alpha_1, \ldots, \alpha_r\}$ denote a set of representatives of $\Gamma\alpha\Gamma$; thus we may write $\Gamma\alpha\Gamma = \bigcup_{i=1}^r \alpha_i \Gamma$ (disjoint union). The Hecke operator $T_G(\alpha)$ on $L_2(\Gamma\backslash G)$ corresponding to α, is defined by

$$(T_G(\alpha)f)(\Gamma g) = \sum_{i=1}^{r} f(\Gamma \alpha_i^{-1} g)\,. \tag{2.3}$$

Using the identification of $L_2^\cdot(X_\Gamma, \mathscr{E})$ with $(L^2(\Gamma\backslash G) \otimes \Lambda^\cdot \mathbf{p}^* \otimes E)^k$, one extends $T_G(\alpha)$ as $T_G(\alpha) \otimes 1 \otimes 1$ to an endomorphism of $L_2^\cdot(X_\Gamma, \mathscr{E})$. We shall also denote this endomorphism by $T_G(\alpha)$.

$T_G(\alpha)$ commutes with D_ρ and therefore preserves $\mathrm{Ker}(D_\rho)$. The Lefschetz number for Hecke operators acting on this signature complex is defined to be

$$L(\alpha, \tau, E) \equiv \text{trace } T_G(\alpha)_{|\mathscr{H}_+} - \text{trace } T_G(\alpha)_{|\mathscr{H}_-} = \text{trace } \tau T_G(\alpha)_{|\mathscr{H}} \tag{2.4}$$

We may also associate a correspondence to $\alpha \in C(\Gamma)$. Let $\Gamma_\alpha := \Gamma \cap \alpha\Gamma\alpha^{-1}$. The space X_{Γ_α} is a finite unramified covering of X_Γ. Let $p_1 : X_{\Gamma_\alpha} \to X_\Gamma$ be the natural projection

$$p_1(\Gamma_\alpha x) = \Gamma x,$$

and let p_α be the projection

$$p_\alpha(\Gamma_\alpha x) = \Gamma\alpha x.$$

The coincidence locus of these projections is defined to be the set

$$F := \{m \in X_{\Gamma_\alpha} : p_1(m) = p_\alpha(m)\}.$$

We recall some facts about the structure of this set from [17b, p. 349]. Given an elliptic element $g \in G$, let X_g denote the subset of X fixed by g, and let F_g denote the image of X_g in X_{Γ_α}. The sets F_{g_1} and F_{g_2} are disjoint unless g_1 is Γ_α conjugate to g_2, in which case $F_{g_1} = F_{g_2}$. Then F may be written as the disjoint union

$$F = \bigcup_{g \in [\Gamma\alpha]_e} F_g,$$

where $[\Gamma\alpha]_e$ denotes a set of representatives of Γ_α conjugacy classes of elliptic elements in $\Gamma\alpha$. Each component F_g is locally symmetric. If we let $\Gamma_{\alpha,g}$ denote the centralizer of g in Γ_α, then

$$F_g \cong \Gamma_{\alpha,g} \backslash G_g / K(g),$$

where $K(g)$ is a maximal compact subgroup of G_g. For $g \in C(\Gamma)$, G_g is an algebraic subgroup of G defined over $\mathbb{Q}$, and $\Gamma_{\alpha,g}$ is an arithmetic subgroup of G_g. (That G_g is defined over $\mathbb{Q}$ is an easy consequence of the following fact proved in [8, §2]. Let f be a $\mathbb{Q}$ morphic function on G. Then ${}^g f(x) = f(g^{-1}xg)$ is also $\mathbb{Q}$-morphic.)

(2.4) Proposition. *F_g has finite volume.*

Proof. F_g has finite volume if G_g does not admit a rational character. Hence, to prove finite volume, it suffices to show that the center of G_g does not contain a nontrivial $\mathbb{Q}$-split torus. Let $P = MAN$ (notation as below) be a rational parabolic subgroup of G with $g \in M$ and M_g admitting no rational character. Then $A \subset G_g$. Choose a maximal compact subgroup of G containing g, and extend A to a fundamental Cartan subgroup T containing g. The equal rank condition implies (see [20] Lemma 11.6 and the subsequent remark) that a^* is spanned by totally real roots of T in N. Hence, N_g and θN_g generate A. So G_g admits no rational characters. □

Let P be a rational parabolic subgroup of G, with rational Langlands decomposition $P = NAM$ (or $N_P A_P M_P$ when dealing with more than one parabolic). Here N denotes the unipotent radical of P, AM is the Levi subgroup stable under the Cartan involution θ_k corresponding to K, and A is the unique θ_k-stable lifting of the identity component of the maximal $\mathbb{Q}$-split torus in the center of P/N. We set

$$\Gamma_P = \Gamma \cap P,\ \Gamma_M = \Gamma \cap N \backslash \Gamma_P,\ K_M = K \cap M, \text{ and } \hat{e}(P) = \Gamma_M \backslash M / K_M.$$

Let $e'(P)$ denote the boundary component of the Borel-Serre compactification of X_Γ, determined by P (see [11]). Let ϕ_P denote the set of weights of A on N, and let $A_t = \{a \in A : a^\alpha \geqq t, \forall \text{ simple } \alpha \in \phi_P\}$. Let ω be a relatively compact subset of $\hat{e}(P)$, let $\tilde{\omega}$ be its lift to $\Gamma_P \backslash NM/K_M$, and let $\hat{\omega}$ be its lift to $\Gamma_P \backslash P$. Then for t sufficiently large, $A_t \times \tilde{\omega}$ (respectively $A_t \times \hat{\omega}$ can be identified with a subset $W'_P(\omega, t)$ (respectively $\hat{W}_P(\omega, t)$) of X_Γ (respectively $\Gamma \backslash G$), called a *special neighborhood*. This identification becomes an isometry if $A_t \times \tilde{\omega}$ is endowed with

a metric which descends from a metric on its universal cover of the form

$$da^2 \oplus dz^2 \oplus \bigoplus_{\beta \in \phi_P} a^{-2\beta} \sigma_\beta(z), \tag{2.5}$$

where da^2 and dz^2 are invariant metrics on A and M/K_M and $\sigma_\beta(z)$ is a smoothly varying metric on the β weight space of $\mathbf{n}$. The complement of a compact set in X_Γ (respectively in $\Gamma \backslash G$) can be covered by sets of this form (see [11]), and it will be useful to adapt our discussion of Hecke operators to functions supported on them. We will denote $W'_P(\omega, t)$ simply by W'_P when t and ω are unimportant to our considerations. It is also convenient to define a good neighborhood of the boundary component $e'(P)$ to be a union of special neighborhoods.

$$W^P = \bigcup_{Q \subset P} W'_Q(\omega_Q, t_Q)$$

such that $W^P \cup e'(P)$ is a neighborhood of $e'(P)$ in the Borel–Serre compactification. An important feature of W^P is the fact that it may be isometrically identified with a subset of X_{Γ_P}(but in general does not, of course, contain a subset of the form $A_t \times e'(P)$ for any t).

Given the rational parabolic subgroup $P = NAM$, we may choose $C(\alpha)$ in such a way that

$$\text{(i) if } \alpha_i \Gamma \cap P \neq \varnothing, \quad \text{then } \alpha_i \in P, \quad \text{and} \tag{2.6}$$

$$\text{(ii) if } \alpha_i \Gamma \cap N \neq \varnothing, \quad \text{then } \alpha_i \in N.$$

One merely replaces α_i by an element of the given intersection. Assume now that $C(\alpha)$ satisfies (2.6.i) and (ii). Let $C(\alpha, P)$ (respectively $C(\alpha, N)$) denote $C(\alpha) \cap P$ (respectively $C(\alpha) \cap N$). Given $\beta \in C(\alpha, P)$, let $C_P(\beta)$ denote a set of representatives for $\Gamma_P \beta \Gamma_P / \Gamma_P$. If $\beta \in C(\alpha, N)$, we shall choose $C_P(\beta) \subset N$. This is clearly possible as P normalizes N; we merely choose representatives of the form $\gamma \beta \gamma^{-1}$, $\gamma \in \Gamma_P$.

(2.7) Lemma. *Let $\beta \in C(\alpha, P)$. The elements of $C_P(\beta)$ represent distinct left Γ cosets.*

Proof. Let $g, h \in C_P(\beta)$, and suppose that $g = hv$, for $v \in \Gamma$. Then $v = h^{-1} g \in P$ implies $v \in \Gamma_P$. □

Hence $C(\alpha)$ may be chosen so that

$$\text{if } \beta \in C(\alpha, P) \text{ then } C_P(\beta) \subset C(\alpha). \tag{2.8}$$

In fact, we see that we may choose a subset $\{\beta_1, \ldots, \beta_f\}$ of $C(\alpha)$ such that

$$C(\alpha, P) = \bigcup_{i=1}^{f} C_P(\beta_i) \text{ (disjoint union).} \tag{2.9}$$

We shall fix such a collection $\{\beta_1, \ldots, \beta_f\}$ which we shall denote $\bar{C}_P(\alpha)$. We set $\bar{C}_N(\alpha) = \bar{C}_P(\alpha) \cap N$.

(2.10) Definition. We say $C(\alpha)$ is *adapted to P* if it is chosen to satisfy (2.6) and (2.8).

(2.11) Lemma. *Let f be a measurable function supported on $\hat{W}_P(\omega, t)$. Then for t sufficiently large, the restriction of $T_G(\alpha)f(\Gamma g)$ to $\hat{W}_P(\omega, t)$ is given by*

$$\sum_{\alpha_i \in C(\alpha, P)} f(\Gamma \alpha_i^{-1} g).$$

Proof. It follows from elementary reduction theory that for t sufficiently large, $\Gamma g \in \hat{W}_P(\omega, t)$ implies $\Gamma \alpha_i^{-1} g \notin \hat{W}_P(\omega, t)$ if $\alpha_i \Gamma \cap P = \varnothing$. This may be seen geometrically in terms of the Borel–Serre compactification by observing that if $\alpha_i \Gamma \cap P = \varnothing$, the correspondence induced by $\Gamma g K \to \Gamma \alpha_i^{-1} g K$ takes $\hat{e}(P)$ to a union of boundary components disjoint from $\hat{e}(P)$. The result then follows from the fact that $\{W'_P(\omega, t)\}_{t \geq t_0}$ is a cofinal system of neighborhoods of a relatively compact subset $\bar{\omega}$ of $\hat{e}(P)$ and the image of $\{W'_P(\omega, t)\}_{t \geq t_0}$ under the correspondence is a cofinal system of neighborhoods of the image of $\bar{\omega}$.

(2.12) Definition. For f a measurable function supported on $\hat{W}_P(\omega, t)$, define

$$T_P(\alpha) f(\Gamma g) = \sum_{\alpha_i \in C(\alpha, P)} f(\Gamma \alpha_i^{-1} g),$$

and extend $T_P(\alpha)$ to $L_2'(W'_P(\omega, t), \mathscr{E})$ in the same manner as $T_G(\alpha)$.

3 Analytic framework

3.1 Local analysis

Set $\Delta = D_\rho^2$, and let $\Delta_\pm$ denote the restriction of Δ to $\Omega_\pm$. We will shift from the $(\Delta + w)^{-k}$ formalism of [26] to the heat equation formalism in order to simplify certain computations and to present arguments in a form more familiar to the reader. Let $e_\pm(t, x, y)$ denote the integral kernel of $e^{-t\Delta_\pm}$, and set $e = e_+ + e_-$. The L_2-projection operator onto $\mathscr{H}_\pm$ is given by $\lim_{t \to \infty} e^{-t\Delta_\pm}$ (see [26, 2.2]). Hence

$$L(\alpha, \tau, E) = \int_{X_\Gamma} \lim_{t \to \infty} [\operatorname{tr}(T_G(\alpha) \circ e_+)(t, x, x) - \operatorname{tr}(T_G(\alpha) \circ e_-)(t, x, x)] dx \tag{3.1.1}$$

$$= \int_{X_\Gamma} \lim_{t \to \infty} \operatorname{tr}(T_G(\alpha) \circ \tau e)(t, x, x) dx\ . \tag{3.1.2}$$

When X_Γ is compact, it is well known that $\int_{X_\Gamma} \operatorname{tr}(T_G(\alpha) \circ \tau e)(t, x, x) dx$ is independent of t and can be computed in terms of the geometry of the coincidence locus by computing the asymptotic expansion of the integrand as $t \to 0$. In particular, if U is a compact subset of X_Γ,

$$\lim_{t \to 0} \int_U \operatorname{tr}(T_G(\alpha) \circ \tau e)(t, x, x) dx \tag{3.1.3}$$

is finite and computed, for example in [*BV*] (see also [17b]). There are minor differences between our context and the situations considered by the above authors. For the convenience of the reader unfamiliar with the standard local calculations of index theory, we shall make several remarks to eliminate these differences.

The signature operator, D_ρ, that we are considering is not a standard twisted Dirac operator. In particular, it is not the generalized signature operator considered by [6]. Let $\{X_j\}_j$ be an orthonormal basis for $\mathbf{p}$, and identify each X_j with an invariant vector field on $\Gamma\backslash G$. Then we may write $D_\rho = D_0 + D_1$, with

$$D_1 = \sum_j C(X_j)X_j, \quad \text{and} \quad D_0 = \sum_j \hat{C}(X_j)\rho(X_j)\,.$$

The operator D_1 is a twisted Dirac operator and is, in fact, the usual generalized (or twisted) signature operator. The operator D_0 anticommutes with D_1 and τ_E and commutes with τ_p. Hence

$$D_\rho^2 = D_0^2 + D_1^2.$$

Let tr_0 denote the (pointwise) trace over the submodule $\mathrm{Ker}\, D_0$. Standard intertwining arguments (see [27, §4]) imply that

$$\mathrm{tr}\,\tau_E\tau_p e^{-t\Delta} = \mathrm{tr}_0\tau_E\tau_p e^{-t\Delta} = \mathrm{tr}_0\tau_E\tau_p e^{-tD_1^2} = \mathrm{tr}\,\tau_E\tau_p e^{-tD_1^2}\,.$$

We rewrite the last expression as

$$\mathrm{tr}\,\tau_E\tau_p e^{-tD_1^2} = \mathrm{tr}\,\tau_p e^{-tD_{E^+}^2} - \mathrm{tr}\,\tau_p e^{-tD_{E^-}^2}\,,$$

where $D_{E^\pm}$ denotes the Dirac operator with coefficients in the K-module given by the ± 1 eigenspace of τ_E. (One can also establish this equality by direct computation using mass considerations; one merely needs to note that D_0^2 has mass zero and hence does not contribute to the local formula.) Thus for X_Γ compact, we are in a subcase of that considered by Moscovici in [17b]. Unfortunately, the computations of [17b] are based on the trace formula, and hence do not explicitly give a formula for (3.1.3) (although the argument can easily be adapted to establish a local version of the results. In fact, the local nature of the heat equation asymptotics allows one to bypass the trace formula aspects of Moscovici's derivation and merely use his computation of elliptic orbital integrals to obtain the local contribution of the coincidence locus to the Lefschetz numbers.)

The analysis of [BV, Theorem 3.32] is already localized, however, so we may apply it with less effort. The result is stated for isometries, but clearly applies directly to correspondences which are generated by local isometries. (The small time asymptotics of the heat operator are "unaware" that a local isometry does not extend to a global isometry. See (3.1.8).) Thus we see that [BV, Theorem 3.32] can be applied directly to compute (3.1.3). More explicitly, let U be a small open neighborhood of a point $x \in p_1(F)$. Let $\{U_i\}_{i=1}^N$ be the components of $p_1^{-1}(U)$. Let U_{i,g_t} denote the intersection with the unique (for U sufficiently small) component F_{g_t} of F with which U_i has nonempty intersection. Then we have the following proposition, where all unexplained notation is as in [BV], extended to local isometries in the obvious fashion.

(3.1.4) Propositions.

$$\lim_{t\to 0}\int_U \mathrm{tr}(T_G(\sigma)\circ\tau e)(t,x,x)dx$$

$$= \sum_i \int_{U_{i,g_t}} [ch(g_i, E_+) - ch(g_i, E_-)]\hat{}\, D^{-1/2}(g_i, NF_{g_t})\hat{}\,\hat{A}(F_{g_t}).$$

In the locally symmetric case, the integrand

$$[ch(g_i, E_+) - ch(g_i, E_-)]\hat{} D^{-1/2}(g_i, NF_{g_i})\hat{} \hat{A}(F_{g_i})$$

is a constant multiple of the volume form of F_{g_i}, which is computed in [17b, (5.3)]. Let c_{g_i} denote this constant.

We return to the situation of interest to us — X_Γ noncompact. Let $\{B_j\}_{j=1}^{\infty}$ be a collection of compact subsets of X_Γ satisfying $B_j \subset B_{j+1}$, $\cup_{j=1}^{\infty} B_j = X_\Gamma$, and ∂B_j intersects the image of the coincidence locus transversely. Define

$$(3.1.5)\quad \ell(\alpha, \tau, E, B_j) \equiv \lim_{t \to 0} \int_{B_j} [\mathrm{tr}(T_G(\alpha)e_+)(t, x, x) - \mathrm{tr}(T_G(\alpha)e_-)(t, x, x)]dx\ ,$$

and set

$$(3.1.6)\qquad \ell(\alpha, \tau, E) \equiv \lim_{j \to \infty} \ell(\alpha, \tau, E, B_j)\ .$$

Let φ be a bounded function of compact support. As we shall recall in (3.1.8) below, the convergence of the small time asymptotic development of $\mathrm{tr}\, T_G(\alpha)\tau e^{-t\Delta}\varphi$ in no way requires X_Γ to be compact, only that supp φ is. (Of course, the rate of convergence depends upon the support of φ. In particular, the zeta function terms studied in §5 of this paper would be zero if the convergence were independent of t.) Hence we may apply the results of (3.1.4) to deduce that $\ell(\alpha, \tau, E, B_j)$ is well defined and finite for fixed j. Moreover, as observed above, it is computed by integrating a locally constant function over $F \cap p^{-1}(B_j)$. As we have noted, F has finite volume; therefore, $\ell(\alpha, \tau, E,)$ exists and is given by $\sum_i c_{g_i} \mathrm{vol}(F_{g_i})$. We gather these remarks in a proposition.

(3.1.7) Proposition. *$\ell(\alpha, \tau, E, B_j)$ is finite for fixed j and converges as $j \to \infty$ to*

$$\ell(\alpha, \tau, E) = \sum_i c_{g_i} \mathrm{vol}(F_{g_i})\ .$$

We will want a somewhat stronger result which estimates the rate at which the above limits converge. Choose B_j as in [26, 3.2], with its injectivity radius greater than or equal to e^{-2j}. Fix $\varepsilon > 0$. Stationary phase arguments (integration by parts) applied to the usual local construction of $e^{-t\Delta}$ easily imply the following proposition. We include a proof for the convenience of readers unfamiliar with singular integral operators.

(3.1.8) Proposition. *If $te^{4j} < t^{\varepsilon}$,*

$$(3.1.9)\qquad \int_{B_j} \mathrm{tr}\,(T_G(\alpha) \circ \tau e)(t, x, x)dx = \ell(\alpha, \tau, E, B_j) + O(t^{\varepsilon})\ .$$

Proof. Let φ be the characteristic function of B_j, and let ψ be a compactly supported function which is identically one in a neighborhood of the support of φ. Suppose further that the injectivity radius of any point in the support of ψ is greater than $e^{-2j}/2$. Fix a contour C_t in the complex plane given by the image of $\mathbb{R}$ under the map $s \to |s| + \text{ is } -t^{-1}$. Observe that for $\lambda \in C_t$, $|\lambda| > t^{-1}/4$, but $-\mathrm{Re}(t\lambda) < 1$.

Moreover for $\lambda \in C_t$, the distance between λ and the spectrum of $\Delta(C[0, \infty))$ is greater than $|\lambda|/2$. Write

$$e^{-t\Delta}\varphi = (2\pi i)^{-1} \int_{C_t} e^{-t\lambda}(\Delta - \lambda)^{-1} d\lambda \varphi \ .$$

Let $P(\lambda)$ be a parametrix for $(\Delta - \lambda)$. By this we mean that $(\Delta - \lambda)P(\lambda) - I \equiv \varepsilon(\lambda)$ is a smoothing operator. By smoothing operator, we do not mean that $\varepsilon(\lambda)$ is infinitely smoothing, but merely that it maps L^2_{loc} to C^N for some finite $N \gg n$. If we assume only that $\varepsilon(\lambda)\varphi$ is smoothing, we shall call $P(\lambda)$ a local parametrix for $(\Delta - \lambda)$. We shall maintain these conventions throughout this paper. Suppose that $\varepsilon(\lambda)$ satisfies $\varepsilon(\lambda)\varphi = \psi\varepsilon(\lambda)\varphi$. Then we may write

$$e^{-t\Delta}\varphi = (2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) d\lambda \varphi - (2\pi i)^{-1} \int_{C_t} e^{-t\lambda}(\Delta - \lambda)^{-1} \varepsilon(\lambda) d\lambda \varphi \ ,$$

and

$$\begin{aligned}
&\operatorname{Tr}(T_G(\alpha)\tau(e^{-t\Delta} - (2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) d\lambda)\varphi \\
&\quad = -\operatorname{Tr}(T_G(\alpha)\tau(2\pi i)^{-1} \int_{C_t} e^{-t\lambda}(\Delta - \lambda)^{-1} \varepsilon(\lambda) d\lambda \varphi \\
&\quad = -\operatorname{Tr}(T_G(\alpha)\tau(2\pi i)^{-1} \int_{C_t} e^{-t\lambda}[P(\lambda) \circ \varepsilon(\lambda) - (\Delta - \lambda)^{-1} \varepsilon(\lambda) \circ \varepsilon(\lambda)] d\lambda \varphi \ .
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left| \operatorname{Tr}(T_G(\alpha)\tau(e^{-t\Delta} - (2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) d\lambda)\varphi \right| \\
&\quad \leqq \left| \operatorname{Tr}(T_G(\alpha)\tau(2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) \circ \varepsilon(\lambda) d\lambda \varphi \right| \\
&\qquad + (2\pi)^{-1} \int_{C_t} |e^{-t\lambda}| |T_G(\alpha)(\Delta - \lambda)^{-1}|_{\text{sup}} |\varepsilon(\lambda)\psi|_{\text{H.S.}} |\varepsilon(\lambda)\varphi|_{\text{H.S.}} |d\lambda| \varphi \ ,
\end{aligned}$$

where $|\cdot|_{\text{sup}}$ and $|\cdot|_{\text{H.S.}}$ denote the supremum and Hilbert-Schmidt operator norms. For $\lambda \in C_t$ and for some $C_\alpha > 0$, $|T_G(\alpha)(\Delta - \lambda)^{-1}|_{\text{sup}} \leqq C_\alpha |\lambda|^{-1}$, and we see that if we construct a parametrix P such that $\varepsilon(\lambda)\psi$ $(\lambda \in C_t)$ and $|\operatorname{Tr}(T_G(\alpha)\tau(2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) \circ \varepsilon(\lambda) d\lambda \varphi|$ *are* $O(t^N)$, then

$$\left| \operatorname{Tr}(T_G(\alpha) \circ \tau(e^{-t\Delta} - (2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) d\lambda \varphi \right| \leqq C t^N \ .$$

Clearly, it suffices to establish rapid pointwise decay for $\varepsilon(\lambda)$ (and its derivatives) in order to obtain these estimates. Therefore, in order to prove the proposition, we must construct $P(\lambda)$ and $\varepsilon(\lambda)$ with the desired properties, and then prove (3.1.9) with $(2\pi i)^{-1} \int_{C_t} e^{-t\lambda} P(\lambda) d\lambda$ substituted for $e^{-t\Delta}$.

The parametrices $P(\lambda)$ may be constructed as follows. Let $\{\varphi_\alpha, \psi_\alpha\}_\alpha$ be a collection of smooth nonnegative functions satisfying:

(i) $\{\varphi_\alpha\}_\alpha$ is a partition of unity.
(ii) $\psi_\alpha = 1$ on the support of φ_α; ψ_α is supported in a coordinate neighborhood V_α.
(iii) For $(x, y) \in$ support $d\psi_\alpha \times$ support φ_α, $d(x, y) \geqq$ (injectivity radius of y)/10.

On each V_α, one constructed a singular integral operator P_α of the form

$$P_\alpha = \sum_{\ell=0}^{N} \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot v} h_\ell^\alpha(x, y, \lambda, v)(\|2\pi v\|^2 - \lambda)^{-\ell-1} dv ,$$

where $\|2\pi v\|^2 = g^{ab}(x)v_a v_b$. (For ℓ small and $x \neq y$, integration by parts transforms the integrands into integrable functions). A recipe for this construction may be obtained from the parametrix construction in the appendix (see A.1), by setting $i = 0$ and putting $h_\ell^\alpha = h_{0,\ell}^{0,\alpha}$. These h_ℓ^α are constructed inductively to satisfy

$$\psi_\alpha(\Delta - \lambda)P_\alpha\psi_\alpha = \psi_\alpha(I + \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot v} I_{N+1}^\alpha(x, y, \lambda, v)(\|2\pi v\|^2 - \lambda)^{-N-1} dv)\psi_\alpha ,$$

where $I_{N+1}^\alpha(x, y, \lambda, v)(\|2\pi v\|^2 - \lambda)^{-1}$ is bounded for $|\lambda| > 1$. Set

$$\varepsilon_\alpha(\lambda) = \psi_\alpha \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot v} I_{N+1}^\alpha(x, y, \lambda, v)(\|2\pi v\|^2 - \lambda)^{-N-1} dv \psi_\alpha .$$

The P_α are patched together to form a parametrix

$$P(\lambda) = \sum_\alpha \psi_\alpha P_\alpha(\lambda)\varphi_\alpha .$$

Consider now

$$(\Delta - \lambda)P(\lambda) = (\Delta - \lambda)\sum_\alpha \psi_\alpha P_\alpha(\lambda)\varphi_\alpha = \sum_\alpha \varepsilon_\alpha(\lambda)\varphi_\alpha + \sum_\alpha [\Delta, \psi_\alpha]P_\alpha(\lambda)\varphi_\alpha .$$

By construction, for N sufficiently large, $|\sum_\alpha \varepsilon_\alpha(\lambda)\varphi_\alpha|_{\text{H.S.}}$ is $O(|\lambda|^{-N/2})$ and therefore $O(t^{N/2})$ for $\lambda \in C_t$; in fact, one can prove (by computing the integrals) that $|\varepsilon_\alpha(x, y, \lambda)| < c_2|\lambda^{-N/2}|e^{-c|\lambda|^{1/2}|x-y|}$, for some constants c_1 and c_2. We are thus left to consider the commutator term $\sum_\alpha [\Delta, \psi_\alpha]P_\alpha(\lambda)\varphi_\alpha$. We will establish a pointwise estimate for the Schwartz kernel of this operator. Estimates for its derivatives follow in the same manner. It is an elementary matter, which we omit, to use this estimate to provide the desired Hilbert-Schmidt and trace estimates (see [F, §3] for related computations). We treat the term

$$\sum_\alpha [\Delta, \psi_\alpha] \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot v} h_0^\alpha(x, y, \lambda, v)(\|2\pi v\|^2 - \lambda)^{-1} dv\varphi_\alpha$$

$$= \sum_\alpha [\Delta, \psi_\alpha] \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot v}(\|2\pi v\|^2 - \lambda)^{-1} dv\varphi_\alpha ;$$

the remaining terms are handled similarly. Observe that by construction, $4|x - y| >$ (injectivity radius of y) on support $[\Delta, \psi_\alpha] \times$ support φ_α. Integrating by parts $2N$ times, we may transform the integral to an integral of the form

$$I = \sum_\alpha [\Delta, \psi_\alpha]p((x - y)^{-1}) \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot}(\|2\pi v\|^2 - \lambda)^{-2N} b(v) dv\varphi_\alpha ,$$

where p and b are polynomials of degree $2N$. One can estimate this integral crudely by making a change of variables to obtain for some $C > 0$, independent of t (depending only on $4N$ derivatives of the metric)

$$|I\psi| \leqq \sum_{\alpha} C(1 + |x - y|^{-2N-2})|\lambda|^{-N+n/2}\psi \ . \tag{3.1.10}$$

A more careful computation of the integral, using its oscillatory nature, allows one to improve this estimate by a factor of $e^{-c|x-y||\lambda|^{1/2}}$, for some $c > 0$, but we will not need this sharper estimate. The estimate (3.1.10) is not useful unless $|\lambda|^{1/2}|x - y|$ is large. Recall that we have assumed that $|\lambda| > t^{-1}/4$, $te^{4j} < t^{\varepsilon}$, and on the support of $\varphi(x)d\psi(y)$, $|x - y| >$ (injectivity radius at y)$/4 \geqq e^{-2j}/4$. Hence,

$$|\lambda|^{1/2}|x - y| \geqq t^{-1/2}e^{-2j}/8 > t^{-\varepsilon/2}/8 \ .$$

Thus $|I\psi| \leqq C_N t^{N\varepsilon/2}|x - y|^{-2}t^{-n/2}\psi$. Taking $N\varepsilon$ large, we see that for every M, there exists a constant $c_M > 0$ depending only on ε, M, and $4M$ derivatives of the metric such that

$$|I\psi| \leqq c_M t^M \ .$$

This gives us the desired pointwise estimate. The remaining estimates follow similarly. Thus in order to prove the proposition, it suffices to show that the trace of $T_G(\alpha)\tau(2\pi i)^{-1}\int_{C_t} e^{-t\lambda}P(\lambda)d\lambda$ converges at the desired rate. This is done in the appendix (A.3.11), where $(2\pi i)^{-1}\int_{C_t} e^{-t\lambda}P(\lambda)d\lambda$ is denoted $H_0^0(t)$. □

Remark. Observe that the hypothesis that t is small entered our estimates only through the choice of the contour C_t. For t large, we are forced to let our contour pass closer to the origin to ensure that $e^{-t\lambda}$ is bounded as $t \to \infty$. Thus, the difficulties associated to computing $e^{-t\Delta}$ for t large are transformed by the functional calculus to the problem of computing $(\Delta - \lambda)^{-1}$ for λ small.

3.2 Global analysis

We now wish to understand $L(\alpha, \tau, E) - \ell(\alpha, \tau, E)$. We observe that because $[T_G(\alpha), D_\rho] = 0$, the arguments in [26, §2] may be modified in an obvious fashion to show that the integral expression for $L(\alpha, \tau, E)$ in (3.1.2) equals

$$\lim_{j \to \infty} \left(\int_{B_j} \operatorname{tr} \tau T_G(\alpha) e(\varepsilon_j, x, x)dx + \int_{\varepsilon_j}^{\infty} \int_{\partial B_j} \operatorname{tr} C(\nu)\tau D T_G(\alpha) e(t, x, x)dx\, dt/2 \right), \tag{3.2.1}$$

where ν is the unit normal to ∂B_j (which may be taken to be piecewise smooth), and $\{\varepsilon_j\}_j$ is any sequence of positive constants. If we select $\varepsilon_j = e^{-4j/(1-\varepsilon)}$, then (3.1.8) implies

$$\int_{B_j} \operatorname{tr} \tau T_G(\alpha) e(\varepsilon_j, x, x)dx = \ell(\alpha, \tau, E, B_j) + O(e^{-\varepsilon 4j/(1-\varepsilon)}) \ . \tag{3.2.2}$$

Inserting this into (3.2.1) yields the following corollary.

(3.2.3) Corollary.

$$L(\alpha,\tau,E)-\ell(\alpha,\tau,E)=\lim_{j\to\infty}\int_{e^{-4j/(1-\varepsilon)}}^{\infty}\int_{\partial B_j}\operatorname{tr}C(v)\tau D(T_G(\alpha)e)(t,x,x)dxdt\ .$$

Remark. Integrating over $[e^{-4j/(1-\varepsilon)},\ \infty]$ instead of $[0,\ \infty]$ eliminates the need for the awkward regularization of the integrand used in [26, §5.1], and consequently simplifies many of our arguments.

Let

$$I_j(t)=\int_{\partial B_j}\operatorname{tr}C(v)\tau D(T_G(\alpha)e)(t,x,x)\,dx\ .$$

In this section we shall discuss how to apply the results and techniques of [26] to compute $\lim_{j\to\infty}\int_{e^{-4j/(1-\varepsilon)}}^{\infty}I_j(t)dt$.

Using the functional calculus, we write

$$I_j(t)=(2\pi i)^{-1}\int_{\partial B_j}\int_{C_t}e^{-t\lambda}\operatorname{tr}C(v)\tau D_\rho T_G(\alpha)(\Delta-\lambda)^{-1}(x,x)\,d\lambda dx\ .$$

As in the case of the small t asymptotics, the basis of our computation is the construction of an approximation to $(\Delta-\lambda)^{-1}$. We let $\tilde{H}(\lambda)$ denote a parametrix for $(\Delta-\lambda)$ and set $\varepsilon(\lambda)=I-(\Delta-\lambda)\tilde{H}(\lambda)$. This is not the same $\varepsilon(\lambda)$ of the previous section. We write

$$I_j=(2\pi i)^{-1}\int_{\partial B_j}\int_{C_t}e^{-t\lambda}\operatorname{tr}C(v)\tau D_\rho T_G(\alpha)[\tilde{H}(\lambda)+(\Delta-\lambda)^{-1}\varepsilon(\lambda)](x,x)d\lambda dx\ .$$

Thus the evaluation of $\int_{e^{-4j/(1-\varepsilon)}}^{\infty}I_j dt$ requires evaluating

$$I_{j,P}:=(2\pi i)^{-1}\int_{e^{-4j/(1-\varepsilon)}}^{\infty}\int_{\partial B_j}\int_{C_t}e^{-t\lambda}\operatorname{tr}C(v)\tau D_\rho T_G(\alpha)\tilde{H}(\lambda)(x,x)d\lambda dx dt\ ,$$

and showing that

$$I_{j,\varepsilon}:=(2\pi i)^{-1}\int_{e^{-4j/(1-\varepsilon)}}^{\infty}\int_{\partial B_j}\int_{C_t}e^{-t\lambda}\operatorname{tr}C(v)\tau D_\rho T_G(\alpha)(\Delta-\lambda)^{-1}\varepsilon(\lambda)(x,x)d\lambda dx dt$$

vanishes in the limit as $j\to\infty$.

In [26, 4.5, 4.7, and 4.9], we constructed integral kernels $h(w, x, y)$ which approximated $(\Delta+w)^{-k}$ near ∂B_j with error vanishing as $j\to\infty$ (uniformly in w and not merely as $w\to\infty$). Replacing k by 1 and $-w$ by $\lambda\in C_t$, this construction immediately yields a construction of a parametrix $\tilde{H}(\lambda)$ with associated $\varepsilon(\lambda)$ having rapidly decreasing trace norm in a neighborhood of ∂B_j, as $j\to\infty$. The vanishing of $\lim_{j\to\infty}I_{j,\varepsilon}$ is demonstrated in [26], primarily in [ibid, 4.8 and 4.9.5]. The proof is unchanged by the above modifications and by the addition of the bounded operator $T_G(\alpha)$. (For example, the only information about τ used in the proof was that it was a bounded operator.) We will not, therefore, reproduce the proof in this paper. We remark, however, that the fundamental property of X_Γ which allows us to estimate $I_{j,\varepsilon}$ is the fact that rank G = rank K. This implies (see [26, 1.2.1)]) that the lower bound of the essential spectrum of Δ is greater than zero. Hence for $\lambda\in C_t$, $D_\rho(\Delta-\lambda)^{-1}$ is a bounded operator with bound independent of t and $\lambda\in C_t$, and

estimates that show that, as $j \to \infty$, $\varepsilon(\lambda)(x, y)$ is rapidly decreasing for y in a small neighborhood of ∂B_j may be used to show that $D_\rho(\Delta - \lambda)^{-1}\varepsilon(\lambda)\varphi$ also has rapidly decreasing trace norm, for φ supported in a small neighborhood of ∂B_j.

The computation of $I_{j,P}$ is carried out in §5, §6, §7, and §8. The construction of $\tilde{H}(\lambda)$ is reviewed and modified for our present purposes in the appendix, but we discuss the pertinent features here.

First we make some remarks about localizing parametrix constructions. Let $\{\varphi_k\}_k$ be a partition of unity for X_Γ, and suppose $\{\varphi_{k,j}\}_k \subset \{\varphi_k\}_k$ is a subset which is a partition of unity for neighborhood of ∂B_j. Let $\tilde{H}'^k(\lambda)$ be a family of integral operators such that $\varepsilon_k := (I - (\Delta - \lambda)\tilde{H}'^k(\lambda))\varphi_k$ is a smoothing operator for all k. Then $\tilde{H}(\lambda) = \sum_k \tilde{H}'^k$ is a parametrix for $(\Delta - \lambda)$, with associated error $\varepsilon(\lambda) := \sum_k \varepsilon_k\varphi_k$. Moreover if we set

$$I_{j,P_k} := (2\pi i)^{-1} \int_{e^{-4j/(1-r)}}^{\infty} \int_{\partial B_j} \int_{C_t} e^{-t\lambda} \operatorname{tr} C(v)\tau D_\rho T_G(\alpha)\tilde{H}'^k(\lambda)\varphi_{k,j}\, d\lambda\, dx\, dt \,,$$

and

$$I_{j,\varepsilon_k} := (2\pi i)^{-1} \int_{e^{-4j/(1-r)}}^{\infty} \int_{\partial B_j} \int_{C_t} e^{-t\lambda} \operatorname{tr} C(v)\tau D_\rho T_G(\alpha)(\Delta - \lambda)^{-1}\varepsilon_k(\lambda)\varphi_{k,j}\, d\lambda\, dx\, dt \,,$$

then $I_{j,P} = \sum_k I_{j,P_k}$ and $I_{j,\varepsilon} = \sum_k I_{j,\varepsilon_k}$. Thus, in order to compute I_j, it suffices to construct $\tilde{H}'^k$ with each $\varepsilon_k\varphi_{k,j}$ (and $\sum_k \varepsilon_k\varphi_{k,j}$) rapidly decreasing. This observation is useful for bookkeeping but does not imply a true localization of the problem of constructing $\tilde{H}(\lambda)$, as we shall now explain. Let U_k be a coordinate neighborhood which contains the support of $\varphi_{k,j}$, and let φ_k be a cutoff function with support on U_k which is identically one on the support of $\varphi_{k,j}$. Suppose we constructed an operator $\tilde{H}'^k$ on $L_2(U_k, E)$ such that $\psi_k[(\Delta - \lambda)\tilde{H}'^k - I]\varphi_{k,j}$ is rapidly decreasing as $j \to \infty$. To extend $\tilde{H}'^k$ to X_Γ, we premultiply by some cutoff function, say ψ_k. Then $((\Delta - \lambda)\psi_k\tilde{H}'^k - I)\varphi_{k,j}$ includes new error terms of the form $[\Delta, \psi_k]\tilde{H}'^k\varphi_{k,j}$ which need not be rapidly decreasing merely because $\psi_k[(\Delta - \lambda)\tilde{H}'^k - I]\varphi_{k,j}$ is. In particular, as $j \to \infty$, $|\nabla\psi_k|$ increases exponentially if we assume that ψ_k is supported in a coordinate neighborhood. We address this problem in the appendix by constructing $\tilde{H}'^k$ as a sum of operators $\tilde{H}^{i,k}$, $i = 1, \ldots, J$, which are defined on larger sets (which are not coordinate neighborhoods) $U_{k,i} \supset U_k$ which admit cutoff functions $\psi_{k,i}$ supported on $U_{k,i}$, with $[\Delta, \psi_{k,i}]\tilde{H}^{i,k}\varphi_{k,j}$ rapidly decreasing as $j \to \infty$.

From the preceding discussion, we see that it suffices to construct a parametrix $\tilde{H}'^k$ with small error in a neighborhood U_k of a point $x \in \partial B_j$. Such parametrices are then patched together using a partition of unity $\{\varphi_{k,j}\}_k$ of a neighborhood of ∂B_j. For j large, ∂B_j is contained in a union of special neighborhoods. Hence we assume that $U_k \subset W'_P(\omega_x, s_x)$, for some rational parabolic subgroup $P = NAM$ depending on x as in (A.1.1). We assume further that $1 \leqq$ diameter of $U_k \leqq 2$. Our construction of $\tilde{H}'^k$ depends on t and a choice of central series for N. In (A.1), we associate to U_k a central series

$$\{\mathrm{id}\} = N_0 \subset \cdots \subset N_J = N \,, \tag{3.2.4}$$

with each N_i normalized by P, and define the corresponding fibrations

$$\bar{\pi}_i : W'_P(\omega, t_P) \to A_{t_P} \times \Gamma_P N_i \backslash NM/K_M, \text{ with fiber } \Gamma_i \backslash N_i \,,$$

where $\Gamma_i := \Gamma \cap N_i$.

Let T_i denote the projection onto N_i invariant forms (as in [26, §4.2]). For each i, we construct an operator $\tilde{H}_1^{i,k}$ which satisfies

$$(T_{i-1} - (\Delta - \lambda)\tilde{H}_1^{i,k})\varphi_k = (T_i + \varepsilon_1^{i,k})\varphi_k ,$$

with $\varepsilon_1^{i,k}$ a rapidly decreasing smoothing error term for all t and all $\lambda \in C_t$. The estimate of the error terms ([26, §4.8]) for $\tilde{H}_1^{i,k}$ is analogous to the small time error estimates, with the growth of Fourier coefficients replacing exploitation of $|\lambda|$ (or more precisely $|\min\{0, \mathrm{Re}(\lambda)\} + \mathrm{Im}(\lambda)|$). The need for a relation analogous to the inequality $t^{1/2} < (\text{injectivity radius})^{1+\varepsilon}$ used in the small time estimates prevents our central series from being taken arbitrarily fine.

We construct a second operator $\tilde{H}_0^{i,k}$ with

$$(T_i - (\Delta - \lambda)\tilde{H}_0^{i,k})\varphi_k = \varepsilon_0^{i,k}\varphi_k ,$$

where $\varepsilon_0^{i,k}$ is smoothing but is no longer uniformly rapidly decreasing independent of t. For t sufficiently small, we do obtain rapidly decreasing error. More precisely, for $\varepsilon > 0$ fixed, $\varepsilon_0^{b,k}$ is rapidly decreasing if

$$t^{1/2}(\text{injectivity radius of } \bar{\pi}_b(x))^{-1} < t^{\varepsilon} . \tag{3.2.5}$$

This can be proved by an argument similar to the proof of (3.1.8). In this range,

$$\left(\sum_{i \leqq b} \tilde{H}_1^{i,k} + \tilde{H}_0^{b,k}\right)\varphi_k$$

gives a good approximation to $(\Delta - \lambda)^{-1}\varphi_k$. (We call an approximation good if the error is rapidly decreasing.) Set

$$H_j^{i,k}(t) = (2\pi i)^{-1} \int_{C_t} e^{-t\lambda}\tilde{H}_j^{i,k} d\lambda, \quad j = 0, 1 . \tag{3.2.6}$$

For large t, where $(\sum_{i \leqq b} \tilde{H}_1^{i,k} + \tilde{H}_0^{b,k})\varphi_k$ is not a good approximation for any $b < J$, we are left to recall approximation $\tilde{H}^{0,k}(\lambda)$ and $H^{0,k}(t)$ to the restrictions of $(\Delta - \lambda)^{-1}$ and $e^{-t\Delta}$ to N-invariant forms with rapidly decreasing error. The operators $\tilde{H}^{0,k}(\lambda)$ and $H^{0,k}(t)$ are constructed using separation of variables on $A_t \times \hat{e}(p)$, which is a metric product (see [26, 4.9]). More precisely, let d_n denote the coboundary operator of $\mathbf{n}$ with coefficients in E, and set $D_n = d_n + d_n^*$. Let $D_Z = d_Z + d_Z^*$ denote the signature operator in $\hat{e}(P)$ with coefficients in the flat vector bundle associated to $\Lambda^{\cdot}\mathbf{n}^* \otimes E$, and set $\hat{D} = D_Z + D_n$. Let $\delta = \sum_{\beta \in \phi_P} \beta/2$, and let $\{\partial/\partial r_i\}$ be an orthonormal basis of $\mathbf{a}$. Let ψ be a cutoff function supported in a good neighborhood of $e'(P)$, which satisfies $d(x, y) \geqq cj^{\varepsilon'}$ for $x \in U_k$, $y \notin \psi^{-1}(1)$, some $\varepsilon' > 0$ independent of x. (That such a ψ always exists follows from our choice of P depending on x. See the remarks following (A.1.1).) Then $\tilde{H}^{0,k}$ is given by $\psi\tilde{H}_P^0(\lambda)\varphi$, where

$$\begin{aligned} \tilde{H}_P^0(\lambda) &= \left(-\sum_i \partial^2/\partial r_i^2 + (\sigma + \rho + \delta)^2(\partial/\partial r_i) + \hat{D}^2 - \lambda \right)^{-1} \\ &= \int_0^{\infty} e^{-t\left(\sum_i \partial^2/\partial r_i^2 + (\sigma + \rho + \delta)^2(\partial/\partial r_i)\right)} e^{-t\hat{D}^2} e^{-t\lambda} dt . \end{aligned} \tag{3.2.7}$$

The error estimates of [26, §4.9] may be improved by our current assumption that the distance between support φ_k and support ψ is $O(j^{\varepsilon'})$ to show that

$(\Delta - \lambda)_{-1}\varphi_k - \tilde{H}_P^0(\lambda)\varphi_k$ is rapidly decreasing. (In fact, a simple maximum principle argument may be used to obtain much sharper bounds on the error.) The desired local parametrix with rapidly decreasing error is thus given by

$$\tilde{H}'^k = \sum_{i \leqq J} \tilde{H}_1^{i,k} + \tilde{H}^{0,k} . \tag{3.2.8}$$

We set $H_P^0(t) = (2\pi i)^{-1} \int_{C_t} e^{-t\lambda} \tilde{H}_P^0(\lambda) d\lambda$.

3.3 Reduction to maximal parabolics

To proceed further, we restrict out choice of exhaustion sequence $\{B_j\}_j$, or equivalently, $\{\partial B_j\}_j$. Let $\{P_i\}_i$ be a collection of representatives of Γ-conjugacy classes of maximal rational parabolic subgroups. For each $P_i = N_i A_i M_i$, let $\{\tilde{\omega}_{i,j}\}_j$ be an exhaustion of $\Gamma_{P_i} \backslash N_i M_i / K_i$ which satisfies:

(i) $[j, \infty) \times \tilde{\omega}_{i,j}$ can be identified isometrically with a subset F_{ij} of X_Γ.
(ii) The F_{ij} are pairwise disjoint for fixed j.
(iii) The set $\cup_i F_{ij}$ can be completed to a hypersurface ∂B_j which satisfies [26, (3.2.1. iv)] (is not too "wild").

Such exhaustions exist. (Use [26, 3.2] and apply a diagonal argument as $t_P = j$ varies.)

Recall that our local parametrix $\tilde{H}'^k \varphi_k$ depends on a choice of rational parabolic subgroup P associated to ($x \in$ the support of) φ_k. One of the fundamental simplifications in [26] is the reduction in [26, (5.1.19)] of the evaluation of the boundary integral in (3.2.3) (with B_j of the above type) to a computation involving only maximal parabolic subgroups. In particular, in the limit as $j \to \infty$, the contribution of $\tilde{H}'^k \varphi_k$ to the trace computation vanishes unless the parabolic subgroup associated to φ_k is maximal. Moreover, if we let F'_{ij} denote the subset of F_{ij} which is covered by φ_k's whose associated parabolic is maximal (and therefore necessarily P_i), then

$$\lim_{j \to \infty} \int_{e^{-4j/(1-\varepsilon)}}^{\infty} \left(\int_{\partial B_j} \operatorname{Tr} C(v) \tau D T_G(\alpha) e(t, x, x) dx - \sum_i \int_{F'_{ij}} \operatorname{Tr} C(v) \tau D T_G(\alpha) H(t, x, x) dx \right) dt = 0 .$$

These results do not hold for spaces which are not $\mathbb{Q}$-irreducible. The boundary contributions in the general Hermitian case take a form similar to products of the contributions from the irreducible case, but we will not prove this here. (See, however, [27b].)

The trace estimates and computations in [26, 4.9 and 5.1] which lead to the reduction to maximal parabolic subgroups require modifications if they are to be applied to the traces of the Hecke operators. The modifications required to estimate the trace for t small are given in the appendix. For large t, we must modify the elementary vanishing argument in [26, Lemma (4.9.4)]. Set $D_A = C(\partial/\partial r_i)\partial/\partial r_i + \hat{C}(\partial/\partial r_i)(\sigma + \rho + \delta)(\partial/\partial r_i)$. Let $e_n \in \mathbf{a}$. The lemma states that if P is not maximal, then $\operatorname{tr} C(e_n)\tau(\hat{D} + D_A) H_P^0(r, x^0; r, x^0) = 0$, where we use the pair (r, x^0) (or sometimes (r_x, x^0)) to denote a point $x \in W_P'(\omega, t) \cong A_t \times \tilde{\omega}$. In order to apply the corresponding analysis to the Hecke operators, we need to establish the vanishing of $\operatorname{tr} C(e_n)\tau(\hat{D} + D_A) T_G(\alpha) H_P^0(r, x^0; r, x^0)$, for r sufficiently large and P not maximal. Lemma (2.11) reduces this to the following.

(3.3.1) Lemma. *If P is not maximal,* $\operatorname{tr} C(e_n)\tau(\hat{D} + D_A)T_P(\alpha)H_P^0 = 0$.

Proof. Because $C(\partial/\partial r_i)$ commutes with D_n and $T_P(\alpha)$, the proof of [26, 4.9.4] implies that

$$\operatorname{tr} C(e_n)\tau(\hat{D} + D_A)T_P(\alpha)H_P^0 = \operatorname{tr} C(e_n)\tau \sum_i C(\partial/\partial r_i)\partial/\partial r_i T_P(\alpha)H_P^0 ,$$

and vanishes if $\dim \mathbf{a} > 2$. In order to prove the vanishing in the case $\dim \mathbf{a} = 2$, we need to exploit an argument introduced in [27a, 4]. Let $\{X_i\}$ be an orthonormal basis for $\mathbf{a} \oplus (\mathbf{p} \cap \mathbf{m})$. Let

$$Q = D_n + \sum_i \hat{C}(X_i)(\sigma(X_i) + \rho(X_i) + \delta(X_i)) . \tag{3.3.2}$$

Q anticommutes with τ and commutes with $C(e_n)C(\partial/\partial r_i)T_P(\alpha)\partial/\partial r_i H_P^0$. According to [20, Proposition (11.1)], $\operatorname{Kernel} Q = \{0\}$. Hence,

$$\begin{aligned}\operatorname{tr} C(e_n)\tau \sum_i C(\partial/\partial r_i)\{\partial/\partial r_i T_P(\alpha)H_P^0 &= \operatorname{tr} Q^{-1}QC(e_n)\tau \sum_i C(\partial/\partial r_i)\partial/\partial r_i T_P(\alpha)H_P^0 \\ &= -\operatorname{tr} Q^{-1}C(e_n)\tau \sum_i C(\partial/\partial r_i)\partial/\partial r_i T_P(\alpha)H_P^0 Q \\ &= -\operatorname{tr} C(e_n)\tau \sum_i C(\partial/\partial r_i)\partial/\partial r_i T_P(\alpha)H_P^0 ,\end{aligned}$$

and the trace vanishes. □

With Lemma (2.11) and (A.4.1), the analysis of [26] modified as above now extends to compute the traces of the Hecke operators in terms of maximal parabolic subgroups. In particular, if we let h_p^1 denote the Schwartz kernel of $\sum_k'(H_1^{1,k} + H_0^{1,k})\varphi_{k,j}$, where the sum runs over $\varphi_{k,j}$ with associated parabolic subgroup P, and let h_P^0 denote the Schwartz kernel of H_P^0, we have

(3.3.3) Proposition.

$$\begin{aligned}&\lim_{j\to\infty} \int_{e^{-4j/(1-\eta)}}^{\infty} \int_{\partial B_j} \operatorname{tr} C(v)\tau D(T_G(\alpha)\circ e)(t,x,x)dx\,dt \\ &= \lim_{j\to\infty}\lim_{r\to\infty} \sum_P \int_{e^{-4j/(1-\eta)}}^{e^{-3j}} \int_{(r,\omega_{j,P})} \operatorname{tr} C(v)\tau D(T_P(\alpha)\circ h_P^1)(t,r,x^0;r,x^0)dx^0\,dt \\ &\quad + \lim_{j\to\infty}\lim_{r\to\infty} \sum_P \int_{e^{-3j}}^{\infty} \int_{(r,\omega_{j,P})} \operatorname{tr} C(v)\tau D(T_P(\alpha)\circ h_P^0)(t,r,x^0;r,x^0)dx^0\,dt ,\end{aligned}$$

where $\{\omega_{j,P}\}_{j=1}^{\infty}$ is an exhaustion of $\Gamma_P\backslash NM/K_M$ by N-invariant compact sets, and the sum $\sum_P$ runs over the Γ-conjugacy classes of rational maximal parabolic subgroups. Here we have identified A_t with a subset of $\mathbb{R}$ in such a way that r acts on N_1 by e^{2r}. We have omitted the integral

$$\int_{e^{-3j}}^{\infty} \int_{(r,\omega_{j,P})} \operatorname{tr} C(v)\tau D(T_P(\alpha)\circ h_P^1(I - T_1))(t,r,x^0;r,x^0)dx^0\,dt$$

because its vanishing is an easy consequence of the explicit form of $h_P^1(I - T_1)$. In particular, for some $c > 0$ this kernel is $O(e^{-cte^{4j}} \leqq e^{-e^j})$ in this region.

(3.3.4) *Remark.* In the above expression, we have used dx^0 to denote an invariant measure on $\omega_{j,P}$. Frequently in the following it will be convenient to introduce invariant measures associated to various factors in the decomposition (4.2) below. When convenient we will denote these by the appropriate subscript (eg. dx_h for an invariant measure on X_h); otherwise, we will denote these by the unmarked dx if the domain of integration is clear. Of course, these measures are only determined up to a constant factor. In our situation, we do not need to fix these constants because the product of the Schwartz kernel and the measure (which is what we seek to compute) is independent of the choice of constant. The measures on the associated groups will be normalized so that the volume of the maximal compact subgroups is one.

4 Maximal parabolic subgroups

In order to exploit fully the reduction to maximal parabolic subgroups, in this section we introduce notation and recall a few facts about the structure of maximal parabolic subgroups of G when G/K is Hermitian.

Let $P = NAM$ be a maximal rational parabolic subgroup of G, $\mathbf{n}$ decomposes as $\mathbf{n} = \mathbf{u} \oplus \mathbf{v}$, with U the center of N, and the central series of (3.4) is given by

$$N_1 = U \text{ and } N_2 = N. \tag{4.1}$$

MA (respectively M) decomposes as the product of two commuting subgroups $G_h G_l$ (respectively $G_h {}^\circ G_l$) with finite intersection. $X_h \equiv G_h/K \cap G_h$ is a lower rank Hermitian symmetric space, and the symmetric space $X_l \equiv G_l/K \cap G_l$ may be identified with a nondegenerate self adjoint homogeneous cone in $\mathbf{u}$. Then X may be written as the product

$$X = N \times X_l \times X_h = A \times N \times {}^\circ X_l \times X_h, \tag{4.2}$$

where ${}^\circ X_l = {}^\circ G_l/K \cap {}^\circ G_l$.

If we denote the Cartan decomposition of $\mathbf{g}_l$ determined by $K_l \equiv K \cap G_l$ by $\mathbf{g}_l = \mathbf{p}_l \oplus \mathbf{k}_l$, then there exists a $\mathbf{k}_l$ equivariant isomorphism $J\colon \mathbf{p}_l \to \mathbf{u}$. Fix an orthonormal basis $\{W_i\}_{i=1}^{d_1}$ of $\mathbf{u}$, the corresponding (under J^{-1}) basis $\{X_i\}_{i=1}^{d_1}$ of $\mathbf{p}_l$, and an orthonormal basis $\{W_i\}_{i=1}^{d_1}$ of $\mathbf{u}$, the corresponding (under J^{-1}) basis $\{X_i\}_{i=1}^{d_1}$ of $\mathbf{p}_l$, and an orthonormal basis $\{V_j\}_{j=1}^{d_2}$ of $\mathbf{v}$.

Let $\Gamma_{W_i} = \Gamma^0_{W_i} + \Gamma^V_{W_i}$, where

$$\Gamma^0_{W_i} = 1/4 \sum_{j,k} C(W_j) C(X_k) \langle W_i, [W_j, X_k] \rangle \quad \text{and}$$

$$\Gamma^V_{W_i} = 1/8 \sum_{j<k} C(V_j) C(V_k) \langle W_i, [V_j, V_k] \rangle$$

(see [26, (4.3.8c) and (4.4.5)]. These are the projections of $\nabla_{W_i} - W_i$ (in the given frame) onto the Clifford algebra. Define $\Gamma_W \cdot \Gamma_W = \sum_i \Gamma_{W_i} \Gamma_{W_i}$, and $Z \cdot \Gamma_W = \sum_i Z_i W_i \Gamma_{W_i}$, for $Z = \sum Z_i W_i \in \mathbf{u}$.

(4.3) Definition. We call a subset ω' of ω *semi-local* if we may identify the Γ_P action on ω with the Γ_U action. We shall identify any semi-local set ω' as a subset of $\Gamma_U\backslash U \times V \times X_l \times X_h$ and write correspondingly $x \in W_P(\omega', R) \subset X_\Gamma$ as $x_u x_v x_l x_h$. Here V denotes $U\backslash N$.

5 The $H_1^1 + H_0^1$ term

Henceforth we assume (when working in a neighborhood associated to P) that we have chosen $C(\alpha)$ adapted to P (see 2.9). With frame fixed in a semi-local neighborhood as in §4, we may identify forms with vector valued functions. We then factor the pullback map associated to α_i^{-1} acting on these vector valued functions into a translation operator $x \to \alpha_i^{-1}x$ and an endomorphism denoted α_i^* of the coefficient bundle.

We isometrically identify U with $\mathbb{R}^{d_1}$ and use additive notation for the group operation whenever convenient. Let Λ denote the lattice in $\mathbb{R}^{d_1}$ corresponding to Γ_U. Let Λ^* denote the lattice dual to Λ. Given $g \in G$, $h \in U$, we denote $g^{-1}hg$ by h^g. For $\mu \in U^*$ we shall denote by $\mu \to \mu^g$ the representation contragredient to $h \to h^g$.

We now recall the form of h_P^1 constructed in (A.1). If x^0 and y^0 do not both lie in some semi-local subset, then

$$h_P^1(t, R, x^0; R', y^0) = 0.$$

Otherwise, we have (with $drdy^0(r)$ an invariant measure on $X \approx \mathbb{R} \times e'(P)$

$$(5.1) \qquad h_P^1(t, R, x^0; R', y^0)\, dy^0(R')$$

$$= e^{R'd_1} \operatorname{vol}(\Gamma_U\backslash U)^{-1} e^{-t\Delta_h}(x_h, y_h) e^{-t\Delta_\Gamma}(R, x_v x_l; R', y_v y_l)$$

$$\times \sum_{\mu \in \Lambda^*} e^{-te^{4R}|\mu^{x^0} - i\Gamma e^{-2R}|^2} e^{2\pi i \mu \cdot (x_u - y_u)}\, dy^0(R') + \text{lower mass terms},$$

where Δ_h is the Laplacian of X_h (with coefficients in E), Δ_E is the usual Laplacian of $\mathbb{R}^{d_1+d_2}$ (and a small subset of $\mathbb{R} \times V \times X_l$ has been identified via normal coordinates with a subset of $\mathbb{R}^{d_1+d_2}$), $|\mu^{x^0} - i\Gamma e^{-2R}|^2 = -e^{-4R}\Gamma_W \cdot \Gamma_W - 2\pi i e^{-2R}\mu^{x^0} \cdot \Gamma_W + |2\pi\mu^{x^0}|^2$, $\Gamma_W(\bar{y})$, and $\mu^y = \mu^{\bar{y}}$, for some $\tilde{y} \in G$ such that $y = \Gamma_P \tilde{y} K$. (We remark that the invariance of Γ_U under Γ_P makes the above expression independent of the choice of representative of $\Gamma_P \tilde{y}$—if the same representative is used for all μ. Using standard linear algebra, the K invariance follows from the fact that K acts orthogonally on $\mathbf{u}$.) Thus, given $x = (R, x^0) \in (R, \omega_{j,P})$ and $\alpha_i \in P$, we choose a representative (if one exists) $\beta \in \alpha_i^{-1}\Gamma_P$ such that x^0 and y^0 are in a semilocal subset of $\omega_{j,P}$, where y^0 is defined by $\beta x = y = (R', y^0)$. If no such representative β exists, then $\operatorname{tr} \alpha_i^* C(v)\tau D h_P^1(t, \alpha_i^{-1}x, x) = 0$. Otherwise, we are to compute

$$(5.2) \qquad \sum_{\mu \in \Lambda^*} \operatorname{tr} \beta^* C(v)\tau D_\rho\, e^{-t\Delta_h} e^{-t\Delta_E} e^{-t\Delta_\mu^x} T_\mu(\beta x, x)\, dx(R),$$

where we let T_μ denote the projection onto the space of forms which satisfy $f(ux) = e^{2\pi i \mu \cdot u} f(x)$, for $u \in U$. T_μ has kernel

$$T_\mu(R, x^0; r, y^0)\, dr\, dy(r) = e^{rd_1} \operatorname{vol}(\Gamma_U\backslash U)^{-1} e^{2\pi i \mu \cdot (x_u - y_u)}$$

Write $\beta = \beta_u\beta_v\beta_l\beta_h$ and $y = \beta x = \beta_u{}^{\beta}x_u\beta_v{}^{\beta}x_v\beta_l x_l\beta_h x_h$. Let $\beta_l = a\tilde{\beta}_l$, $a \in A$, $\tilde{\beta}_l \in M$, and let s be defined by $aWa^{-1} = e^{2s}W$, for $W \in \mathbf{u}$. Then (5.2) may be written (see (A.4.1) and [26, §5.2])

$$\text{(5.3)}\qquad \operatorname{tr}\beta^* C(v)\tau\, e^{-t\Delta_h}(\beta_h x_h, x_h)\, e^{-t\Delta_E}(R+s, y_v y_l; R, x_v x_l))$$

$$\times \sum_{\mu\in\Lambda^*}\sum_r C(W_r)(\Gamma_{W_r} + i2\pi e^{2(R+s)}\mu_r^{y^0})\, e^{-te^{4(R+s)}|\mu^{y^0} - i\Gamma e^{-2(R+s)}|^2}\, T_\mu(\beta_u{}^{\beta}x_u x_u)\, dx(R)$$

$+$ terms which (when integrated over $(R, \omega_{j,P})$) are rapidly decreasing as $R \to \infty$.

We are actually interested in

$$\text{(5.4)}\qquad \lim_{R\to\infty} \int_{(R,\omega_{J,P})} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \operatorname{tr}\beta^* C(v)\tau e^{-t\Delta_h}(\beta_h x_h, x_h)$$

$$\times e^{-t\Delta_E}(R+s, y_v y_l; R, x_v x_l) \sum_{\mu\in\Lambda^*} C(W_r)(\Gamma_{W_r} + i2\pi e^{2(R+s)}\mu_r^{y^0})$$

$$\times e^{-te^{4(R+s)}|\mu^{y^0} - i\Gamma e^{-2(R+s)}|^2}\, T_\mu(\beta_u{}^{\beta}x_u, x_u)\, dt dx(R).$$

As noted in the beginning of §A.3, (5.4) is rapidly decreasing unless, after a change of base point in $\omega_{j,P}$, $\beta = \beta_u\tilde{\beta}_l\beta_h$. We henceforth assume that β is of this form.

In order to compute (5.4), we exploit (4.2). $C(v)\tau$ may be decomposed as $\tau = \tau_h\tau_l\tau_n\tau_E$, where τ_h (respectively τ_l and τ_n) is the involution given by Clifford multiplication by the volume element of X_h (respectively ${}^0X_l$ and N). Let tr_h (respectively tr_l) denote the trace over $\Lambda^\cdot T^*X_h \otimes \mathscr{E}$ (respectively $\Lambda^\cdot T^*N^0X_l$). Similarly decompose β^* as $\beta^* = \beta_h^* \otimes \beta_l^*$ where β_h^* is a map on the fibers of $\Lambda^\cdot T^*X_h \otimes \mathscr{E}$ and β_l^* is a map of the fibers of $\Lambda^\cdot T^*N^0X_l$. From the fact that the trace of a tensor product is the product of the traces), we obtain the following lemma.

(5.5) Lemma. *Let ω' be a compact semilocal subset of $\omega_{J,P}$ of the form $\omega' = \mathcal{O}_h \times \omega''$, where $\mathcal{O}_h$ may be identified as a subset of X_h and ω'' may be identified as a subset of ${}^0X_l \times \Gamma_U \backslash N$. Let $I_h(\beta)$ be the measure*

$$I_h(\beta) := \lim_{\varepsilon\to 0} \operatorname{tr}_h \beta_h^* \tau_h e^{-\varepsilon\Delta_h}(\beta_h x_h, x_h)\, dx_h.$$

Then

$$\lim_{R\to\infty} \int_{(R,\omega')} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \operatorname{tr}\beta^* C(v)\tau e^{-t\Delta_h}(\beta_h x_h, x_h)\, e^{-t\Delta_E}(R, y_v y_l; R, x_v x_l))$$

$$\times \sum_{\mu\in\Lambda^*}\sum_r C(W_r)(\Gamma_{W_r} + i2\pi e^{2R}\mu_r^{y^0})\, e^{-te^{4R}|\mu^{y^0} - i\Gamma e^{-2R}|^2}\, T_\mu(\beta_u{}^{\beta}x_u, x_u)\, dt dx(R)$$

$$= \lim_{R\to\infty} \int_{\mathcal{O}_h I_h(\beta)} \int_{(R,\omega'')} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \operatorname{tr}_l \beta_l^* \tau_l\tau_n$$

$$\times e^{-t\Delta_E}(R, y_v y_l; R, x_v x_l) \sum_{\mu\in\Lambda^*}\sum_r C(W_r)(\Gamma_{W_r} + i2\pi e^{2R}\mu_r^{y^0})$$

$$\times e^{-te^{4R}|\mu^{y^0} - i\Gamma e^{-2R}|^2}\, T_\mu(\beta_u{}^{\beta}x_u, x_u)\, dt\, dx'(R).$$

where we define $dx'(R)$ by $dx(R) = dx_h dx'(R)$.

Proof. As shown in (A.3.11), $\mathrm{tr}_h \beta_h^* \tau_h e^{-t\Delta_h}(\beta_h x_h, x_h)$ is bounded in L^1_{loc} as $t \to 0$ and $\int_{\mathscr{C}_h} I_h(\beta)$ exists and is finite. Moreover,

$$\int_{\mathscr{C}_h} \mathrm{tr}_h \beta_h^* \tau_h e^{-t\Delta_h}(\beta_h x_h, x_h) dx_h = \int_{\mathscr{C}_h} I_h(\beta) + O(t^{1/2}) .$$

From (A.4.1), we know that

$$\int_{(R,\omega'')} \mathrm{tr}_l \beta_l^* \tau_l \tau_n e^{-t\Delta_E}(R y_v y_l; R, x_v x_l) \sum_{\mu \in \Lambda'} \sum_r C(W_r)(\Gamma_{W_r} + i2\pi e^{2R} \mu_r^{y^0})$$
$$\times e^{-te^{4R}|\mu^{y^0} - i\Gamma e^{-2R}|^2} T_\mu(\beta_u^\beta x_u, x_u) dx'(R) = O(t^{-5/4}).$$

Hence, if we replace $\int_{\mathscr{C}_h} \mathrm{tr}_h \beta_h^* \tau_h e^{-t\Delta_h}(\beta_h x_h, x_h) dx_h$ by $\int_{\mathscr{C}_h} I_h(\beta)$ then the error is $O(t^{-3/4})$ and its t-integral over $[e^{-4R/1-\varepsilon}, e^{-3R}]$ vanishes in the limit as $R \to \infty$. □

Henceforth ω' will denote a compact semilocal subset of $\omega_{j,P}$ of the form described in Lemma 5.5.

To simplify further, we must first integrate in the $\Gamma_U \backslash U$ fiber. Let M be a matrix such that $\Lambda = {}^t M^{-1} Z^{d_1}$. Set

$$\chi_\beta(\mu) = \mathrm{Vol}(\Gamma_U \backslash U)^{-1} e^{2\pi i \mu \cdot \beta_u} \int_{\Gamma_U \backslash U} e^{2\pi i \mu \cdot (\beta x_u - x_u)} dx_u$$

$$= e^{2\pi i \mu \cdot \beta_u} \int_{Z^{d_1} \backslash \mathbb{R}^{d_1}} e^{2\pi i M^{-1}(\mu^\beta - \mu) \cdot x} dx.$$

Fix an orthonormal basis of Z^{d_1}, and let μ_j (respectively μ_j^β) denote the corresponding jth coordinate of $M^{-1}\mu$ (respectively $M^{-1}\mu^\beta$). Then

$$\chi_\beta(\mu) = e^{2\pi i \mu \cdot \beta_u} \prod_{j=1}^{d_1} \frac{e^{2\pi i \mu_j^\beta} - 1}{2\pi i(\mu_j^\beta - \mu_j)} . \tag{5.6}$$

Here we are adopting the convention that any factor of the form 0/0 is taken to be one.

We return to the consideration of Hecke operators. Set $\Lambda(\beta) = \Lambda \cap \beta^{-1} \Lambda \beta$. Let $\{w_j\}_{j=0}^{c(\beta)-1}$ be a set of representatives for $\Lambda/\Lambda(\beta)$. Here we have set $c(\beta)$ equal to the number of elements of $\Lambda/\Lambda(\beta)$. Let $\bar{\beta} = \{(\beta w_j)^{-1}\}_{j=0}^{c(\beta)-1}$.

(5.7) Lemma. $\bar{\beta} \subset \Gamma_P \beta^{-1} \Gamma_P$ and represents distinct left Γ_P cosets.

Proof. It is clear that $w_j^{-1} \beta^{-1} \in \Gamma_P \beta^{-1} \Gamma_P$. Suppose that $w_j^{-1} \beta^{-1} = w_k^{-1} \beta^{-1} \gamma$ for some $\gamma \in \Gamma_P$, and some $j \neq k$. Then $\gamma = \beta w_k w_j^{-1} \beta^{-1}$. Hence, $\gamma \in \beta \Lambda \beta^{-1} \cap \Lambda$. This implies that $w_k w_j^{-1} = \beta^{-1} \gamma \in \Lambda(\beta)$. This contradicts the assumption that w_j and w_k are distinct mod $\Lambda(\beta)$.

We now further restrict the choice of $C_P(\beta)$ so that $\bar{\beta} \subset C_P(\beta)$. The preceding lemma shows that this is possible. Let $\chi_{\bar{\beta}}(\mu)$ denote the sum over $\theta^{-1} \in \bar{\beta}$ of $\chi_\beta(\mu)$. Then

$$\chi_{\bar{\beta}}(\mu) = e^{2\pi i \mu \cdot \beta_u} \sum_{k=0}^{c(\beta)-1} e^{2\pi i \mu \cdot \beta w_k} \prod_{j=1}^{d_1} \frac{e^{2\pi i \mu_j^\beta} - 1}{2\pi i(\mu_j^\beta - \mu_j)} \tag{5.8}$$

(5.9) Lemma. $\chi_{\bar\beta}(\mu) = 0$ if $\mu \neq \mu^\beta$, and $\chi_{\bar\beta}(\mu) = c(\beta)\, e^{2\pi i \mu \cdot \beta_u}$ if $\mu = \mu^\beta$.

Proof. It is easy to see that the function $\sum_{\beta \in \bar\beta} e^{2\pi i \mu \cdot \beta_u} e^{2\pi i \mu^\beta \cdot x_u}$ is Γ_U-invariant. Hence, $\sum_{\beta \in \bar\beta} e^{2\pi i \mu \cdot \beta_u} e^{2\pi i (\mu^\beta - \mu)\cdot x_u}$ is an eigenfunction of the Laplace operator on $\Gamma_U \backslash U$ which has nonzero eigenvalue unless $\mu^\beta = \mu$. Using the orthogonality of distinct eigenspaces, we see that $\chi_{\bar\beta}(\mu)$, which is a multiple of the inner product of this eigenfunction with one, is zero if $\mu^\beta \neq \mu$.

Define Λ_β^* to be the subgroup of Λ^* on which β acts trivially. Let d_1^β denote the rank of Λ_β^*. We shall use β subscripts to denote subspaces fixed by β. For example $\mathbf{u}_\beta$ denotes the subspace of $\mathbf{u}$ fixed by β. For each $\bar\beta \subset C_P$, we are left to study

$$\lim_{R\to\infty} \int_{\mathcal{C}_h} I_h(\beta) \int_{(R, U\backslash\omega'')} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} (4\pi t)^{-(d_1+d_2)/2} \operatorname{tr}_l \beta_l^* \tau_l \tau_n \tag{5.10}$$

$$\times (4\pi t)^{-(d_1+d_2)/2}\, e^{-t\Delta_E}(R, y_v y_l, R, x_v x_l) \sum_{\mu \in \Lambda_\beta^*} c(\beta)\, e^{2\pi i \mu \cdot \beta_u}$$

$$\times \sum_r C(W_r)(\Gamma_{W_r} + i2\pi e^{2R} \mu_r^{x^0})\, e^{-te^{4R}|\mu^{x^0} - i\Gamma e^{-2R}|^2}\, dt d\bar x(R).$$

As in (A.1.6), this expression vanishes in the complement of a neighborhood of the subset $F_{\beta,U}$ of $(R, U\backslash\omega'')$ fixed by β. Let d_2^β denote the dimension of $\mathbf{v}_\beta$. Then $2d_1^\beta + d_2^\beta$ is the dimension of $\mathbf{p}_{l\beta} \oplus \mathbf{n}_\beta$. Here we use the fact that β_l is elliptic and that $\mathbf{p}_l$ is K_l isomorphic to $\mathbf{u}$ to deduce that $\dim \mathbf{p}_{l\beta} = d_1^\beta$. Let $F^\perp$ denote the orthogonal complement of $\mathbf{p}_{l\beta} \oplus \mathbf{n}_\beta$ in $\mathbf{p}_l \oplus \mathbf{n}$. In a neighborhood of $F_{\beta,U}$, mass $\beta^* = 2d_1 - 2d_1^\beta + d_2 - d_2^\beta$, and the maximal mass terms in (5.10) are those in which β_l^* contributes all the Clifford factors from $F^\perp$ (and no others) and the Γ's contribute all the Clifford factors from $\mathbf{p}_{l\beta} \oplus \mathbf{n}_\beta$ (and no others). Write $\tau_l \tau_n = \tau_{l,\beta}\, \tau_{n,\beta}\, \tau^\perp$, where $\tau_{l,\beta}$ (respectively $\tau_{n,\beta}$) is Clifford multiplication by the volume form of $p_{l\beta}$ (respectively $\mathbf{n}_\beta$) and $\tau^\perp$ is Clifford multiplication by the volume form of the $F^\perp$ (with compatible choices of orientations). Set $\tau_\beta = \tau_{l,\beta}\, \tau_{n,\beta}$ and let tr_β denote the trace over $\Lambda^*(\mathbf{p}_{l\beta} \oplus \mathbf{n}_\beta)^*$. Then we may factor out the $\beta_l^*\, \tau^\perp$ contribution to the trace. Set $\sigma^\perp(\beta) = \operatorname{tr}_{\Lambda^*(F^\perp)^*} \beta_l^*\, \tau^\perp$. Let Γ^β denote the restriction of Γ to $\mathbf{p}_{l\beta}^* + \mathbf{n}_\beta^*$. With this notation, (5.10) reduces to

$$\lim_{R\to\infty} \int_{\mathcal{C}_h} I_h(\beta) \int_{(R, U\backslash\omega'')} \sigma^\perp(\beta) \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} (4\pi t)^{-(d_1+d_2)/2} \operatorname{tr}_\beta \tau_\beta \tag{5.11}$$

$$\times (4\pi t)^{-(d_1+d_2)/2}\, e^{-t\Delta_E}(R, y_v y_l, R, x_v x_l) \sum_{\mu \in \Lambda_\beta^*} c(\beta)\, e^{2\pi i \mu \cdot \beta_u}$$

$$\times \sum_r C(W_r)(\Gamma_{W_r}^\beta + i2\pi e^{2R} \mu_r^{x^0})\, e^{-te^{4R}|\mu^{x^0} - i\Gamma^\beta e^{-2R}|^2}\, dt d\bar x(R).$$

Arguing as in (5.5), we may now replace the term

$(4\pi t)^{-(d_1+d_2-d_1^\beta-d_2^\beta)/2}\, e^{-t\Delta_E}(R, y_v y_l, R, x_v x_l)$ in the above expression by its limiting measure $I_l(\beta) := \lim_{t\to 0} (4\pi t)^{-(d_1+d_2-d_1^\beta-d_2^\beta)/2}\, e^{-t\Delta_E}(R, y_v y_l, R, x_v x_l)\, d\bar x(R)$. This measure is clearly the current dual to the quotient by U of the fixed point set of $\beta_l \beta_v$ intersected with the level set $r = R$. (We must sum over all $\bar\beta \in \Gamma_P \beta^{-1} \Gamma_P$ before

we can define this measure globally.) Now (5.11) becomes

$$\lim_{R\to\infty} \int_{\mathcal{O}_h} I_h(\beta) \int_{(R,U\setminus\omega'')} \sigma^{\perp}(\beta)c(\beta) \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \mathrm{tr}_{\beta}\tau_{\beta}\,(4\pi t)^{-(d_1^{\beta}+d_2^{\beta})/2} \tag{5.12}$$

$$\times \sum_{\mu\in\Lambda_{\beta}^{*}} e^{2\pi i\mu\cdot\beta_u} \sum_{r} C(W_r)(\Gamma_{W_r}^{\beta} + i2\pi e^{2R}\mu_r^{x^0})\, e^{-te^{4R}|\mu^{x^0} - i\Gamma^{\beta}e^{-2R}|^2}\, dt I_l(\beta)$$

$$= \int_{\mathcal{O}_h} I_h \int_{N\setminus\omega''} \sigma^{\perp}(\beta)\kappa_{\beta}c(\beta) I_l(\beta) Z_P(x',\bar{\beta}), \tag{5.13}$$

where we have set

$$Z_P(x',\bar{\beta}) \equiv \lim_{R\to\infty} e^{-Rd_2^{\beta}} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \mathrm{tr}_{\beta}\tau_{\beta}\,(4\pi t)^{-(d_1^{\beta}+d_2^{\beta})/2} \tag{5.14}$$

$$\times \sum_{\mu\in\Lambda_{\beta}^{*}} \sum_{r} C(W_r)(\Gamma_{W_r}^{\beta} + i2\pi e^{2R}\mu_r^{x^0})\, e^{-te^{4R}|\mu^{x^0} - i\Gamma^{\beta}e^{-2R}|^2}\, e^{2\pi i\mu\cdot\beta_u}\, dt.$$

The factor $\kappa_{\beta}e^{-Rd_2^{\beta}}$ enters when we integrate along the fiber of $\omega'' \to N\setminus\omega''$. Here κ_{β} denotes $\mathrm{Vol}(\Gamma_{N_{\beta}}\setminus N_{\beta})/\mathrm{Vol}(\Gamma_{U_{\beta}}\setminus U_{\beta})$.

6 Zeta functions

Let $Z_P(g,\bar{\beta})$ denote the function on M defined by lifting $Z_P(x',\bar{\beta})$.

$$Z_P(g,\bar{\beta}) = \lim_{R\to\infty} \sum_{\beta\in\bar{\beta}} e^{-Rd_2^{\beta}} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \mathrm{tr}_{\beta}\tau_{\beta} \sum_{\mu\in\Lambda_{\beta}^{*}} \sum_{r} C(W_r)(\Gamma_{W_r}^{\beta} + i2\pi e^{2R}\mu_r^{g}) \tag{6.1}$$

$$\times e^{-t|\Gamma^{\beta}+i2\pi e^{2R}\mu_R|^2}(4\pi t)^{-(d_1^{\beta}+d_2^{\beta})/2} e^{2\pi i\mu\cdot\beta_u}\, dt.$$

The first goal of this section is to analyze $Z_P(g,\bar{\beta})$. We give in (6.17) a simple local formula for it.

The contribution of $Z_P(g,\bar{\beta})$ to the limit as $j\to\infty$ in (3.2.1) is easily seen from (5.13) to be given by

$$\tilde{Z}_P(\beta) \equiv \int_{\Gamma_M\setminus M} \sum_{\bar{\theta}\subset C_P(\beta)} I_h(\beta)I_l(\beta)\sigma^{\perp}(\beta)c(\beta)\kappa_{\beta}Z_P(m,\bar{\theta})dk(m). \tag{6.2}$$

The measure $\sum_{\theta\subset C_P(\beta)} I_h(\beta)I_l(\beta)$ is supported in the image of the coincidence locus $F_M = \{x\in(\Gamma_M\cap\beta_h\beta_l\Gamma_M\beta_l^{-1}\beta_h^{-1})\setminus M/K_M : \Gamma_M x = \Gamma_M\beta_h\beta_l x\}$, and we may re-express $\tilde{Z}_P(\beta)$ as an integral over the components F_M (but now multiplicities may be introduced – see (7.6)). Recall from §2 that each component of F_M is of the form $\Gamma_{M\mu}\setminus M_{\mu}/K_{M\mu}$ for some elliptic $\mu\in M$. Passing to a finite cover and introducing rational multiplicities, we may replace the integral over $\Gamma_{M\mu}\setminus M_{\mu}/K_{M\mu}$ by an integral over $\Gamma_{h\mu_h}\setminus G_{h\mu_h}/K_{h\mu_h}\times\Gamma_{l\mu_l}\setminus{}^{\circ}G_{l\mu_l}/K_{l\mu_l}$. The integral $\tilde{Z}_P(\beta)$ then reduces to a sum of integrals (with multiplicity) of the form

$$\int_{\Gamma_{h_\mu}\setminus G_{h_\mu}\setminus K_{h_\mu}} I_h(\mu)\times\sigma^{\perp}(\mu)\kappa_{\mu}c(\mu) \int_{\Gamma_{l_\mu}\setminus{}^{\circ}G_{l_\mu}} Z_p(g,\bar{\mu})dg.$$

The first integral factor is then the local data associated to a component of the coincidence locus of a Hecke operator on $\Gamma_h \backslash G_h / K_h$. The second goal of this section is to analyze

$$Z_P(\bar{\mu}) \equiv \int_{\Gamma_{l_\mu} \backslash {}^\circ G_{l_\mu}} Z_P(g, \bar{\mu}) dg.$$

In particular, we will discuss the relation of $Z_P(\bar{\beta})$ to Sato Shintani zeta functions.

Let m_β be a matrix such that $\Lambda_\beta^* = m_\beta Z^{d_1^\beta}$, and set $m = m_{\beta_u}$. We use the Poisson summation formula to rewrite $Z_P(g, \bar{\beta})$ as

(6.3)

$$\begin{aligned} Z_P(g, \bar{\beta}) = & \lim_{R \to \infty} e^{-Rd_2^\beta} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \mathrm{tr}_\beta \tau_\beta \sum_{k \in Z^{d_1^\beta}} \int_{\mathbb{R}^{d_1^\beta}} (4\pi t)^{-(d_1^\beta + d_2^\beta)/2} \sum_r C(W_r) \\ & \times (i2\pi e^{2R}(m_\beta x)_r^g + \Gamma_{W_r}^\beta) e^{-t|i\Gamma^\beta + 2\pi e^{2R}(m_\beta x)^g|^2} e^{2\pi i m_\beta x \cdot (\beta_u + {}^t m_\beta^{-1} k)} dx dt \\ = & \lim_{R \to \infty} e^{-R(2d_1^\beta + d_2^\beta)} \det m_\beta^{-1} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \mathrm{tr}_\beta \tau_\beta \sum_{k \in \Lambda_\beta} \int_{\mathbb{R}^{d_1^\beta}} \sum_r C(W_r) \\ & \times (i2\pi x_r + \Gamma_{W_r}^\beta) e^{-t|i\Gamma^\beta + 2\pi x|^2} e^{2\pi i e^{-2R} x^{g^{-1}} \cdot (\beta_u + k)} dx (4\pi t)^{-(d_1^\beta + d_2^\beta)/2} dt, \end{aligned}$$

where Λ_β denotes the subgroup of Λ dual to Λ_β^*. We may make a formal change of variables modulo commutator terms which vanish in the $R \to \infty$ limit to obtain

(6.4)

$$\begin{aligned} Z_P(g, \bar{\beta}) = & \lim_{R \to \infty} e^{-R(2d_1^\beta + d_2^\beta)} \det m_\beta^{-1} \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \mathrm{tr}_\beta \tau_\beta \sum_{k \in \Lambda_\beta + \beta_u} \int_{\mathbb{R}^{d_1^\beta}} \sum_r C(W_r) \\ & \times i2\pi x_r e^{-t|2\pi x|^2} e^{2\pi i e^{-R}(x - i\Gamma^\beta/2\pi) \cdot k^g} (4\pi t)^{-(d_1^\beta + d_2^\beta)/2} dx dt. \end{aligned}$$

Performing the x integration, we have $Z_P(g, \bar{\beta}) =$

$$\begin{aligned} (6.5) \quad & \lim_{R \to \infty} e^{-R(2d_1^\beta + d_2^\beta)} \det m_\beta^{-1} \sum_{k \in \Lambda_\beta + \beta_u} \mathrm{tr}_\beta \tau_\beta \sum_r e^{-2R} k_r^g C(W_r) e^{e^{-2R} \Gamma^\beta \cdot k^g} \\ & \times \int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} e^{-e^{-4R}|k^g|^2/4t} (4\pi t)^{-(2d_1^\beta + d_2^\beta)/2} dt/2t \\ = & \lim_{R \to \infty} e^{2R(d_1^\beta + d_2^\beta/2)} \det m_\beta^{-1} \sum_{k \in \Lambda_\beta + \beta_u} \mathrm{tr}_\beta \tau_\beta \sum_r e^{-2R} k_r^g C(W_r) e^{e^{-2R} \Gamma^\beta \cdot k^g} \\ & \times (\pi |k_g|^2)^{-(d_1^\beta + d_2^\beta/2)} (d_1^\beta + d_2^\beta/2 - 1)!/2 \\ = & \det m_\beta^{-1} \sum_{k \in \Lambda_\beta + \beta_u} \mathrm{tr}_\beta \tau_\beta \sum_r C(W_r) k_r^g (\Gamma^\beta \cdot k^g)^{d_1^\beta + d_2^\beta/2 - 1} (\pi |k^g|^2)^{-(d_1^\beta + d_2^\beta/2)} \end{aligned}$$

When $d_2^\beta \neq 0$, the sum is (6.5) is absolutely convergent. To make sense of this expression when $d_2^\beta = 0$, we rewrite expression (6.4) as

$$(6.6) \quad Z_P(g, \bar{\beta}) = \lim_{R \to \infty} e^{-2Rd_1^\beta} \det m_\beta^{-1} \sum_{k \in \Lambda + \beta_u} \mathrm{tr}_\beta \tau_\beta \sum_r e^{-2R} k_r^g C(W_r) e^{e^{-2R} \Gamma^\beta \cdot k_g}$$

$$\times \int_{e^{4R/L}}^{e^{4R/(l-\varepsilon)}} e^{-te^{-4R}|k^g|^2/4}(t/4\pi)^{d_1^\beta} dt/2t + \lim_{R\to\infty} \int_{e^{-4R}L}^{e^{-3R}} \mathrm{tr}_\beta \tau_\beta \sum_{\mu\in\Lambda_\beta^*} \sum_r C(W_r)(\Gamma_{W_r}^\beta + i2\pi e^{2R}\mu_r^g)$$

$$\times e^{2\pi i\mu\cdot\beta_u} e^{-t|\Gamma^\beta + i2\pi e^{2R}\mu^g|^2}(4\pi t)^{-d_1^\beta/2}\, dt$$

$$= \det m_\beta^{-1} \sum_{k\in\Lambda_\beta+\beta_u} \mathrm{tr}_\beta\tau_\beta \sum_r k_r^g C(W_r)(-\Gamma^\beta\cdot k^g)^{d_1^\beta-1} \int_{1/4L}^{\infty} e^{-t|k^g|^2}(t/\pi)^{d_1^\beta} dt/2t$$

$$+ \lim_{R\to\infty} e^{2R(d_1^\beta-1)} \int_L^\infty \mathrm{tr}_\beta\tau_\beta \sum_{\substack{\mu\in\Lambda_\beta^* \\ \mu\neq 0}} \sum_r C(W_r)(e^{-2R}\Gamma_{W_r}^\beta + i2\pi\mu_r^g)$$

$$\times e^{2\pi i\mu\cdot\beta_u} e^{-t|e^{-2R}\Gamma^\beta + i2\pi\mu^g|^2}(4\pi t)^{-d_1^\beta/2}\, dt$$

$$+ \lim_{R\to\infty} \int_{e^{-4R}L}^{e^{-3R}} \mathrm{tr}_\beta\tau_\beta \sum_r C(W_r)\Gamma_{W_r}^\beta e^{-t|\Gamma^\beta|^2}(4\pi t)^{-d_1^\beta/2}\, dt.$$

(6.7) Lemma. *When* $d_2^\beta = 0$,

$$\lim_{R\to\infty} \int_{e^{-4R}L}^{e^{-3R}} \mathrm{tr}_\beta\tau_\beta \sum_r C(W_r)\Gamma_{W_r}^\beta e^{-t|\Gamma^\beta|^2}(4\pi t)^{-d_1^\beta/2}\, dt = 0, \quad \textit{and}$$

$$\lim_{R\to\infty} e^{2R(d_1^\beta-1)} \int_L^\infty \mathrm{tr}_\beta\tau_\beta \sum_{\mu\in\Lambda_\beta^*} \sum_r C(W_r)(e^{-2R}\Gamma_{W_r}^\beta + i2\pi\mu_r^g)$$

$$\times e^{2\pi i\mu\cdot\beta_u} e^{-t|e^{-2R}\Gamma^\beta + i2\pi\mu^g|^2}(4\pi t)^{-d_1^\beta/2}\, dt$$

$$= \int_L^\infty \mathrm{tr}_\beta\tau_\beta \sum_{\substack{\mu\in\Lambda_\beta^* \\ \mu\neq 0}} e^{2\pi i\mu\cdot\beta_u} e^{-t|2\pi\mu^g|^2} \sum_r C(W_r) 2\pi i\mu_r^g$$

$$\times \sum_{b\geqq(d_1^\beta-1)/2}^{d_1^\beta-1} (-t)^b (4\pi t)^{-d_1^\beta/2}\, dt(\Gamma^\beta\cdot\Gamma^\beta)^{d_1^\beta-1-b}(\Gamma^\beta\cdot 2\pi i\mu^g)^{2b-d_1^\beta+1}/(d_1^\beta - 1 - b)!$$

$$\times (2b - d_1^\beta + 1)! = O(L^{-1}).$$

Proof. From [26, Proposition 4.4.9iii and Remark 4.4.11], we see that when $d_2^\beta = 0$.

$$\mathrm{mass}(C(\partial/\partial r)\sum_r C(W_r)\Gamma_{W_r}^\beta(\Gamma^\beta\cdot\Gamma^\beta) \leqq 2d_1^\beta - 1.$$

Hence, expanding the exponential, we find

$$\mathrm{tr}_\beta\tau_\beta \sum_r C(W_r)\Gamma_{W_r}^\beta e^{-t\Gamma^\beta\cdot\Gamma^\beta} = 0 \text{ for all } t.$$

This implies the vanishing of the first limit. The second result also follows from mass considerations. We compute:

$$\lim_{R\to\infty} e^{2R(d_1^\beta-1)} \int_L^\infty \mathrm{tr}_\beta \tau_\beta \sum_{\substack{\mu\in\Lambda_\beta^* \\ \mu\neq 0}} \sum_r C(W_r)(e^{-2R}\Gamma_{W_r}^\beta + i2\pi\mu_r^g)$$

$$\times e^{2\pi i\mu\cdot\beta_u} e^{-t|2\pi\mu^g|^2} \sum_{b=0}^\infty (-te^{-2R}\Gamma^\beta\cdot(e^{-2R}\Gamma^\beta + 2\pi i\mu^g))^b (4\pi t)^{-d_1^\beta/2}\, dt/b!$$

$$= \lim_{R\to\infty} \int_L^\infty \mathrm{tr}_\beta \tau_\beta \sum_{\substack{\mu\in\Lambda_\beta^* \\ \mu\neq 0}} e^{2\pi i\mu\cdot\beta_u} e^{-t|2\pi\mu^g|^2} \sum_r C(W_r)(e^{-2R}\Gamma_{W_r}^\beta + 2\pi i\mu_r^g)$$

$$\times \sum_{b=0}^\infty (-t)^b \sum_{c=0}^b \{e^{-2R(b+c+1-d_1^\beta)}(\Gamma^\beta\cdot\Gamma^\beta)^c(\Gamma^\beta\cdot 2\pi i\mu^g)^{b-c}/c!(b-c)!\}(4\pi t)^{-d_1^\beta/2}\, dt.$$

$$\mathrm{Mass}\{C(W_r)(e^{-2R}\Gamma_{W_r}^\beta + 2\pi i\mu_r^g)(\Gamma^\beta\cdot\Gamma^\beta)^c(\Gamma^\beta\cdot 2\pi i\mu^g)^{b-c}\} \leqq 1 + 2b + 2c.$$

Hence the above trace vanishes except for those b and c with $b + c \geqq d_1^\beta - 1$, and we are left with

$$\int_L^\infty \mathrm{tr}_\beta \tau_\beta \sum_{\substack{\mu\in\Lambda_\beta^* \\ \mu\neq 0}} e^{2\pi i\mu\cdot\beta_u} e^{-t|2\pi\mu^g|^2} \sum_r C(W_r) 2\pi i\mu_r^g (4\pi t)^{-d_1^\beta/2}$$

$$\times \sum_{b\geqq(d_1^\beta-1)/2}^{d_1^\beta-1} (-t)^b (\Gamma^\beta\cdot\Gamma^\beta)^{d_1^\beta-1-b}(\Gamma^\beta\cdot 2\pi i\mu^g)^{2b-d_1^\beta+1}/(d_1^\beta-1-b)!(2b-d_1^\beta+1)!\, dt.$$

(6.8) Corollary. *When $d_2^\beta = 0$, the limit defining $Z_P(g,\bar\beta)$ exists and is finite. Moreover,*

$$Z_P(g,\bar\beta) = \lim_{L\to\infty} \det m_\beta^{-1} \sum_{k\in\Lambda_\beta+\beta_u} \mathrm{tr}_\beta\tau_\beta \sum_r k_r^g C(W_r)(-\Gamma^\beta\cdot k^g)^{d_1^\beta-1}$$

$$\times \int_{L^{-1}}^\infty e^{-t|k^g|^2} (t/\pi)^{d_1^\beta}\, dt/2t.$$

(6.9) Definition. For $\mathrm{Re}(s) > 0$, set

$$Z_P(g,\bar\beta,s) = \det m_\beta^{-1} \sum_{k\in\Lambda_\beta+\beta_u} \mathrm{tr}_\beta\tau_\beta \sum_r k_r^g C(W_r)(-\Gamma^\beta\cdot k^g)^{d_1^\beta-1}$$

$$\times \int_0^\infty e^{-t|k^g|^2} (t/\pi)^{d_1^\beta+s}\, dt/2t.$$

(6.10) Proposition. *$Z_P(g,\bar\beta,s)$ can be analytically continued to an entire function, and $Z_P(g,\bar\beta,0) = Z_P(g,\bar\beta)$.*

Proof. The proof follows readily from the preceding analysis. Set

$$I(t) := \sum_{k\in\Lambda_\beta+\beta_u} \mathrm{tr}_\beta\tau_\beta \sum_r k_r^g C(W_r)(-\Gamma^\beta\cdot k^g)^{d_1^\beta-1} \int_{L^{-1}}^\infty e^{-t|k^g|^2/4} (t/4\pi)^{d_1^\beta}\, dt/(d_1^\beta-1)!2.$$

Applying the Poisson summation formula to $I(t)$, we obtain

$$\begin{aligned} I(t) &= \sum_{k \in \Lambda_\beta^*} \int_{\mathbb{R}^{d_1^\beta}} \mathrm{tr}_\beta \tau_\beta \sum_r (x+\beta_u)_r^g C(W_r)(-\Gamma^\beta \cdot (x+\beta_u)^g)^{d_1^\beta - 1} \\ &\quad \times e^{-t|(x+\beta_u)^g|^2/4} e^{2\pi i x \cdot k} t^{d_1^\beta - 1} dt \\ &= \sum_{k \in \Lambda_\beta^*} e^{-2\pi i \beta_u \cdot k} \int_{\mathbb{R}^{d_1^\beta}} \mathrm{tr}_\beta \tau_\beta \sum_r x_r C(W_r)(-\Gamma^\beta \cdot x)^{d_1^\beta - 1} e^{-t|x|^2/4} e^{2\pi i x \cdot k^g} dx t^{d_1^\beta} dt/t \\ &= \int_{\mathbb{R}^{d_1^\beta}} \mathrm{tr}_\beta \tau_\beta \sum_r x_r C(W_r)(-\Gamma^\beta \cdot x)^{d_1^\beta - 1} e^{-|x|^2/4} dx dt/2t + O(e^{-\varepsilon(g)/t}), \end{aligned}$$

for some positive number $\varepsilon(g)$ depending on g. We may express the above as

$$\begin{aligned} I &= \int_{\mathbb{R}^{d_1^\beta}} \mathrm{tr}_\beta \tau_\beta (-\Gamma^\beta \cdot x)^{d_1^\beta - 1} \sum_r C(W_r)(-\partial/\partial x_r) e^{-|x|^2} dx 4^{d_1^\beta} dt/2t + O(e^{-\varepsilon(g)/t}) \\ &= \int_{\mathbb{R}^{d_1^\beta}} \mathrm{tr}_\beta \tau_\beta \, 2^{-1}(d_1^\beta - 1) \sum_r C(W_r) \Gamma_r^\beta (-\Gamma^\beta \cdot x)^{d_1^\beta - 2} \\ &\quad \times e^{-|x|^2} dx 4^{d_1^\beta} dt/t + O(e^{-\varepsilon(g)/t}) = O(e^{-\varepsilon(g)/t}), \end{aligned}$$

because mass $\sum_r C(W_r)\Gamma_r^\beta(-\Gamma^\beta \cdot x)^{d_1^\beta - 2} \leqq 2d_1 - 3$, when $d_2^\beta = 0$. Thus we see that $t^s I$ is integrable for all s, and $Z_P(g, \bar{\beta}, s)$ is entire. Clearly, $Z_P(g, \bar{\beta}) = Z_P(g, \bar{\beta}, 0)$. □

(*6.11*) *Remark.* The preceding proposition addresses the pointwise convergence of $Z_p(g, \bar{\beta})$. The fact that $Z_p(g, \bar{\beta})$ is $L^1(\Gamma_M \backslash M)$ (in fact rapidly decreasing) can be shown by using partial Poisson summation as in the proof of (A.2.2). Equivalently, near a cusp of $\Gamma_M \backslash M$, one may compute the trace contribution with an approximation to $e^{-t\Delta}$ constructed using a finer central series corresponding to a smaller parabolic subgroup. This trace and the difference between the two approximations can be shown to be rapidly decreasing using (A.4.1).

The formula (6.5) is already much simpler than the one obtained in [26, §5]. We may now use the Jordan algebra structures to further simplify this expression. Via the isomorphism J of §3, we identify k^g with an element of $\mathfrak{p}_{l\beta}$. Hence, we may fix a maximal torus in $\mathfrak{p}_{l\beta}$ which contains k^g. Let $r(p, \theta)$ denote the $\mathbb{R}$-rank of the centralizer $G_{l\bar{\theta}}$ of θ_l in G_l, $\theta \in \bar{\theta}$. Using the correspondence between real split tori in $\mathfrak{p}_{l\beta}$ and sets of mutually orthogonal idempotents $\{\varepsilon_i\}_{i=1}^{r(P,\beta)}$ in $\mathfrak{u}_\beta$ (see [2, p. 90]), we see that we may write (as an element) of $\mathfrak{u}$)

$$k^g = c(P) \sum_{i=1}^{r(P,\beta)} k_i^g \varepsilon_i. \tag{6.12}$$

and

$$k^g \cdot \Gamma^\beta = c(P) \sum_{i=1}^{r(P,\beta)} k_i^g \Gamma_{\varepsilon_i}. \tag{6.13}$$

where $c(P) = \|\varepsilon_i\|^{-1}$. Here $\|\cdot\|^2$ denotes one half the metric on $\mathfrak{u}$ induced by the Killing form and the Cartan involution on G corresponding to K. The factor of one half enters from the explicit expression given in [9] for the metrics $\sigma_\beta(z)$ appearing in (2.5). Henceforth $\mathfrak{u}$ will be endowed with this metric.

The Γ_{ε_i} have particularly nice expressions. Fix coordinates γ_i corresponding to the ε_i as in [2, Chap. II, §3, Proposition 8]. Let $\omega_{i\beta}^v$ denote left Clifford multiplication by the Kahler form on the $\gamma_i/2$ weight space $\mathbf{v}_{i\beta}$, with respect to its natural complex structure and the above metric. (Here we follow the convention that the Kahler form on $\mathbb{C}$ is $dx \wedge dy$.) Similarly, let $\omega_{ij\beta}$ denote left Clifford multiplication by the Kahler form on $\mathbf{u}_{ij\beta} \oplus J^{-1}\mathbf{u}_{ij\beta}$, where $\mathbf{u}_{ij\beta}$ is the subspace of $\mathbf{u}$ with weight $(\gamma_i + \gamma_j)/2$. The metric determining this Kahler form is given by the given metric on $\mathbf{u}_{ij\beta}$ and the metric induced by the Killing form on $J^{-1}\mathbf{u}_{ij\beta}$. Then (see [21, p. 102, Lemma 3.2])

$$\Gamma_{\varepsilon_t}^{\beta} = 1/2\omega_{ii\beta} + 1/8 \sum_{j \neq i} \omega_{ij\beta} + 1/4\omega_{i\beta}^v, \text{ and} \tag{6.14}$$

$$k^g \cdot \Gamma^\beta = c(P)/2\{k_i^g(\omega_{ii\beta} + 4^{-1}\omega_{i\beta}^v) + 1/2 \sum_{j \neq i} \omega_{ij\beta}(k_i^g + k_j^g)\}.$$

In order to establish (6.14) one uses the correspondence for $x \in \mathbf{p}_l$ and $y \in \mathbf{u}$,

$$2[x, y] = Jx \cdot y \tag{6.15}$$

where $Jx \cdot y$ denotes Jordan multiplication of Jx and y. Clearly, then

(6.16)

$$\operatorname{tr}_\beta \tau_\beta \sum_{r=1}^{d_1} C(W_r) k_r^g (k^g \cdot \Gamma^\beta)^{(d_1^\beta + d_2^\beta/2 - 1)} = \operatorname{tr}_\beta \tau_\beta \sum_{r=1}^{d_1} C(W_r) k_r^g [c(P)\{k_i^g(\omega_{ii\beta} + 4^{-1}\omega_{i\beta}^v)/2 + 1/2 \sum_{j \neq i} \omega_{ij\beta}(k_i^g + k_j^g)\}]^{(d_1^\beta + d_2^\beta/2 - 1)}.$$

Let $l_1(P, \beta)$ denote the common dimension of the $\gamma_i + \gamma_j$, $i \neq j$ weight spaces. Let $l_2(P, \beta) = d_2^\beta / r(P, \beta)$ denote the common dimension of the r_i weight spaces. Then (6.16) equals

$$c(P)^{d_1^\beta - 1 + d_2^\beta/2^{d_1^\beta - d_2^\beta/2}} (d_1^\beta - 1 + d_2^\beta/2)! \prod_i (k_i^g)^{l + l_2(P,\beta)/2} \prod_{i<j} (k_i^g/2 + k_j^g/2)^{l_1(P,\beta)},$$

and we obtain the formula

$$Z_P(g, \bar{\beta}) = \det m_\beta^{-1} c(P)^{d_1^\beta - 1 + d_2^\beta/2^{-d_2^\beta - 1}} (d_1^\beta - 1 + d_2^\beta/2)! \times \sum_{k \in \Lambda + \beta_u} |k^g \pi^{1/2}|^{-2d_1^\beta - d_2^\beta} \prod_i (2k_i^g)^{1 + l_2(P,\beta)/2} \prod_{i<j} (k_i^g + k_j^g)^{l_1(P,\beta)}. \tag{6.17}$$

To obtain a coordinate independent expression, the factor

$$\prod_i (2k_i^g) \prod_{i<j} (k_i^g + k_j^g)^{l_1(P,\beta)}$$

may be viewed as the determinant of the endomorphism on u_β corresponding to Jordan multiplication by k^g.

We now briefly indicate the relation between $Z_P(\bar{\beta})$ and the zeta functions of Sato and Shintani associated with prehomogeneous vector spaces.

Let f denote the Schwartz function on $\mathbf{u}_\beta$ given by

$$f(x) = r(P,\beta)\det m_\beta^{-1}\pi^{-d_1^\beta - d_2^\beta/2}c(P)^{d_1^\beta - 1 + d_2^\beta/2^{2-d_2^\beta}} \times \prod_i (2x_i)^{1+l_2(P,\beta)/2}\prod_{i<j}(x_i+x_j)^{l_1(P,\beta)}e^{-|x|^2}. \tag{6.18}$$

Here we decompose x as in (6.7) with $x = c(P)\Sigma x_i\varepsilon_i$ and with the ε_i determined by x. We identify G_l with $\mathbb{R}^+ \times {}^\circ G_l$, where we normalize the isomorphism between $\mathbb{R}^+$ and the $\mathbb{Q}$-split torus in the center of G_l so that $s \in \mathbb{R}^+$ acts on x as

$$x^{sg} = s^2 x^g = \aleph(s)^{1/r(P,\beta)}x^g, \tag{6.19}$$

where $\aleph(g)$ denotes a rational character of $G_{l\beta}$. We may express $Z_P(g,\bar\beta)$ in terms of f as

$$Z_P(g,\bar\beta) = \int_0^\infty \sum_{x\in\Lambda_\beta+\beta_u} f(x^{sg})\aleph(s)^{(d_1^\beta - r(P,\beta)+d_2^\beta/2)/r(P,\beta)}d\aleph(s). \tag{6.20}$$

In fact, replacing $\aleph(s)^{2/r(P,\beta)}$ by t, this is the expression we obtained from the heat equation.

Let $\Gamma_{1\beta}$ denote $\Gamma \cap G_{l\beta}$. Then we may write

$$\begin{aligned} Z_P(\beta) &= \int_{\Gamma_{l\beta}\backslash {}^\circ G_{l\beta}} Z_P(g^0,\bar\beta)dg^0 \\ &= \int_{\Gamma_{l\beta}\backslash {}^\circ G_\beta}\int_0^\infty \sum_{k\in\Lambda_\beta+\beta_u} f(x^{sg^0})\aleph(s)^{(d_1^\beta - r(P,\beta)+d_2^\beta/2)/r(P,\beta)}d\aleph(s)dg^0, \\ &= \int_{\Gamma_{l\beta}\backslash G_{l\beta}} \sum_{k\in\Lambda_\beta+\beta_u} f(x^g)\aleph(s)^{(d_1^\beta + d_2^\beta/2)/r(P,\beta)}dg, \end{aligned}$$

where the measure dg is normalized so that

$$\int_{\Gamma_{l\beta}\backslash G_{l\beta}} F(g)dg = \int_{{}^\circ G_\beta\backslash G_\beta} |\aleph(\dot g)|^{-1}d\aleph(\dot g)\int_{\Gamma_{l\beta}\backslash {}^\circ G_{l\beta}} F(\dot g g^0)dg^0, \text{ for all } F\in L^1(\Gamma_{l\beta}\backslash G_{l\beta}).$$

(*6.22*) *Remark.* Observe that the integration along the time parameter in the heat equation is now reinterpreted as integration along the torus $A \subset G_l$.

We now recall briefly the basic definitions and results of [24] and [23] on zeta functions associated to prehomogeneous vector spaces. Let H be a reductive complex linear algebraic group and (φ, V) a finite dimensional rational representation of H on V. The triple (H,φ,V) is called a prehomogeneous vector space if there exists a proper algebraic subset S of V such that $V - S$ is a single H orbit. We will assume that S is $\mathbb{Q}$-irreducible. Then there exists a homogeneous polynomial B and a rational character $\tilde\aleph$ such that

$$S = \{x\in V \mid B(x) = 0\} \text{ and } B(\varphi(h)x) = \tilde\aleph(h)B(x).$$

Assume also that H and V have real structures such that $\varphi(H_\mathbb{R}) \subset GL(V_\mathbb{R})$. Let $V_\mathbb{R} - S = V_1 \cup \ldots \cup V_l$ be the decomposition of $V_\mathbb{R} - S$ into its connected components. Let Γ_H be an arithmetic subgroup of $H_\mathbb{R}$, and let $L' = L - S\cap L$ for some Γ_H-invariant lattice L in $V_\mathbb{R}$. Set $L_i = L \cap V_i$, and let $\Gamma_H\backslash L_i$ denote a set of

representatives of Γ_H orbits of L_i. Define

$$\phi_i(f, s) = \int_{V_i} f(x)|B(x)|^s dx,$$

and

$$Z(f, L, s) = \int_{\Gamma_H \backslash H_{\mathbb{R}}^+} \sum_{k \in L'} f(\varphi(g)k)\tilde{\aleph}(g)^s dg. \tag{6.23}$$

Here dx is a Euclidean measure and $H_{\mathbb{R}}^+$ denotes the connected component of the identity element of $H_{\mathbb{R}}$. Then there exist Dirichlet series $\xi_i(s, L)\,(1 \leqq i \leqq l)$ such that

$$Z(f, L, s) = \sum_{i=1}^{l} \xi_i(s, L)\phi_i(f, s - n/d), \tag{6.24}$$

where $n = \dim V$ and $d = \text{degree}\, B$. The Dirichlet series is of the form

$$\xi_i(s, L) = \sum_{x \in \Gamma_H \backslash L_i} \mu(x)|B(x)|^{-s},$$

where $\mu(x) = \text{Vol}\{(\Gamma_H)_x \backslash (H_{\mathbb{R}}^+)_x\}$ for a suitable choice of measure on $(\Gamma_H)_x \backslash (H_{\mathbb{R}}^+)_x$.

Let V^* be the vector space dual to V, L^* the lattice dual to L, and φ^* the representation contragredient to φ. Then (H, φ^*, V^*) is also a prehomogeneous vector space, and one may define the associated dual zeta functions $Z^*(f^*, L, s)$, Dirichlet series $\xi_i^*(s, L^*)$, and $\phi_i^*(f^*, s)$. Let $\hat{f}$ denote the Fourier transform of f. Then under certain additional technical assumptions (see [24, (2.6) and (2.13)] one has the functional equation

$$Z(f, L, s) = \text{vol}(L^* \backslash V_{\mathbb{R}}^*) Z^*(\hat{f}, L^*, n/d - s),$$

and related functional equations for the Dirichlet series and ϕ_i.

We now identify the correspondence between the $Z_P(\beta)$ and the Sato-Shintani zeta functions. Recall that U is a Q-simple Jordan algebra with identity e. Changing base points, we may assume that $\beta e \beta^{-1} = e$. It is then clear that U_β can be given the structure of a Jordan subalgebra, which need not be Q-simple even if U is. Let φ denote the adjoint representation of $G_{1\beta}$ on U_β. The identity component of $G_{1\beta}$ is isomorphic to the identity component of the structure group $G(U_\beta)$ of U_β (see [12, 1] for the definition of the structure group and its natural action on U and see [2, p. 247] for the isomorphism). This isomorphism preserves the actions on U_β. The triple $(G_{1\beta}, \varphi, U_\beta)$ has the structure of a (real) prehomogeneous vector space determined by its Jordan algebra structure. Given the above isomorphisms, it suffices to indicate, following [12], the prehomogeneous space structure associated to a Q-simple Jordan algebra and its structure group. The general case is readily derived from the simple case.

Let $\tilde{U}$ be a Q-simple Jordan algebra of dimension d and rank r. Let B denote the reduced norm of $\tilde{U}$. B is a $\mathbb{Q}$-irreducible homogeneous polynomial determined by the properties $B(e) = 1$ and

$$B(gx) = \det(g)^{r/d} B(x) = \tilde{\aleph}(g)B(x),$$

for $x \in \tilde{U}$ and g in the identity component $G(\tilde{U})^0$ of the structure group $G(\tilde{U})$ of $\tilde{U}$. Then the complexification $G(\tilde{U})_{\mathbb{C}}$ acts transitively on $\tilde{U}_{\mathbb{C}} - S$, where

$$S = \{x \in \tilde{U}_{\mathbb{C}} \colon B(x) = 0\}.$$

$\tilde{U}$-$S_{\mathbb{R}}$ is the set of invertible elements in $\tilde{U}$ and decomposes into the disjoint union of $r+1$ open $G(\tilde{U})^0$ orbits

$$\tilde{U} - S_{\mathbb{R}} = \bigcup_{i=0}^{\tilde{r}} \tilde{U}_i,$$

where (see [12]) when $\tilde{U}$ is simple, $\tilde{r} = r$ and $\tilde{U}_i$ is the set of elements of signature $(r-i, i)$. With this structure, we have for $U = \tilde{U}$,

$$B(\varphi(g)x) = \aleph(g)B(x), \text{ with } \aleph(g) \text{ as in (6.19).}$$

With these preliminaries, we have

$$Z_P(\bar{\beta}) = Z(f, \Lambda_\beta + \beta_u, (d_1^\beta + d_2^\beta/2)/r(P, \beta)), \tag{6.27}$$

with f as defined in (6.18) and $Z(f, \Lambda_\beta + \beta_u, s)$ is the zeta function associated to the prehomogeneous space $(G_{l\beta}, \varphi, U_\beta)$ as in (6.23). (Because B is a factor of f, we need not distinguish between the sum over L and L'.)

The choice of f that is forced upon us by the index computations leads to surprising simplifications in the computation of $\phi_i(f, s)$ which we shall now indicate in the case $U = \tilde{U}$ is simple.

Recall that $\phi_i(f, s) = \int_{U_i} f(x)|B(x)|^s\, dx$. We shall compute $\phi_i(f, s)$ in terms of the gamma function associated to the self adjoint homogeneous cone U_0. Let $H \subset U$ denote the hypersurface defined by $\prod_{i<j}(x^i + x^j) = 0$. Each $U_i - H$ decomposes further as the disjoint union of open sets:

$$U_i - H = \bigcup_{j=0}^{r_j} U_{ij}. \tag{6.28}$$

The sign of $B(x)$ restricted to U_i is given by $(-1)^i$. Let ε_{ij} denote the sign of $\prod_{i<j}(x^i + x^j)^{l_1}$ restricted to U_{ij}.

The fundamental step in our computation is to observe that $\prod_i (2x^i) \prod_{i<j}(x^i + x^i)^{l_1}$ is the determinant of the Jacobian of the transformation $x \to x \cdot x$, (where $x \cdot x$ now denotes the Jordan multiplication). Moreover, it is easy to check that $U_0 - y(H \cap U_i)$ is the image of U_{ij} under this map. This follows immediately from the facts

(i) $y: U_i \to U_0$ is onto, and
(ii) codimension $y(H) > 1$.

Let n_{ij} denote the number of elements in the fiber of $y: U_{ij} \to U_0 - H$. Thus

$$\begin{aligned} \phi_i(f, s) &= \int_{U_i} f(x)|B(x)|^s\, dx = \sum_j \int_{U_{ij}} f(x)|B(x)|^s\, dx \\ &= r(-1)^{i + il_2/2} \det m^{-1} \pi^{-d_1 - d_2/2} c(P)^{d_1 - 1 + d_2} 2^{2 - d_2} \\ &\quad \times \sum_j \varepsilon_{ij} n_{ij} \int_{U_0} |B(x)|^{s/2 + l_2(P)/4} e^{-tr(x)}\, dx \end{aligned} \tag{6.29}$$

$$= r(-1)^{i+il_2/2} \det m^{-1} \pi^{-d_1-d_2/2} c(P)^{d_1-1+d_2/2} 2^{2-d_2}$$

$$\times \sum_j \varepsilon_{ij} n_{ij} \Gamma_{U_0}(s/2 + l_2/4 + d_1/r),$$

where the gamma function $\Gamma_{U_0}(s)$ associated to the cone U_0 is defined to be

$$\Gamma_{U_0}(s) = \int_{U_0} B(x)^{s-d_1/r} e^{-tr(x)} dx,$$

and $\mathrm{tr}(x)$ denotes the (regular) Jordan algebra trace (see [22, (1.2)]). In particular, we have

(6.30) $$\phi_i(f, d_2/2r) = r(-1)^{i+il_2/2} \det m^{-1} c(P)^{d_1-1+d_2/2} 2^{2-d_2}$$

$$\times \sum_j \varepsilon_{ij} n_{ij} \pi^{-d_1-d_2/2} \Gamma_{U_0}(d_2/2r + d_1/r).$$

Here we have used $l_2 = d_2/r$.

As the following lemma shows, the expression (6.30) is readily computable.

(6.31) Lemma. (See e.g. [12, (3)]).

$$\Gamma_{U_0}(s) = (2\pi)^{(d_1-r)/2} \prod_{j=1}^{r} \Gamma(s - (d_1 - r)(j-1)/r(r-1)),$$

where $\Gamma(s)$ denotes the usual Euler's gamma function.

When β = the identity, one can use the integrality of the index and the rationality of the volume of arithmetic varieties to investigate the rationality of certain linear combinations of special values of the Dirichlet series associated to the Sato-Shintani zeta functions. (See also [22]). In fact, given the expression (6.30) for $\phi_i(f, s)$, it seems most appropriate to consider special values of the Dirichlet series ξ_i^* associated to the dual lattice. We first recall the functional equation for $Z(f, L, s)$ in our context ($V = \tilde{U}$ is simple, $\beta = 1$).

$$Z(f, L, s) = \det m Z^*(\hat{f}, L^*, d_1/r - s),$$

where, as in (6.24), $Z^*(\hat{f}, L^*, s)$ can be decomposed as

(6.32) $$Z^*(\hat{f}, L^*, s) = \sum_i \xi_i^*(s, L^*) \phi_i^*(\hat{f}, s - d_1/r).$$

The $\phi_i^*(\hat{f}, s)$ satisfy the functional equation (see [12])

(6.33) $$\phi_i^*(\hat{f}, -s) = (2\pi)^{sr-d_1} e^{\pi i(sr-d_1)/2} \Gamma_{U_0} \sum_j u_{ij}(d_1/r - s) \phi_j(f, s - d_1/r),$$

where the $u_{ij}(s)$ are polynomials in $e^{-i\pi s}$, which have been computed in [12] for the simple Jordan algebras. In particular, for $s = d_1/r + d_2/2r$, we obtain

$$\phi_i^*(\hat{f}, -d_1/r - d_2/r) = (2\pi)^{d_2/2} e^{\pi i d_2/4} \Gamma_{U_0}(-d_2/2r) \Sigma u_{ij}(-d_2/2r) \phi_j(f, d_2/r).$$

Inserting the value for $\phi_i(f, d_2/r)$ obtained in (6.30), we obtain,

$$\phi_k^*(\hat{f}, -d_1/r - d_2/r) \tag{6.34}$$
$$= e^{\pi i d_2/4}\, \Gamma_{U_0}(-d_2/2r)\, \Sigma u_{ki}(-d_2/2r)\, r(-1)^{i+il_2/2} \det m^{-1}\, c(P)^{d_1-1+d_2/2}$$
$$\times 4 \sum_j \varepsilon_{ij} n_{ij}\, \pi^{-d_1}\, \Gamma_{U_0}(d_2/2r + d_1/r).$$

We will henceforth work mod $\mathbb{Q}^*$. We will write $a \tilde{\mathbb{Q}} b$ when $a/b \in \mathbb{Q}^*$. From [22, §1] one has the relation

$$\Gamma_{U_0}(-d_2/2r)\, \Gamma_{U_0}(d_2/2r + d_1/r) \tilde{\mathbb{Q}}$$
$$(\pi i)^{d_1}\, e^{-\pi i d_1 d_2/2r}\, (e^{\pi i d_1 d_2/r^2} - 1)^{-r},$$

if $2(d_1 - r)/r(r-1)$ is even, and

$$\Gamma_{U_0}(-d_2/2r)\, \Gamma_{U_0}(d_2/2r + d_1/r) \tilde{\mathbb{Q}}$$
$$(\pi i)^{d_1}\, e^{-\pi i d_1 d_2/2r}\, (e^{\pi i d_1 d_2/r^2} + 1)^{-r/2}\, (e^{\pi i d_1 d_2/r^2} - 1)^{-(r+1)/2},$$

if $2(d_1 - r)/r(r-1)$ is odd.

When $d_1 d_2/2r$ and $d_1 d_2/r^2 \in Z$, as will be the case in the examples considered below, (see, [21, pp. 115–116])

$$\Gamma_{U_0}(-d_2/2r)\, \Gamma_{U_0}(d_2/2r + d_1/r) \tilde{\mathbb{Q}} (\pi i)^{d_1},$$

and

$$\phi_k^*(\hat{f}, -d_1/r - d_2/r) \tilde{\mathbb{Q}} \tag{6.35}$$
$$\sum u_{ki}(-d_2/2r)\, (-1)^{i+il_2/2} \sum_j \varepsilon_{ij} n_{ij} \det m^{-1}\, c(P)^{d_1-1+d_2/2}.$$

Inserting this into (6.32) we obtain

$$Z(f, L, d_1/r + d_2/2r) = \det m Z^*(\hat{f}, L^*, -d_2/2r) \tilde{\mathbb{Q}} \tag{6.36}$$
$$\sum_k \xi_k^*(-d_2/r, L^*) \sum_i u_{ki}(-d_2/2r)(-1)^{i+il_2/2} \sum_j \varepsilon_{ij} n_{ij} c(P)^{d_1-1+d_2/2}.$$

When $r = 1$, we merely obtain special values of the Riemann zeta function from the above. As an illustration of what sort of data we obtain, we now consider the case $r = 2$ arising in those groups for which the necessary technical conditions for the Sato–Shintani theory have been previously worked out in [24]. These include:

(6.37) (i) $G = \mathrm{SO}(p, 2)$, $G_l = \mathbb{R}^+ \times \mathrm{SO}(p-1, 1)$,

(ii) $G = \mathrm{SU}(p, q)$, $G_l = \{g \in Gl(2, \mathbb{C}) : \det g \in \mathbb{R}^+\}$, $2 \leqq \min\{p, q\}$.

Let Herm(2) denote the space of Hermitian 2×2 matrices. Then the representations that arise in the above cases are:

(6.38) (i) $V_{\mathbb{R}} = \mathbb{R}^p$, $\varphi(g)v = gv$ (the standard representation),

(ii) $V_{\mathbb{R}} = \mathrm{Herm}(2)$, $\varphi(g)v = gv^t\bar{g}$.

The corresponding polynomials and characters are given by:

(6.39) (i) $B(x) = {}^t xSx$, where S is the diagonal $p \times p$ matrix with entries

$$(1, -1, \ldots, -1), \aleph(g) = \frac{-1}{(p-2)} \operatorname{tr}{}^t gSg,$$

(ii) $B(x) = \det(x)$, $\aleph(g) = \det(g)^2$.

We now tabulate the constants that arise in (6.36) for the groups considered above. In each of these cases, $U - S_{\mathbb{R}} = U_0 \cup U_1 \cup U_2$, with $U_0 \cap H = \phi = U_2 \cap H$, and $U_1 - U_1 \cap H = U_{11} \cup U_{12}$. In all cases $n_{ij} = 1$. Moreover, from [21, pp. 115–116], one obtains

(6.40) (i) $G_l = \mathbb{R}^+ \times \mathrm{SO}(2p-1, 1), l_1 = 2p-2, \varepsilon_{ij} = 1, d_2 = 0$.

(ii) $G_l = \{g \in Gl(2, \mathbb{C}) : \det g \in \mathbb{R}^+\}, l_1 = 2, \varepsilon_{ij} = 1$.

We have

(6.41) $Z(f, L, d_1/2 + d_2/2r)_{\tilde{\mathbb{Q}}}$

$$\sum_k \xi_k^*(-d_2/2, L^*)(u_{k0}(-d_2/4) - 2u_{kl}(-d_2/4)(-1)^{d_2/4} + u_{k2}(-d_2/4))\, c(P)^{d_1 - 1 + d_2/2}.$$

In order to analyze

$$U_k(s) = u_{k0}(s) - (-1)^{d_2/4}\, 2u_{k1}(s) + u_{k2}(s),$$

we use the explicit formula for $u_{ij}(s)$ given in [12, (23)]:

(6.42) $u_{\mu 0}(s) - (-1)^{d_2/4}\, 2u_{\mu 1}(s) + u_{\mu 2}(s)$

$$= e^{-i\pi s} \sum_{\varepsilon \in E_\mu} e^{\pi i l_1(\varepsilon_1 - \varepsilon_2)/4} (e^{\pi i(\varepsilon_1 + \varepsilon_2)s} + e^{-\pi i(\varepsilon_1 + \varepsilon_2)s} + (-1)^{1 + d_2/4}\, 2e^{\pi i(\varepsilon_1 - \varepsilon_2)s}),$$

where $E_0 = \{(1, 1)\}, E_1 = \{(1, -1), (-1, 1)\}$, and $E_2 = \{(-1, -1)\}$.

Hence

$$u_{00}(s) - (-1)^{d_2/4}\, 2u_{01}(s) + u_{02}(s) = u_{20}(s) - (-1)^{d_2/4}\, 2u_{21}(s) + u_{22}(s)$$
$$= e^{-i\pi s}(e^{\pi i 2s} + e^{-\pi i 2s} - (-1)^{d_2/4} 2),$$

and
$$u_{10}(s) - (-1)^{d_2/4}\, 2u_{11}(s) + u_{12}(s)$$
$$= e^{-i\pi s}\{e^{-\pi i l_1/2}(2 - (-1)^{d_2/4}\, 2e^{-2\pi i s}) + e^{\pi i l_1/2}(2 - (-1)^{d_2/4}\, 2e^{2\pi i s})\}.$$

All of the poles of ξ_i^* are simple (see [24]). $U_0(s)/s$, and $U_2(s)/s$ vanish at $s = 0$. Hence, we find for the rank two cases under consideration:

(6.43) Proposition. *$Z(f, L, d_1/2) = 0$, and the zeta function associated to the minimal boundary components make no contribution to the index computation for the groups* $\mathrm{SO}(2p, 2)$ *and* $\mathrm{SU}(2, 2)$ *with the standard rational structures.*

(For $\mathrm{SU}(p, q) p \neq q$, d_2 is never zero. For all the spaces under consideration, d_2 can vanish only for minimal boundary components.)

For $SU(p, q)$, $d_2/4 = r(p + q - 2r)/2 \in Z$, when $r = 2$. Hence

$$U_0(d_2/4) = U_2(d_2/4) = 4, \text{ and } U_1(d_2/4) = 8, \text{ if } d_2/4 \text{ is odd}.$$

$$U_k(d_2/4) = U_k'(d_2/4) = 0, \text{ if } d_2/4 \text{ is even}.$$

(6.44) $$Z(f, L, d_1/2 + d_2/4) = 0, \text{ if } d_2/4 \text{ is even, and}$$

$$Z(f, L, d_1/2 + d_2/4)_{\tilde{\mathbb{Q}}}$$

$$4(\xi_0^*(-d_2/2, L^*) + 2\xi_1^*(-d_2/2, L^*) + \xi_2^*(-d_2/2, L^*)\, c(P)^{2d_1 - 2 + d_2},$$

if $d_2/4$ is odd. More precisely, we have for $d_2/4$ odd and $G = SU(p, q)$,

(6.45) $$Z(f, L, d_1/2 + d_2/4) = \Gamma_{U_0}(-d_2/4) 2^5 \pi^{-d_1} \Gamma_{U_0}(d_2/4 + d_1/2)$$

$$\times (\xi_0^*(-d_2/2, L^*) + 2\xi_1^*(-d_2/2, L^*) + \xi_2^*(-d_2/2, L^*)\, c(P)^{d_1 - 1 + d_2/2}.$$

For $SU(2p, 2)$, $d_2/4$ is always even for the minimal boundary component, hence we have the following.

(6.46) Proposition. *The zeta function associated to the minimal boundary component of* $SU(2p, 2)$ *vanishes at* $d_1/2 + d_2/4$. *In particular, the zeta function associated to the minimal boundary components make no contribution to the* L^2*-signature computation for* $SU(2p, 2)$ *with the standard rational structure.*

In order to obtain rationality results for combinations of special values of zeta functions, one may consider groups of the form $G = R_{k/\mathbb{Q}} G'$, where $k \neq \mathbb{Q}$ is a totally real algebraic number field and $R_{k/\mathbb{Q}}$ is the restriction of scalars functor. In this case, [27, Theorem 5.1] implies that the eta invariant terms arising in the L^2-signature theorem of [26] (see also Theorem 7.10 below) vanish. The eta invariant terms and the combinations of special values of zeta functions appearing in the index formula are the only terms which are not obviously rational. Hence, the vanishing of the eta invariants implies the rationality of the combinations of special values of zeta functions.

7 The H^0 contribution

In order to describe the form of H_p^0 corresponding to a maximal parabolic, we must first introduce a plethora of notation (from [27]) relating to maximal parabolics. Let $P = NAM$ be a maximal rational parabolic subgroup of G, and recall the operators d_n, D_n, D_Z, and $\hat{D}$ defined in §3. We may decompose $\Lambda^{\cdot}\mathbf{n}^* \otimes E$ into eigenspace of D_n^2: $\Lambda^{\cdot}\mathbf{n}^* \otimes E = \otimes_\mu E_\mu$, where $D_\mu^2 \varphi = \mu\varphi$ for all $\varphi \in E_\mu$. Let $\mathscr{E}_\mu$ denote the associated flat vector bundle over $\hat{e}(P)$. For $\mu \neq 0$, we may write $E_\mu = E_\mu^+ \oplus E_\mu^-$, where $E_\mu^\pm$ are the $\pm \mu^{1/2}$ eigenspaces of $\tau_n \tau_E D_n$. Let $\mathscr{E}_\mu = \mathscr{E}_\mu^+ \oplus \mathscr{E}_\mu^-$ be the corresponding decomposition of $\mathscr{E}_\mu$. Define the involution τ_μ on E_μ by letting $E_\mu^\pm$ be the ± 1 eigenspaces of τ_μ. Let D_μ^+ denote the corresponding signature operator on $\hat{e}(P)$ with coefficients in E_μ, and let $\Delta^\mu = D_\mu^2$.

With these notations we decompose the operator $H_P^0(t)$ as

$$H_P^0(t) = \sum_\mu e^{t\partial^2/\partial r^2}\, e^{-t(\Delta^\mu + \mu + w^2)},$$

where $w = (\sigma + \rho + \delta)(\partial/\partial r)$. Intertwining arguments allow us to omit the e^{-tw^2} factor. Hence, the contribution of H_p^0 to the computation of $L(\alpha, \tau, E)$ in (3.3.3) is given by

$$(7.1) \qquad \sum_j \sum_{b \in C_P(\beta_j)} \sum_{\mu} \int_{\hat{e}(P)} \int_0^{\infty} \operatorname{tr} C(v) \tau D_\rho b^* e^{t(\partial^2/\partial r^2)} e^{-t(\Delta^\mu + \mu)}(x, bx)\, dt\, dx .$$

Fix some $\beta = \beta_j$. Let $\{b_1, \ldots, b_l\} = C_P(\beta)$, and set $b_i = n_i m_i a$. Write $x = n' m' r K_M$, and let $s = \log a^\delta$. Using the notation of §5, we write $b_i x = (r + s, n_i n'^{m_i a} m_i m' K_M)$. The results of [26, Lemma 4.9.4] and the proof of [27, Proposition 4.12] imply that D_ρ may be replaced by D_n in (7.1); thus, the summand in (7.1) corresponding to β equals

$$(7.2) \qquad \sum_i \sum_{\mu} \int_{\hat{e}(P)} \int_0^{\infty} e^{-s^2/4t} (4\pi t)^{-1/2} \operatorname{tr} b_i^* \tau_\mu \tau_h \mu^{1/2} e^{-t(\Delta^\mu + \mu)}(m', m_i m')\, dt\, dm' .$$

Moreover, from [27, Lemmas 5.1 and 5.2] (whose proofs do not depend on evaluation along the diagonal) we may deduce that (7.2) vanishes unless

(7.3) (i) G is simple and
(ii) $\hat{e}(P)$ is an equal rank maximal boundary component for some

Satake compactification of X_Γ.

In order to proceed further, we now assume that $\hat{e}(P)$ satisfies (7.3ii) and introduce the Hecke operators induced on $L_2^\cdot(\Gamma_M \backslash M/K_M, \mathscr{E}_\mu)$ by the image of β in Γ_M. Let $\beta \in \bar{C}_P(\alpha)$, and write $C_P(\beta) = \{b_1, \ldots, b_l\}$ as above. $C_P(\beta)$ induces an endomorphism $T_P(\beta)$ of the complex $C^\cdot(X, \mathscr{E}_\mu)^{\Gamma_P}$ of continuous left Γ_P-invariant $\mathscr{E}_\mu$-valued forms on X, given by

$$(7.4) \qquad T_P(\beta) f(x) = \sum_i f(b_i^{-1} x) .$$

The space $C^\cdot(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$ of continuous $\mathscr{E}_\mu$-valued forms on $\Gamma_M \backslash X_M$ may be identified with the subset of $C^\cdot(X, \mathscr{E}_\mu)^{\Gamma_P}$ consisting of forms left invariant under NA. Thus we may restrict the action of $T_P(\beta)$ to this space.

(7.5) Lemma. $T_P(\beta): C^\cdot(\Gamma_M \backslash X_M, \mathscr{E}_\mu) \to C^\cdot(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$.

Proof. Let $f \in C^\cdot(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$, and identify f with an element of $C^\cdot(X, \mathscr{E}_\mu)^{\Gamma_P}$ left invariant under NA as above. We have

$$T_P(\beta) f(nax) = \sum_i f(b_i^{-1} nax) = \sum_i f(b_i^{-1} nb_i a a^{-1} b_i^{-1} ax) = \sum_i f(b_i^{-1} x),$$

since $b_i^{-1} nb_i \in N$, and $a^{-1} b_i^{-1} a = n' b_i^{-1}$ for some $n' \in N$. □

The operator $T_P(\beta)$ defined above extends in an obvious way to an endomorphism $\hat{T}_P(\beta)$ of $L_2^\cdot(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$. At this point we would like to deduce directly that $\hat{T}_P(\beta)$ is a linear combination of Hecke operators on $L_2^\cdot(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$. When Γ is an appropriate congruence subgroup of the Siegel modular group and X is the Siegel upper half space, this is known to be true (see [1, 4.2.4]). Unfortunately, a direct proof escapes us, and we must give a somewhat clumsy argument to establish the following proposition.

(7.6) Proposition. *If $\beta = nam$, then $\hat{T}_P(\beta) = q_P(\beta)\, T_M(m)$, where $T_M(m)$ is the Hecke operator on $L^{\cdot}_2(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$ associated to m and $q_P(\beta)$ is a positive integer.*

Proof. Identifying forms on $\Gamma_M \backslash X_M$ with left Γ_M invariant forms on X_M, we have

$$\hat{T}_P(\beta) f(x) = \sum_i f(m_i^{-1} x).$$

Clearly $\Gamma_P \beta \Gamma_P = \bigcup_i b_i \Gamma_P$ implies that $\Gamma_M m \Gamma_M = \bigcup_i m_i \Gamma_M$. In general, the latter union is not disjoint. We pick a subset $\{m_{i_1}, \ldots, m_{i_r}\}$ of $\{m_1, \ldots, m_l\}$ such that $\Gamma_M m \Gamma_M = \bigcup_{i_k} m_{i_k} \Gamma_M$ is a disjoint union. Then

$$\hat{T}_P(\beta) f(x) = \sum_{i_k} q_{i_k} f(m_{i_k}^{-1} x),$$

where q_{i_k} equals the number of m_j such that $m_j \Gamma_M = m_{i_k} \Gamma_M$. Suppose that $q_{i_1} \neq q_{i_2}$. Fix x and let f be a left Γ_P invariant form such that $f(m_{i_j}^{-1} x) = 1$ and $f(m_{i_j}^{-1} x) = 0$, for $j > 1$. Let $\gamma \in \Gamma_M$ such that $\gamma m_{i_1} \Gamma_M = m_{i_2} \Gamma_M$. Then

$$\hat{T}_P(\beta) f(x) = \sum_{i_k} q_{i_k} f(m_{i_k}^{-1} x) = q_{i_1}, \text{ but}$$

$$\hat{T}_P(\beta) f(\gamma x) = \sum_{i_k} q_{i_k} f(m_{i_k}^{-1} \gamma x) = q_{i_2}.$$

Hence, $\hat{T}_P(\beta) f(x) \neq \hat{T}_P(\beta) f(\gamma x)$ which is a contradiction. Thus all q_{i_k} are equal and we denote their common value by $q_P(\beta)$. □

We may now rewrite (7.2) as

$$\sum_\mu \int_{\hat{e}(P)} \int_0^\infty e^{-s^2/4t} (4\pi t)^{-1/2} \operatorname{tr} \tau_\mu \tau_h \mu^{1/2} q_P(\beta)\, T_M(m)\, e^{-t(\Delta^\mu + \mu)}\, dt\, dm'. \tag{7.7}$$

We define

$$\eta_P(\hat{D}, t, \beta, m') \tag{7.8}$$

$$\equiv \sum_\mu e^{-s^2/4t} (4\pi t)^{-1/2} \operatorname{tr} \tau_\mu \tau_h \mu^{1/2} e^{-t\mu} q_P(\beta)\, T_M(m)\, e^{-t\Delta^\mu}(m' K_M, m' K_M).$$

As in (2.4), we let $L(m, \tau_\mu \tau_h, E_\mu)$ denote the Lefschetz number corresponding to $T_M(m)$ but now acting on $L^{\cdot}_2(\Gamma_M \backslash X_M, \mathscr{E}_\mu)$. Continuing our analysis as in [27, (5.7)], we set $\eta_P(\hat{D}, t, \beta) = \int_{\hat{e}(P)} \eta_P(\hat{D}, t, \beta, m')\, dm'$. Recall that we are assuming that $\hat{e}(P)$ satisfies condition (7.3). Then we may show exactly as in the proof of [27, Theorem 5.9] that

$$\begin{aligned}\eta_P(\hat{D}, t, \beta) = {} & \sum_\mu q_P(\beta)(4\pi t)^{-1/2} e^{-s^2/t} \mu^{1/2} e^{-t\mu} L(m, \tau_\mu \tau_h, E_\mu) \\ & + \sum_\mu q_P(\beta)(4\pi t)^{-1/2} e^{-s^2/t} \mu^{1/2} e^{-t\mu} \\ & \times \int_\alpha^t \frac{d}{du} \int_{\hat{e}(P)} \operatorname{tr} \tau_h \tau_\mu (T_M(m) \circ e^{-u\Delta^\mu})(m', m')\, dm'\, ds\end{aligned}$$

$$= \sum_{\mu} q_P(\beta)(4\pi t)^{-1/2} e^{-s^2/t} \mu^{1/2} e^{-t\mu} \Big\{ L(m, \tau_\mu \tau_h, E_\mu) + \int_{\infty}^{t} \sum_{Q} \sum_{\beta' \in \bar{C}_Q(m_\beta)} \eta_Q(\hat{D}_\mu, w, \beta')\, dw \Big\}.$$

Here the second sum runs over the Γ_M-conjugacy classes of maximal rational parabolic subgroups of M, and m_β is the projection of β to M. Inductively, this solves our problem. Observe that

$$\int_0^\infty (4\pi t)^{-1/2} e^{-s^2/t} \mu^{1/2} e^{-t\mu} dt = e^{-\mu|s|}/2.$$

Hence "the" contribution from a maximal boundary component can be expressed as

$$\sum_{\mu} 2^{-1} q_P(\beta)\, e^{-\mu|s|} L(m, \tau_\mu \tau_h, E_\mu).$$

Of course, one may replace $L(m, \tau_\mu \tau_h, E_\mu)$ by $\ell(m, \tau_\mu \tau_h, E_\mu)$ in this expression by comparing $e^{-t\Delta^\mu}$ in the expression for η_P to $\lim_{\varepsilon \to 0} e^{-\varepsilon \Delta^\mu}$. This brings in additional zeta function terms.

We summarize.

(7.9) Theorem. *If $\hat{e}(P)$ is not an equal rank maximal boundary component for some Satake compactification, then $\eta_P(\hat{D}, t, \beta) = 0$,* otherwise,

$$\eta_P(\hat{D}, t, \beta) = \sum_{\mu} q_P(\beta)(4\pi t)^{-1/2} e^{-s^2/t} \mu^{1/2} e^{-t\mu} \Big\{ L(m, \tau_\mu \tau_h, E_\mu) + \int_{\infty}^{t} \sum_{Q} \sum_{\beta' \in \bar{C}_Q(m_\beta)} \eta_Q(\hat{D}_\mu, w, \beta')\, dw \Big\}.$$

With the notations introduced above, we have the following theorem.

(7.10) Theorem.

$$L(\alpha, \tau, E) = \ell(\alpha, t, E) + \sum_{P} \Big\{ \tilde{Z}_P(\beta) + \sum_{\beta' \in \bar{C}_P(\alpha)} \int_0^\infty \eta_P(\hat{D}, t, \beta)\, dt \Big\}.$$

8 The Gauss–Bonnet complex

The index theorem in [26] is stated for generalized signature operators, rather than the somewhat more basic Dirac operators. We recall that this restriction was imposed because our technique for computing the index of a Dirac type operator D requires D to be Fredholm. Borel and Casselman [10], proved this for the signature operators, but it is false for an arbitrary Dirac operator. The algebraic considerations of [26] were in most other respects reliant only on the more general Dirac module structure. In this section, we will sketch the extension of the index theorem and the above Lefschetz theorem to the Gauss–Bonnet complex.

Consider the involution $\tau_p \hat{\tau}_p$ acting on $L_2^{\cdot}(X_\Gamma, \mathscr{E})$. Let Ω^e and Ω^o denote respectively the $+1$ and -1 eigenspaces of this involution. Then Ω^e and Ω^o are the even and odd forms respectively. Hence, $d + d^*$ anti-commutes with $\tau_p \hat{\tau}_p$, and we denote by D^e the restriction of $d + d^*$ to Ω^e. The L^2-index of D^e thus computes the Euler characteristic of the L^2 cohomology. Similarly, suppose that (ρ, E) is a finite dimensional complex representation of G endowed with an admissible inner product as in §2, but we no longer assume (ρ, E) to be irreducible. We again denote by D_ρ the associated signature operator. Let τ_E^e be an involution of E which commutes with both D_ρ and ρ, and let $E = E^+ \oplus E^-$ denote the corresponding decomposition of E into the ± 1 eigenspaces of τ_E^e. Set $\tau^e = \tau_p \hat{\tau}_p \tau_E^e$. We define the operator D_ρ^e to the restriction of D_ρ to the $+1$ eigenspace of τ^e. The index of D_ρ^e computes the difference of the Euler characteristic of the L^2-cohomology with coefficients in E^+ and E^-.

As the operator D_ρ^e is a signature operator (restricted to a subspace), the above mentioned results of Borel and Casselman imply that it is Fredholm and that the techniques of [26] and [27] apply to extend the index theorem and the above Lefschetz theorem to this case. Two fundamental differences occur. The first is a drastic algebraic simplification obtained when dealing with the involution τ^e and the associated extra vanishing implied by (1.3ii).

In the inductive construction of the operator H_1, each step in the induction increased both the mass and mass_e by at most 4. Hence, in the case currently under consideration, $n/2$ iterations (of the inductive construction) are required to obtain a nonvanishing trace. (Whereas it required a great deal of work to show that more than $n/4$ iterations were required for nonvanishing in the signature case). The inequalities of [26, §5.1] imply that the trace of H_1 terms arising from more than $n/2 - 1$ iterations is exponentially decreasing. Hence, we conclude:

(8.1) Proposition. *The H^1 terms make no contribution to the computation of the index or the Lefschetz number of the Gauss–Bonnet complex.*

(8.2) Remark. The index theorem of [26] is stated for $\mathbb{Q}$-irreducible spaces X. This restriction arose because of the difficulty in controlling the contribution of the H^1 term associated with nonmaximal parabolics. In the case of the Gauss–Bonnet complex, we see that these terms vanish; hence in this case we may remove the restriction that X be $\mathbb{Q}$-irreducible.

The second fundamental difference between the Gauss–Bonnet complexes and the signature complexes arises in computing the trace of $C(v)\tau D H_P^0$, P maximal. In [26, §4.9] D_ρ is written as a sum of two operators $D_\rho = \hat{D} + D_A$, with $D_A = C(\partial/\partial r)\partial/\partial r + (\hat{C}(\partial/\partial r) W(\partial/\partial r)$, where $\partial/\partial r$ is a unit vector in $\mathbf{a}$, $W(\partial/\partial r) - \delta_P(\partial/\partial r)$ computes the A-weight of an element of $\Lambda^* \mathbf{n} \otimes E$, and δ_P denotes one half the sum of the weights of A on $\mathbf{n}$. In [26, Lemma 4.9.4] (see also [27, (1.9)], we showed that $\operatorname{tr} C(v)\tau D H_\rho^0 = \operatorname{tr} C(v)\tau \hat{D} H_\rho^0$, for τ the involution associated to a signature complex. This lemma is no longer valid for a Gauss–Bonnet complex. Instead, we have the following.

(8.3) Proposition.

$$\operatorname{tr} C(v)\tau^e D H_P^0 = \operatorname{tr} C(v)\tau^e \hat{C}(\partial/\partial r) W(\partial/\partial r) H_P^0.$$

Proof. In order to have nonzero trace, DH_P^0 must contribute a $\hat{C}(v)$ factor to cancel with the $\hat{C}(v)$ factor in $C(v)\tau^e$. The only term with such a factor is $\hat{C}(\partial/\partial r)\, W(\partial/\partial r)$. □

We now simplify the expression in (8.3). Elementary computations show that

$$\begin{aligned}(8.4)\qquad &\operatorname{tr} C(v)\tau^e \hat{C}(\partial/\partial r)\, W(\partial/\partial r)\, H_P^0 \\ &\quad = \operatorname{tr}(\varepsilon(\partial/\partial r)\varepsilon^*(\partial/\partial r)) - (\varepsilon^*(\partial/\partial r)\varepsilon(\partial/\partial r))\, W(\partial/\partial r)\tau^e H_P^0 \\ &\quad = -\operatorname{tr} W(\partial/\partial r)\tilde{\tau}^e H_{\tilde{P}}^0,\end{aligned}$$

with $\tilde{\tau}^e = \tau_{\mathbf{p}_m}\hat{\tau}_{\mathbf{p}_m}\tau_{\mathbf{n}}\hat{\tau}_{\mathbf{n}}\tau_E^e$ the involution determined by the Gauss–Bonnet complex of $T^*\hat{e}(P)\otimes \Lambda^{\cdot}\mathbf{n}^*$. Here $\mathbf{m} = \mathbf{p}_m \oplus \mathbf{k}_m$ is the Cartan decomposition of $\mathbf{m}$ determined by K_M. We expand 8.4 as

$$-\operatorname{tr} W(\partial/\partial r)\tilde{\tau}^e e^{t\partial^2/\partial r^2} e^{-t(\Delta_Z + \Delta_n + W(\partial/\partial r)^2)},$$

where $\Delta_Z = D_Z^2$. D_n anti-commutes with $\tilde{\tau}^e$ and commutes with $W(\partial/\partial r)\, e^{t\partial^2/\partial r^2} e^{-t(\Delta_Z + \Delta_n + W(\partial/\partial r)^2)}$, hence the above trace equals

$$-\operatorname{tr} W(\partial/\partial r)\,\tilde{\tau}^e e^{t\partial^2/\partial r^2} e^{-t(\Delta^P + W(\partial/\partial r)^2)}\, dt,$$

with $\Delta^P = (D^P)^2$ and $D^P = F_\Gamma^P(D)$ where F_Γ^P is the map which assigns to a signature operator on X_Γ with coefficients in E the signature operator on $\hat{e}(P)$ with coefficients in $H^{\cdot}(\mathbf{n}, E)$. The above trace is taken over this reduced space. Hence, arguing as in §7, if we define $\eta_P^e(D^P, t, x)$ by

$$\begin{aligned}(8.5)\qquad \eta_P^e(D^P, t, x) &= -\operatorname{tr} W(\partial/\partial r)\tilde{\tau}^e e^{t\partial^2/\partial r^2} e^{-t(\Delta^P + W(\partial/\partial r)^2)}(x, x) \\ &= -\operatorname{tr} W(\partial/\partial r)\tilde{\tau}^e (4\pi t)^{-1/2} e^{-t(\Delta^P + W(\partial/\partial r)^2)},\end{aligned}$$

then we have

$$(8.6)\qquad \int_{X_I} \operatorname{tr}\tau^e e^{-t\Delta}(x, x)\, dx = \int_{X_I} \lim_{t\to 0} \operatorname{tr}\tau^e e^{-t\Delta}(x, x)\, dx + \sum_P \int_0^t \int_{\hat{e}(P)} \eta_P^e(D^P, s, x)\, dx\, ds,$$

where $\sum_P$ runs over the Γ-conjugacy classes of maximal rational parabolics of G. As before, we may also use this formula to compute $\eta_P^e(D^P, t) \equiv \int_{\hat{e}(P)} \eta_P^e(D^P, x, t)\, dx$. We may decompose $H^{\cdot}(\mathbf{n}, E)$ as $H^{\cdot}(\mathbf{n}, E) = \oplus H_\mu$, where $W(\partial/\partial r)\varphi = \mu\varphi$ for $\varphi \in H_\mu$. Further decompose H_μ as $H_\mu^+ \oplus H_\mu^-$, where $H_\mu^\pm$ denotes the ± 1 eigenspace of $\tau_{\mathbf{n}}\hat{\tau}_{\mathbf{n}}\tau_E^e$ acting on H_μ. Let D_μ^e denote the signature operator on $\hat{e}(P)$ with coefficients in H_μ, and let $\Delta_\mu^e = (D_\mu^e)^2$. Then we obtain the following proposition.

(8.7) Proposition. *Let $c_n(P)$ denote the Chern class of $\hat{e}(P)$. With the notation and assumptions as above we have*

$$\begin{aligned}\eta_P^e(D^P, t) &= -(4\pi t)^{-1/2} \sum_\mu \mu e^{-t\mu^2} \left\{ \int_{\hat{e}(P)} \lim_{s\to 0} \operatorname{tr}\tilde{\tau}\, e^{-s\Delta_z^e}(x, x) + \sum_Q \int_0^t \eta_Q^e(D_\mu^P, s)\, ds \right\} \\ &= -(4\pi t)^{-1/2} \sum_\mu \mu e^{-t\mu^2} \left\{ \int_{\hat{e}(P)} c_n(P)(d_\mu^+ - d_\mu^-) + \sum_Q \int_0^t \eta_Q^e(D_\mu^Q, s)\, ds \right\},\end{aligned}$$

where $d_\mu^\pm = \dim H_\mu^\pm$ and $D_\mu^Q = F_{\Gamma_M}^Q(D_\mu^e)$.

(*8.8*) *Remark.* Lemma 5.2 in [27] is no longer completely valid in this context when $G_l' = \mathrm{Sl}(2, \mathbb{R})$ (the algebra is slightly different). Hence, in the case of $Sp(n, \mathbb{R})$, we may not conclude that $\eta_P^e = 0$ if $\hat{e}(P)$ is not a maximal boundary component of the Bailey–Borel compactification. That part of Lemma 5.2 in [27] which states that $\eta_P^e = 0$ if rank $G_l'(P) \neq$ rank $K_l'(P)$ remains valid.

With the above proposition, the following L^2-Gauss–Bonnet theorem (see also [3], [4], and [13] for earlier treatments of this special case) obtained by letting $t \to \infty$ in (8.6) becomes effectively computable.

(8.9) Theorem. Index $(D^e) = [\dim(E^+) - \dim(E^-)] \int_{X_\Gamma} c_n(X_\Gamma) + \sum_P \int_0^\infty \eta_P^e(D^P, s)\, ds$, *where* $c_n(X_\Gamma)$ *is the top Chern class of* X_Γ.

This result may be extended to a Lefschetz theorem as above, although now we may drop the term associated to the zeta function. Set

$$\eta_P^e(D^P, t, \beta) = \int_{\hat{e}(P)} -\operatorname{tr} W(\partial/\partial r)\, \tilde{\tau}^e (4\pi t)^{-1/2}\, \hat{T}_P(\beta)\, e^{-t(\Delta^P + W(\partial/\partial r)^2)}(x, x)\, dx\,.$$

Then we have the following theorem. (See also [13b].)

(8.10) Theorem. $L(\alpha, \tau^e, E) = \ell(\alpha, \tau^e, E) + \sum_P \sum_{\beta \in \bar{C}_P(\alpha)} \int_0^\infty \eta_P^e(D^P, t, \beta)\, dt$, *and*

$$\eta_P^e(D^P, t, \beta)$$

$$= -(4\pi t)^{-1/2}\, e^{-s^2/t}\, q_P(\beta) \sum_\mu \mu e^{-t\mu^2} \left\{ \ell(\alpha, \tau^e, E_\mu) + \sum_Q \sum_{\beta' \in \bar{C}_Q(m_\beta)} \int_0^t \hat{\eta}_Q^e(D_\mu, s, \beta')\, ds \right\}.$$

Here m_β *denotes the image of* β *in* M.

We now compute a simple rank two example (cf. [13b]). Let G be $Sp(4, \mathbb{R})$, let Γ be a neat arithmetic subgroup of $Sp(4, \mathbb{Z})$, and let E be trivial. Because $\mathrm{SL}(2, \mathbb{R})$ is equal rank, the localization of the Lefschetz formula for the Gauss–Bonnet complex lives on the "reductive Borel–Serre compactification" (in the terminology of [13a, b]) rather than on the Baily–Borel–Satake compactification. The proper boundary components of X_Γ with respect to this compactification are of three types:

(8.11)
(i) points
(ii) quotients of the upper half plane of complex type,
(iii) quotients of the upper half plane of real type.

We have introduced the terminology complex (respectively real) type to refer to those boundary components which are not (respectively are) collapsed in the passage to the Baily–Borel–Satake compactification. In particular, the boundary components of real (respectively complex) type correspond to maximal parabolic subgroups with abelian (respectively nonabelian) unipotent radical. Using the equality

$$\int_0^\infty (4\pi t)^{-1/2}\, e^{-s^2/t}\, \mu e^{-t\mu^2}\, dt = \operatorname{sign}(\mu)\, e^{-|\mu||s|}/2\,,$$

we obtain,

$$L(\alpha, \tau^e, E) = \ell(\alpha, \tau^e) - \sum_P \Bigg\{ \sum_{\beta \in \bar{C}_P(\alpha)} q_P(\beta) \sum_\mu \operatorname{sign}(\mu)\, e^{-|\mu||s|}\, \ell(\alpha, \tau^e, E_\mu)/2$$

$$- \sum_Q q_P(\beta) \sum_{\beta \in \bar{C}_Q(m_\beta)} q_Q(\beta') \sum_{\mu_1, \mu_2} \mu_1 \mu_2 \int_0^\infty \int_0^{t_1} (4\pi t_1)^{-1/2}\, e^{-s_1^2/t_1}\, e^{-t_1 \mu_1^2}$$

$$\times\, (4\pi t_2)^{-1/2}\, e^{-s_2^2/t_2}\, e^{-t_2 \mu_2^2}\, dt_2\, dt_1 \Bigg\},$$

where μ_1 (respectively μ_2) denotes the A_P (respectively A_Q) weight. Similarly, s_1 and s_2 ($= s_i(\beta)$) denote the lengths of the projections of β onto A_P and A_Q. For simplicity, consider the case where $\alpha = 1$. Then we have

$$L^2\text{-Euler characteristic} = \int_{X_\Gamma} c_n(X_\Gamma) - \sum_P \Bigg\{ \sum_\mu \operatorname{sign}(\mu) c_n(P)(d_\mu^+ - d_\mu^-)/2$$

$$- \sum_Q \sum_\mu \mu_1 \mu_2 \int_0^\infty \int_0^{t_1} (4\pi t_1)^{-1/2}\, e^{-t_1 \mu_1^2} (4\pi t_2)^{-1/2}\, e^{-t_2 \mu_2^2}\, (-1)^{\text{degree}(\mu)}\, dt_2\, dt_1 =$$

(8.12) (i) $$\int_{X_\Gamma} c_n(X_\Gamma) - \sum_P \Bigg\{ \sum_\mu \operatorname{sign}(\mu) c_n(P)(d_\mu^+ - d_\mu^-) +$$

(ii) $$- \sum_Q \sum_\mu \operatorname{sign}(\mu_1 \mu_2) \tan^{-1}(\mu_2/\mu_1)(-1)^{\text{degree}(\mu)}/2\pi \Bigg\}.$$

Recall that the Q in the above expression parametrizes rational boundary components of $\hat{e}(P)$. These boundary components of maximal (proper) boundary components correspond to point boundary components in the Borel–Serre compactification of X_Γ. In particular, each point boundary component enters the above expression twice – once as the boundary component of a type (ii) boundary component and once as the boundary component of a type (iii) boundary component. We list here the collection of triples ($\mu_1 \mu_2$, degree (μ)) which occur in the contribution of a single point boundary component to (8.13). Arising from the boundary of a type (ii) boundary component we have $(-2, -1, 0)$, $(-1, -2, 1)$, $(-2, 1, 1)$, $(-1, 2, 2)$, $(1, -2, 2)$, $(2, -1, 3)$, $(1, 2, 3)$, $(2, 1, 4)$. Plugging these numbers into (8.12.ii) we obtain

$$2(\tan^{-1}(2) - \tan^{-1}(1/2))/\pi\,. \tag{8.13}$$

From the boundary of a boundary component of type (iii), we have $(-3, -1, 0)$, $(-3, 1, 1)$, $(-1, -3, 1)$, $(1, -3, 2)$, $(-1, 3, 2)$, $(1, 3, 3)$, $(3, -1, 3)$, $(3, 1, 4)$, which contribute

$$2(\tan^{-1}(3) - \tan^{-1}(1/3))/\pi\,,$$

and the contribution from a point boundary component is

$$2(\tan^{-1}(2) - \tan^{-1}(1/2) + \tan^{-1}(3) - \tan^{-1}(1/3))/\pi = 1\,.$$

It is easy to compute the sum $\sum_\mu \operatorname{sign}(\mu)(d_\mu^+ - d_\mu^-)$ arising in (8.12.i), and we may reorganize (8.12) as

$$(8.14)\qquad L^2\text{-Euler characteristic} = \chi(X_\Gamma) - 2\sum_P{}^{(\mathrm{iii})}\,\chi(\hat{e}(P)) - \sum_P{}^{(\mathrm{ii})}\,\chi(\hat{e}P) + \sum_P{}^{(\mathrm{i})}\,1\ ,$$

where $\sum_P{}^{(a)}$ denotes the sum over rational boundary components of type (a). Here we have used the equality due to Harder [14]:

$$\int_{X_\Gamma} c_n = \chi(X_\Gamma)\ ,$$

for any locally symmetric manifold X_Γ, where χ denotes the topological Euler characteristic.

9 The holomorphic Lefschetz theorem – A sketch

Consider now the Dolbeault complex $L_2^{0,\cdot}{}^*(X_\Gamma, \mathscr{E})$ consisting of E-valued forms of type $(0, *)$. Let $\bar\partial\colon L_2^{0,\cdot}{}^*(X_\Gamma, \mathscr{E}) \to L_2^{0,\cdot}{}^{*+1}(X_\Gamma, \mathscr{E})$ denote the usual (flat) exterior derivative for this complex, and let $\bar\partial^*$ denote its formal adjoint. Let τ_R denote the involution of $L_2^{0,\cdot}{}^*(X_\Gamma, \mathscr{E})$ given by $\tau_R\varphi = (-1)^p$, for $\varphi \in L_2^{0,\cdot}{}^p(X_\Gamma, \mathscr{E})$. Let V be a subspace of T^*X_Γ such that $T^*X_\Gamma = V \oplus jV$, where j denotes the complex structure operator. Let $\{X_i\}_{i=1}^m$ be an orthonormal basis of V, let $Y_i = jX_i$, and set $Z_i = 2^{-1/2}(X_i + ijX_i)$. The involution τ_R may be realized as

$$(9.1)\qquad \tau_R = C(\bar Z_1)\ldots C(\bar Z_m)\hat C(\bar Z_1)\ldots\hat C(\bar Z_m)\ .$$

Here we remark that the $\varepsilon^*(\bar Z_i)$ in the definitions of $C(\bar Z_i)$ and $\hat C(\bar Z_i)$ is the Hermitian adjoint. In particular, $C(i\bar Z_i) = i\hat C(\bar Z_i)$, and $\hat C(i\bar Z_i) = iC(\bar Z_i)$.

Let $D_{\mathbb{C}}$ denote the (flat) spin^c Dirac operator with coefficients in $\mathscr{E}$ given (on $(L^2(\Gamma\backslash G)\otimes \Lambda^\cdot \rho^* \otimes E)^K$ with respect to invariant horizontal vector fields) by (see [15])

$$\begin{aligned}(9.2)\qquad D_{\mathbb{C}} &= \sum C(\bar Z_i)X_i + C(i\bar Z_i)Y_i + \hat C(\bar Z_i)\rho(X_i) + \hat C(i\bar Z_i)\rho(Y_i)\\ &= \sum C_R(X_i)X_i + C_R(Y_i)Y_i - i[C_R(Y_i)\rho(X_i) + C_R(X_i)\rho(Y_i)]\ ,\end{aligned}$$

where $C_R(X_i) = C(\bar Z_i)$, $C_R(Y_i) = C(i\bar Z_i)$, $\hat C_R(X_i) = \hat C(\bar Z_i)$, and $\hat C_R(Y_i) = \hat C(i\bar Z_i)$. It is easy to see that the above considerations for the signature and Gauss-Bonnet complexes can be extended to this case. Because both $C(\bar Z)$ and $\hat C(\bar Z)$ appear in τ_R, we see that we obtain two eta invariant terms – one as in the signature complex and one as in the Gauss-Bonnet complex. The $\hat C$ factors do not lead to the cancellation of the zeta term in this case, however, because τ_R can still be interpreted as Clifford multiplication by the volume element, using the C_R representation of Clifford multiplication. Let $L(\alpha, \tau_R, E)$ denote the L^2-holomorphic Lefschetz number defined analogously to $L(\alpha, \tau, E)$ with Λ^* replaced by $\Lambda^{0,\cdot}{}^*$. Similarly define $\ell(\alpha, \tau_R, E)$. We record here without proof (and without computation of associated constants) the L^2-Riemann–Roch and Lefschetz theorems one obtains in this case. All unexplained notation will be analogous to that of the preceding sections.

(9.3) Theorem.

$$L(\alpha, \tau_R, E) =$$

$$\ell(\alpha, \tau_R, E) + \sum_P \left\{ \sum_{\beta \in \bar{C}_U(\alpha)} \int_{\hat{e}(P)} \sigma^{\perp}(\beta) c(\beta) \kappa_\beta I_{h,R}(\beta) I_{l,R} Z(x, \beta) dx + \sum_{\beta \in \bar{C}_P(\alpha)} \int_0^\infty \eta_P^R(t, \beta) dt \right\}.$$

Here $\eta_P^R(t, \beta) \equiv \int_{\hat{e}(P)} \operatorname{tr} C(v) \tau_R D_{\mathbb{C}} \hat{T}_P(\beta) e_{\mathbb{C}}^{-tD^2}(x, x) dx$, *with* tr *now denoting trace over* $\Lambda^{0,*}$ *now rather than* Λ^*. $I_{h,R}$ *and* $I_{l,R}$ *denote the measures defined analogously to* I_h *and* $I_{l,R}$ *with* τ_h *and* τ_l *replaced by the corresponding (j-stable) restriction of* τ_R.

The term η_P^R in the above theorem naturally decomposes as a sum of two eta invariants – one like that arising in the signature complex and one like that arising in the Euler complex. One can then reduce the computation of these terms to lower rank Lefschetz computations as in the preceding section. We leave this as an "exercise" for the interested reader. When $\alpha = 1$, the above theorem yields an L^2-Riemann–Roch theorem as a special case.

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Appendix

A.1 In this appendix, we modify aspects of the parametrix constructions of [26, §4] in order to simplify and to adapt arguments to the computation of Lefschetz numbers. In particular, we eliminate the awkward concept of the regularized trace introduced in [26]. We will not discuss the error estimates as these do not differ essentially from those of [26]. In order to conserve notation, we will initially consider the evaluation of $\operatorname{tr} T_G(\alpha) \tau e^{-t\Delta}$ instead of $\operatorname{tr} C(e_n) T_G(\alpha) \tau D_\rho e^{-t\Delta}$. The proofs are the same and the former case has independent interest. The latter case is merely notationally more complicated; moreover, it has already been treated once in [26].

We first recall some concepts and notation from [26, §4]. Let $Q = N_Q A_Q M_Q$ be a parabolic subgroup of G and $W'_Q(\omega_Q, t_Q)$ an associated special neighborhood. Let ϕ_Q denote the set of weights of A_Q on N_Q. Given $x = (a(x), x^0) \in W'_Q(\omega_Q, t_Q)$, we order the elements of ϕ_Q according to the rule $\alpha > \beta$ if $a(x)^\alpha > a(x)^\beta$. Label the weights $\{\alpha_1, \ldots, \alpha_N\}$, with $\alpha_j \geqq \alpha_{j+1}$. Fix $\sigma \ll (\dim G)^{-3}$. Passing to a larger rational parabolic subgroup $P \supset Q$ if necessary, we shall assume that

$$a(x)^\alpha > a(x)^{\sigma \dim(G) \alpha_1}, \tag{A.1.1}$$

for all $\alpha \in \phi_P$. Then it is easy to check that we may pick P so that for some $\omega_x \subset \hat{e}(P)$ and $t_x > 0$ depending on x and σ, $x \in W'_P(\omega_x, t_x)$, and the distance from x to the complement of $W'_P(\omega_x, t_x)$ is greater than or equal to $\sigma^2 \log a(x)^{\alpha_1}$.

We associate to σ a partition of ϕ_P into a disjoint union $\bigcup_i \phi_\sigma^i$, with the ϕ_σ^i uniquely determined by the conditions below.

$$\alpha_1 \in \phi_\sigma^1, \quad \text{and}$$

$$\text{if } \alpha_j \in \phi_\sigma^i, \text{ then } \alpha_{j+1} \in \phi_\sigma^{i+1} \text{ if } a(x)^{\alpha_j} \geqq a(x)^{\alpha_{j+1}^{(1+\sigma')}},$$

$$\text{otherwise } \alpha_{j+1} \in \phi_\sigma^i .$$

This partition determines a corresponding central series $N(\sigma)$ for N:

$$\{id\} = N_0 \subset N_1 \subset \ldots \subset N_J = N ,$$

with $\bigcup_{i \leqq j} \phi_\sigma^i$ the set of A-weights on $\mathbf{n}_j$. Let $d_i = \dim N_i$, and $d^i = d_i - d_{i-1}$. $N(\sigma)$ induces a compatible series for $\Gamma_N \equiv \Gamma \cap N$:

$$\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_J = \Gamma_N, \text{ with } \Gamma_i \equiv \Gamma \cap N_i .$$

Set $N^i = N_{i-1}/N_i$ and let Γ^i denote the image of Γ_i in N^i.

Let $\bar{\pi}_i$: $W'_P(\omega_P, t_P) \to N_i \backslash W'_P(\omega_P, t_P)$ denote the natural projection, where the right hand side is defined by identifying $W'_P(\omega_P, t_P)$ with a subset of $\Gamma_P \backslash X$. Let $\mathcal{O}_\alpha^i$ be a coordinate neighborhood in $N_i \backslash W'_P(\omega_P, t_P)$. Then with frame chosen as in [26, §4.3], $f \in L_2^\cdot(\bar{\pi}_i^{-1}(\mathcal{O}_\alpha^i), \mathscr{E})^{N_{i-1}}$ has a Fourier expansion as in [26, (4.2.7)]:

$$f(u, x) = \sum_\mu e^{i\mu \cdot u} f_\mu(x) .$$

Let $T_{\mu,i}$ denote the projection onto the span of $e^{i\mu \cdot u}$. We now construct an approximation $\tilde{H}_\mu^{i,\alpha}(\lambda)$ for $(D_\rho^2 - \lambda)^{-1} T_{\mu,i}$ in a neighborhood of $y \in \bar{\pi}_i^{-1}(\mathcal{O}_\alpha)$. When $i = 0$, we adopt the convention that $\mu = 0$ and $T_{\mu,0} =$ the identity $= \bar{\pi}_0$. Let $x' = \bar{\pi}_i(y)$, and fix a corresponding product structure for $y \in \bar{\pi}_i^{-1}(\mathcal{O}_\alpha)$ as in [26, 4.3] with $y = (u', x')$.

$T_{\mu,i}$ is given by:

$$\text{(A.1.2)} \qquad T_{\mu,i} f(u', x') = \operatorname{Vol}(\Gamma^i \backslash N^i)^{-1} \int_{\Gamma^i \backslash N^i} f(u, x') e^{i\mu \cdot (u' - u)} du$$

$$= \operatorname{Vol}(\Gamma_i \backslash N_i)^{-1}(x') \int_{\Gamma_i \backslash N_i} f(u, x') e^{i\mu \cdot (u' - u)} du_{x'} .$$

Here $\operatorname{Vol}(\Gamma_i \backslash N_i)(x')$ denotes the volume of $\Gamma_i \backslash N_i$ induced by viewing it as the fiber $\bar{\pi}_i^{-1}(x')$, and du_x is the corresponding volume element. Let $g_i(x')$ denote $\operatorname{Vol}(\Gamma_i \backslash N_i)^{-1}(x')$, with the convention that $g_0(x') \equiv 1$.

$\tilde{H}_\mu^{i,\alpha}(\lambda)$ will be the operator with kernel

$$\tilde{h}_\mu^{i,\alpha}(x, u; x', u', \lambda) = \sum_{\ell=0}^{N} \int_{\mathbb{R}^{n-d_i}} g_i(x') e^{i\mu \cdot (u - u')} e^{2\pi i (x - x') \cdot v} \tilde{h}_{\mu,\ell}^{i,\alpha}(x, x', \lambda, v)$$

$$\times (\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-\ell - 1} dv .$$

Here $\|v\|^2 = \sum_{l,j} g^{lj} u_l u_j$. The $\tilde{h}_{\mu,\ell}^{i,\alpha}$ are constructed inductively as follows (see [26]).

Set $\tilde{h}_{\mu,-1}^{i,\alpha} = 0$, $\tilde{h}_{\mu,0}^{i,\alpha} =$ Identity of $\Lambda^\cdot T_y^*(\Gamma_P \backslash X) \otimes E$. (The choice of frame allows us to identify $\Lambda^\cdot T_y^*(\Gamma_P \backslash X)$ and $\Lambda^\cdot T_{y'}^*(\Gamma_P \backslash X)$, for $y, y' \in \bar{\pi}_i^{-1}(\mathcal{O}_\alpha)$.)

$$(D_\rho^2 - \lambda)[g_i(x') e^{i\mu \cdot (u - u')} e^{2\pi i (x - x') \cdot v} \tilde{h}_{\mu,\ell}^{i,\alpha}(x, x', \lambda, v) (\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-\ell - 1}]$$

$$= g_i(x') e^{i\mu \cdot (u - u')} e^{2\pi i (x - x') \cdot v} \tilde{h}_{\mu,\ell}^{i,\alpha}(x, x', \lambda, v) (\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-\ell}$$

$$- I_\ell e^{i\mu \cdot (u - u')} e^{2\pi i (x - x') \cdot v} g_i(x') (\|2\pi v\|^2 + |\mu(x) - i\Gamma|^2 - \lambda)^{-\ell - 1},$$

with I_ℓ defined by the equality.

Set $\tilde{h}^{i,\alpha}_{\mu,\ell+1} = I_\ell$. Then we see that locally,

$$(D_\rho^2 - \lambda) \sum_{\ell=0}^{N} \int_{\mathbb{R}^{n-d_i}} g_i(x') e^{i\mu(u-u')} e^{2\pi i(x-x')\cdot v} \tilde{h}^{i,\alpha}_{\mu,\ell}(x, x', \lambda, v)$$
$$\times (\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-\ell-1} dv$$
$$= T_{\mu,i} + \int_{\mathbb{R}^{n-d_i}} g_i(x') e^{i\mu(u-u')} e^{2\pi i(x-x')\cdot v} I_{N+1} (\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-N-1} dv.$$

The assertion, for two operators A and B, that $A = B$ "locally" means that $\psi A\varphi = \psi B\varphi$, for any pair of cutoff functions ψ and φ supported in the appropriate subdomain – here $\bar{\pi}_i^{-1}(\mathcal{O}_\alpha)$. Set

$$\tilde{H}_1^{i,\alpha}(\lambda) = \textstyle\sum_{\mu \neq 0} \tilde{H}_\mu^{i,\alpha}(\lambda).$$

We next use the $\tilde{H}_\mu^{i,\alpha}(\lambda)$ to construct an approximation $H_\mu^{i,\alpha}(t)$ for $e^{-tD^2}\rho T_{\mu,i}$ in $\bar{\pi}_i^{-1}(\mathcal{O}_\alpha)$. Let C_t be the curve in $\mathbb{C}$ given by the image of $\mathbb{R}$ under the map $s \to |s| + is - 1/t$. This curve surrounds the spectrum of D_ρ^2 and satisfies $\mathrm{Re}(t\lambda) \geqq -1$ for every $\lambda \in C_t$. Set

$$H_\mu^{i,\alpha}(t) = (2\pi i)^{-1} \int_{C_t} e^{-t\lambda} \tilde{H}_\mu^{i,\alpha}(\lambda) d\lambda \,. \tag{A.1.3}$$

This is the operator with kernel

$$h_\mu^{i,\alpha}(x, x', t) = (2\pi i)^{-1} \int_{C_t} \tilde{h}_\mu^{i,\alpha}(x, x', \lambda) e^{-t\lambda} d\lambda \,.$$

Let $H_1^{i,\alpha}(t) = \sum_{\mu \neq 0} H_\mu^{i,\alpha}(t) T_{\mu,i}$. Fix a smooth bounded function φ supported in $W'_P(\omega_P, t_P)$. Choose ρ_α^i and φ_α^i as in [26, (4.7.1)], with $\sum \varphi_\alpha^i = 1$ on the support of φ. The $H_j^{i,\alpha}(t)$ can be patched together to form $H_j^i := \sum_\alpha \rho_\alpha^i H_j^{i,\alpha} \varphi_\alpha^i$, $j = 0, 1$. Similarly, define $\tilde{H}_j^i$. Then $H_1^i \varphi$ (respectively $H_1^i \varphi$) is a good approximation (in the sense of §3.2) to $e^{-t\Delta}\varphi$ (respectively $(\Delta - \lambda)^{-1}\varphi$) restricted to the intersection of the N_{i-1}-invariant forms and the orthogonal complement of the N_i invariant forms. Similarly, the sum $\sum_{i>0} H_1^i \varphi$ is a good approximation to $e^{-t\Delta}\varphi$ restricted to the orthogonal complement of the N-invariant forms.

(A.1.4) Remark. It is important for our applications to note that if $a(x)^{-\nu(1+\sigma^{i+1})} \geqq t^{1/2}$ for every $\nu \in \phi_\sigma^{i+1}$, (or if $t \leqq a(x)^{-\varepsilon\nu}$ for all $\nu \in \phi_\sigma^i$ and some $\varepsilon > 0$ if $N_{i+1} = N$), then $(H_0^i(t) + \sum_{j \leqq 1} H_1^j(t)))\varphi$ is a good approximation to $e^{-t\Delta}\varphi$. In particular the error terms are rapidly decreasing. This follows from the same arguments as the proof of (3.1.8). On the other hand, for all $b > i$, $(H_0^b(t) + \sum_{j \leqq b} H_1^j(t))\phi$ gives an even better approximation. For t in the given range, however, we will use the first choice of approximation because its trace is easier to compute than the new infinite sums arising in the trace of the second choice. In fact the (asymptotic) equality of the two traces which follows from the error estimates, can be used to establish the convergence of the new infinite sums.

Tailoring our parametrix constructions in this fashion to a given value of t allows us to avoid regularizing traces as in [26]. For t outside the ranges considered, we will use the more global construction indicated in (§3.2) to obtain an approximation to $e^{-t\Delta}\varphi$ on the N-invariant forms. This approximation will not be discussed further in this appendix.

We now return to the analysis of the terms in $H_1^{i,\alpha}$ needed for our trace computations. As the patching plays no further role in our computations, we henceforth suppress the superscript α when no confusion is likely to result. We may write

$$\tilde{h}^i_{\mu,\ell}(x, x', \lambda, v) = \sum_{s,s'} \tilde{h}^i_{\mu,\ell,s,s'}(x, x', v) P_{s'}(\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-s},$$

where $P_{s'}$ is a polynomial in derivatives of $(\|v\|^2 + |\mu^x - i\Gamma|^2)$, homogeneous of degree $s' \leqq s$.

We compute

$$\begin{aligned}(2\pi i)^{-1} \int_{C_t} & \tilde{h}^i_{\mu,\ell,s,s'}(x, x', v) P_{s'}(\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-s-\ell-1} d\lambda \\ &= (2\pi i)^{-1}[(\ell + s)!]^{-1} \int_{C_t} (-t)^{\ell+s} \tilde{h}^i_{\mu,\ell,s,s'}(x, x', v) P_{s'} \\ &\quad \times (\|2\pi v\|^2 + |\mu^x - i\Gamma|^2 - \lambda)^{-1} e^{-t\lambda} d\lambda \\ &= [(\ell + s)!]^{-1}(-t)^{\ell+s} \tilde{h}^i_{\mu,\ell,s,s'}(x, x', v) P_{s'} e^{-t(\|2\pi v\|^2 + |\mu^x - i\Gamma|^2)}.\end{aligned}$$

We may further decompose $\tilde{h}^i_{\mu,\ell,s,s'}$ as

$$\tilde{h}^i_{\mu,\ell,s,s'} = \sum_{A,B} \tilde{h}^i_{\ell,s,s',A,B}(x, x')(2\pi v)^A (\mu^x)^B,$$

where the sum runs over a finite number of multi-indices A and B. Write $P_{s'} = P_{s',S_2}(2\pi v)^{S_2}$, where P_{s',S_2} is a polynomial in derivatives of $|\mu^x - i\Gamma|^2$, homogeneous of degree $s' - |S_2|/2$. Integrating with respect to v we obtain

$$\begin{aligned}(\text{A.1.5}) \quad \int_{\mathbb{R}^{n-d_t}} & [(\ell + s)!]^{-1}(-t)^{\ell+s} \tilde{h}^i_{\ell,s,s',A,B}(x, x') e^{2\pi i(x-x')\cdot v} \\ &\quad \times P_{s',S_2}(2\pi v)^{S_2 \cup A} \mu(x)^B e^{-t(\|2\pi v\|^2 + |\mu^x - i\Gamma|^2)} dv \\ &= t^{(2\ell + 2s - n + d_t - |A| - |S_2|)/2} \tilde{h}^i_{\ell,s,s',A,B}(x, x') \\ &\quad \times [(\ell + s)!]^{-1}(-1)^{\ell+s} F_{A,S_2}(x, x', t)(\mu^x)^B P_{s',S_2} e^{-t|\mu^x - i\Gamma|^2},\end{aligned}$$

where

$$\begin{aligned}F_{A,S_2} &= \int_{\mathbb{R}^{n-d_t}} e^{2\pi i(x-x')\cdot vt^{-1/2}} (2\pi v)^{A \cup S_2} e^{-\|2\pi v\|^2} dv \\ &= (4\pi)^{-(n-d_t)/2}(-it^{1/2}\partial/\partial x)^{A \cup S_2}(\det H) e^{-|H(x-x')|^2/4t},\end{aligned}$$

for some symmetric matrix H such that $H = I + O(|x - x'|^2)$ and $H^{-1}v \cdot H^{-1}v = \|v\|^2$. Expanding the derivative, we may rewrite this expression as

$$Q_{A,S_2}((x - x')t^{-1/2})(1 + O(|x - x'|) + O(t^{1/2}))e^{-|H(x-x')|^2/4t},$$

with Q_{A,S_2} a polynomial of degree $|A| + |S_2|$.

Fix some $\varepsilon > 0$. Suppose now that $x' = B(x) + B_0$, for some map B which fixes the origin in $\mathbb{R}^{n-d_i}$, and some vector B_0 which either vanishes or satisfies $|B_0| > t^{1/2-\varepsilon}$. Let dB denote the linearization of B at 0, and assume that dB is orthogonal and dB fixes B_0. Write $\mathbb{R}^{n-d_i} = \mathbb{R}^f \oplus \mathbb{R}^{n-d_i-f}$, where $\mathbb{R}^f$ is the $+1$ eigenspace of dB, and let dB_2 denote the restriction of dB to $\mathbb{R}^{n-d_i-f}$. Consider an open set of the form $E_1 \times E_2$, where E_1 and E_2 are neighborhoods of the origin in $\mathbb{R}^f$ and $\mathbb{R}^{n-d_i-f}$ respectively, with E_2 containing a ball of radius $> t^{1/2-\varepsilon}$. Write $Q_{A,S_2}(x) = Q_{A,S_2}(x_1, x_2)$, for $x = (x_1, x_2) \in E_1 \times E_2$. Consider the integral

$$\int_{E_1 \times E_2} Q_{A,S_2}(t^{-1/2}(x - B(x) - B_0))e^{-|H(x-B(x)-B_0)|^2/4t}dx$$
$$= \int_{E_1 \times E_2} Q_{A,S_2}(t^{-1/2}B_0, t^{-1/2}(I - B)(x_2))e^{-|H(I-B)x_2|^2/4t}$$
$$\times e^{-|B_0|^2/4t}dx_1\,dx_2 + O(t^{1/2}e^{-|B_0|^2/t}).$$

This integral is clearly $O(\mathrm{Vol}(E_1)e^{-t^{-\varepsilon}/2})$ unless $B_0 = 0$. In the latter case, we reduce to

$$\int_{E_1 \times E_2} Q_{A,S_2}(t^{-1/2}(I - B))(x))\mathrm{e}^{-|H(I-B)(x)|^2/4t}dx_1\,dx_2$$
$$= \mathrm{Vol}(E_1)t^{(n-d_i-f)/2}\det(I - dB_2)^{-1}q_{A,S_2} + O(t^{(1+n-d_i-f)/2}),$$

where $q_{A,S_2} = \int_{\mathbb{R}^{n-d_i-f}} Q_{A,S_2}(x)e^{-|x|^2/4}dx$.

Let $\beta \in P$, and write $\beta = \beta_n\beta_a\beta_m$. Let $x \in W_P'$, and write $x = (n, a(x), mK_M)$. Then we may write $\beta x = (\beta_n n^{\beta_a\beta_m}, \beta_a a(x), \beta_m mK_M)$. Let $\tilde{\beta}$ denote the image of β in $N_i \backslash P$. B is a model for $\tilde{\beta}$. The affine approximation of the action of $\tilde{\beta}$ has a nonzero vector B_0 in the preceding notation, unless $\beta_a = 1$ and β_m in elliptic. Let λ_+^i and λ_-^i denote respectively the largest and smallest weights of A on $\mathbf{n}^i$ (the ordering being determined by $a(g)^{\lambda_+^i} > a(g)^{\lambda_-^i}$, for some choice of $g \in \bar{\pi}_i^{-1}(\theta_\alpha)$. For $t^{1/2} < a^{-\lambda_-^{i+1}(1+\sigma^{i+1})}$ and ε a function of σ, if there is a not a nonzero B_0 as above, then after a change of base point for W_P', we may assume that $\beta_n \in N_i$. From these considerations we may deduce the following lemma. Suppose that E_1 is a coordinate neighborhood in an f-dimensional fixed submanifold of $\tilde{\beta}$. Fix a product structure on a tubular neighborhood of E_1 in $N_i \backslash X$, and consider a neighborhood of the form $E_1 \times E_2$.

(A.1.6) Lemma. *For* $t^{1/2} < \inf_{x \in E}\{a(x)^{-\lambda_-^{i+1}(1-\sigma^{i+1})} : x \in E\}$, *the integral* $\int_{E_1 \times E_2}(x - \tilde{\beta}x)^R F_{A,S_2}(x, \tilde{\beta}x, t)dx$ *is* $O(e^{-t^{-\varepsilon}/2})$ *unless* $\beta_a = 1$ *and a basepoint can be chosen so that* $\beta_n \in N_i$ *and* $\beta_m \in K_M$. *If all these conditions are satisfied, then*

$$\int_{E_1 \times E_2}(x - \tilde{\beta}x)^R F_{A,S_2}(x, \tilde{\beta}x, t)dx$$
$$= t^{(n-d_i-f+|R|)/2}\left(\lim_{t \to 0} t^{-(n-d_i-f+|R|)/2}\right)\int_{E_1 \times E_2}(x - \tilde{\beta}x)^R F_{A,S_2}(x, \tilde{\beta}x, t)dx$$
$$+ O(t^{(n-d_i-f+|R|+1)/2})$$
$$= O(\mathrm{Vol}(E_1)t^{(n-d_i-f+|R|)/2}).$$

A.2. Now we estimate the infinite sums arising in $H_1^{i,\alpha} + H_0^{i,\alpha}$. Assume that $a(g)^{-2\lambda'_-(1+(d'+2)\sigma')} < t$, and recall that $a(g)^{-2\lambda'_-(1+(d'+2)\sigma')} < a(g)^{-2\lambda'_+}$. Let L^i denote the lattice dual to Γ^i, and let L^i_β denote $Ad\beta$ invariant sublattice. Set

$$f_{B,\beta}(g) = \sum_{\mu \in L^i_\beta} (t^{1/2}\mu^g)^B e^{-t|\mu^g|^2} e^{i2\pi\mu\cdot\beta_u}, \qquad \text{and}$$

$$f^1_{B,\beta}(g) = f_{B,\beta}(g) - \delta_{B,0},$$

It is useful for our computations to estimate $|f_{B,\beta}|$ and $|f^1_{B,\beta}|$ in terms of t, given the above constraints on t. We consider the 3 cases.

(A.2.1) (i) $t \geqq a(g)^{-2\lambda'_-}$,

(ii) $a^{-2\lambda'_-(1+(d'+2)\sigma')} < t \leqq a(g)^{-2\lambda'_+}$,

(iii) $a(g)^{-2\lambda'_+} < t < a(g)^{-2\lambda'_-}$.

In case (i),

$$|f_{B,\beta}| \leqq c,$$

for some positive constant c, and $|f^1_{B,\beta}|$ is $O(t^N)$, for all N. In case (ii), we apply the Poisson summation formula to find

$$|f_{B,\beta}| \leqq c \det(g)^{-1} t^{-d'/2} \leqq c a^{-\lambda'_- d'} t^{-d'/2}$$

$$\leqq c t^{-\sigma' d'(d'+2)/(1+(d'+2)\sigma')}.$$

Consider now case (iii). Write $N^i_\beta{}^* = \oplus N^i_{\beta\lambda}$, where $a(g)$ acts on $N^i_{\beta\lambda}$ by $a(g)^\lambda$. Let $N^i_{\beta+}$ (respectively $N^i_{\beta-}$) denote the direct sum of eigenspaces $N^i_{\beta\lambda}$, with $ta(g)^{2\lambda} \geqq 1$ (respectively $ta(g)^{2\lambda} < 1$). Let $l^i_{\beta-} = \dim N^i_{\beta-}$. Let $L^i_{\beta+} := L^i_\beta \cap N^i_{\beta+}$ and $L^i_{\beta-} := L^i_\beta \cap N^i_{\beta-}$. The lattice $L^i_{\beta+} \oplus L^i_{\beta-}$ is of finite index in L^i_β (because the decomposition is with respect to a rational torus action). Hence we may apply the Poisson summation formula to $f_{B,\beta}$ viewed as a sum over $L^i_{\beta-}$ to obtain

$$|f_{B,\beta}| \leqq c a^{-\lambda'_- l'_\beta -} t^{-l'_-/2} \leqq c t^{-\sigma' l'_\beta - (d'+2)/(1+(d'+2)\sigma')}.$$

Thus, we may conclude:

(A.2.2) Lemma. *If* $a(g)^{-2\lambda'_-(1+(d'+2)\sigma')} < t$, *then*

$$|f_{B,\beta}(g)| \leqq c t^{-\sigma' d'(d'+2)/(1+(d'+2)\sigma i)},$$

for some positive constant c.

A.3. Let $\beta \in P$. We now exploit the above analysis to compute the trace of our approximation $C(e_n)\tau\beta^* D(H_1^{i,\alpha}(t) + H_0^{i,\alpha}(t))$. If $\beta = \beta_n\beta_a\beta_m$ is the decomposition of β corresponding to $P = NAM$, then (A.1.6) implies that the trace is rapidly decreasing unless $\beta_a =$ identity and β_m, a base point may be chosen so that $\beta_m \in K_m$

and $\beta_n \in N_i$. From (5.9) we have further reduction

$$\sum_{\beta\in\bar{\beta}} \sum_{\mu\in L^i} \int_{\Gamma_i\setminus N_i} \operatorname{tr} C(e_n)\tau\beta^* DH^i_\mu(t,\beta x,x)dx_n = \sum_{\beta\in\bar{\beta}} \sum_{\mu\in L^i_{\bar{\beta}}} \int_{\Gamma_i\setminus N_i} \operatorname{tr} C(e_n)\tau\beta^* DH^i_\mu(t,\beta x,x)dx_n,$$

We return to mass considerations.

(A.3.1) Lemma. *In a neighborhood of an f-dimensional fixed submanifold of β, β^* has mass $\leqq n-f$.*

Proof. β^* fixes the tangent space of the fixed submanifold. Hence the expression of the action of β^* in terms of Clifford multiplication contains no factors corresponding to tangent directions. The result now follows because the action of β^* ($\approx \mathrm{ad}(\beta)$) is locally constant.

(A.3.2) Claim. The mass of $\beta^* \tilde{h}^i_{\ell,s,s',A,B} \leqq 2l - s - |A| - 2|B| + n - f$.

The mass of $\beta^* \tilde{h}^0_{\ell,s,s',A,0} \leqslant 2\ell - s - |A| + n - f$, with equality only if $s = 0$. β^* may then increase the mass by $n - f$. This implies the first claim.

Proof. (cf. [26, (4.5.2)]). This follows from an easy induction. One merely observes that $\tilde{h}^i_{\mu,\ell}$ is constructed from $\tilde{h}^i_{\mu,\ell-1}$ by an operator derived from a conjugate of $(\Delta - |\mu - i\Gamma|^2)$ that raises mass by at most two. The terms in $(\Delta - |\mu - i\Gamma|^2)$ of degree k in μ have mass $\leqq 1 - k$; moreover, one derivative (mass one) is required to add each factor of v and each extra (s) factor of $(\|v\|^2 + |\mu - i\Gamma|^2 - \lambda)^{-1}$. β^* may then increase the mass by $n - f$. This implies the first claim. The second follows from the observation that for $i = 0$, $(\|v\|^2 + |\mu - i\Gamma|^2 - \lambda)^{-1}$ reduces to $(\|v\|^2 - \lambda)^{-1}$. As our coordinate system is normal, differentiating $(\|v\|^2 - \lambda)^{-1}$ once lowers the mass. To restore this loss of mass, absorbs at least one derivative. The second claim follows.

Next we shall estimate the trace of the term arising in (A.1.5):

(A.3.3)
$$t^{(2\ell+2s-n+d_i-|A|-|S_2|)/2} \operatorname{tr} \beta^* \tau F_{A,S_2} \tilde{h}^i_{\ell,s,s',A,B}(x,x') \sum_\mu (\mu^x)^B P_{s',S_2} e^{-t|\mu^x - i\Gamma|^2} e^{2\pi i \mu\cdot\beta_u}.$$

Define $\hat{\mathbf{n}}_i$ and $\hat{d}_i$ as in [26, (4.5.4)] (where $\hat{\mathbf{n}}_i$ was denoted $\hat{\mathbf{u}}_i$). $\hat{\mathbf{n}}_i$ is a subspace of $\mathbf{n} \cap \mathbf{n}_i^\perp$ consisting of vectors which do not annihilate the dual of $\mathbf{n}_i$ under the coadjoint representation. Let $\{W_a\}$ be a basis of $\mathbf{n}_i$ and $\{Z_b\}$ a basis of $\mathbf{n}_i \oplus \hat{\mathbf{n}}_i$; then $\hat{d}_i \leqq \dim \hat{\mathbf{n}}_i$ is defined so that

(A.3.4) $$\text{mass} \prod_{a\in A} \Gamma_{W_a} \leqq |A| + \min\{|A|, d_i + \hat{d}_i/2\}, \text{ and}$$

(A.3.5) $$\text{mass} \prod_{j\in J} C(Z_j) \prod_{i\in I} \Gamma_{W_i} \leqq |J| + |I| + \min\{|I|, d_i + \hat{d}_i/2 - |J|\}.$$

We will use these relations repeatedly without further reference.

Write the dimension f of the fixed point set under β as

(A.3.6) $$f = \varphi + d_i^\beta + \hat{d}_i^\beta,$$

where d_i^β (respectively $\hat{d}_i^\beta$) denotes the dimension of the subspace of $\mathfrak{n}_i$ (respectively $\hat{\mathfrak{n}}_i$) fixed by $\mathrm{Ad}(\beta)$. Expand P_{s',S_2} as

$$P_{s',S_2} = P_{s',S_2,F,K}\,\Gamma^F(\mu^g)^K, \text{ with } |F| + |K| = 2s' - |S_2|\,,$$

and write

$$\begin{aligned} e^{-t|\mu^g - i\Gamma|^2} &= \sum_{b=0}^{\infty} e^{-t|\mu^g|^2} t^b (2i\mu^g\cdot\Gamma + \Gamma\cdot\Gamma)^b/b! \\ &= \sum_{b=0}^{\infty} e^{-t|\mu^g|^2} t^b \sum_{r=0}^{\infty} (2i\mu^g\cdot\Gamma)^r(\Gamma\cdot\Gamma)^{b-r}/r!(b-r)! \\ &= \sum_{b=0}^{\infty} e^{-t|\mu^g|^2} t^b \sum_{r=0}^{b}\sum_{|R|=r} (\mu^g)^R G_R(\Gamma\cdot\Gamma)^{b-r}/r!(b-r)!\,, \end{aligned}$$

where G_R is a homogeneous polynomial in Γ of degree $|R|$. Then (A.3.3) equals

$$\begin{aligned} \text{(A.3.7)}\quad &\operatorname{tr}\beta^*\tau F_{A,S_2}\tilde{h}^i_{\ell,s,s',A,B}\, t^{(2\ell+2s-n+d_i-|A|-|S_2|)/2} \\ &\quad\times\sum_{\mu\in L} P_{s',S_2,F,K}\,\Gamma^F(\mu^g)^{B\cup K} e^{-t|\mu^g - i\Gamma|^2} e^{2\pi\imath\mu\cdot\beta_u} \\ &= \operatorname{tr}\beta^*\tau F_{A,S_2}\tilde{h}^i_{\ell,s,s',A,B}\, t^{(2\ell+2s-n+d_i-|A|-|S_2|)/2} \\ &\quad\times P_{s',S_2,F,K}\,\Gamma^F\sum_{b=0}^{\infty}\sum_{\mu\in L} e^{2\pi\imath\mu\cdot\beta_u} e^{-t|\mu^g|^2}\sum_{r=0}^{b} t^{b-(r+|B|+|K|)/2} \\ &\quad\times\sum_{|R|=r}(t^{1/2}\mu^g)^{R\cup B\cup K} G_R(\Gamma\cdot\Gamma)^{b-r}/r!(b-r)! \\ &= \operatorname{tr}\beta^*\tau F_{A,S_2}\tilde{h}^i_{\ell,s,s',A,B} P_{s',S_2,F,K}\,\Gamma^F\sum_{b=0}^{\infty}\sum_{r=0}^{b}\sum_{|R|=r} f_{R\cup B\cup K} \\ &\quad\times G_R(\Gamma\cdot\Gamma)^{b-r} t^{(2\ell+2s-n+d_i-|A|+2b-r-|B|-|K|-|S_2|)/2}/r!(b-r)!\,. \end{aligned}$$

From (A.2.2), we have for $t > a(g)^{-2\lambda^i_{\beta-}(1+(d'+2)\sigma')}$

$$\text{(A.3.8)}\qquad a^{-\hat{d}_i^\beta\lambda^i_{\beta-}/2}|f_{R\cup B\cup K}| \leqq c_3 t^{\hat{d}_i^\beta/4 - 2\sigma'(d')^2)}$$

Hence, in estimating the integral of (A.3.7) over $\Gamma_N N_i\backslash N$, we may estimate $\mathrm{Vol}(\Gamma_N N_i\backslash N_\beta)(x)|f_{R\cup B\cup K}|$ by the above power of t. The factor $a(x)^{-\hat{d}_i^\beta\lambda^i_{\beta-}/2}$ enters as an upper bound on the volume of $\mathrm{Vol}(\Gamma_N N_i\backslash N_\beta)(x)$.

Recall (A.3.2) that

$$\operatorname{mass}\beta^*\tilde{h}^i_{\ell,s,s',A,B} \leqq 2\ell - s - |A| - 2|B| + \varphi + d_i^\beta + \hat{d}_i^\beta\,.$$

Hence,

$$\text{(A.3.9)}\qquad \text{mass of } \beta^*\tilde{h}^i_{\ell,s,s',A,B} P_{s',S_2,F,K}\,\Gamma^F G_R(\Gamma\cdot\Gamma)^{b-r} \leqq m\,,$$

where

$$m = 2\ell - s - |A| - 2|B| + n - f + 2b - r + |F| + \min\{2b - r + |F|, d_i^\beta + \hat{d}_i^\beta/2\}\,.$$

Thus,

$$\operatorname{tr} \tau\beta^* \tilde{h}^i_{\ell, s, s', A, B} P_{s', S_2, F, K} \Gamma^F G_R(\Gamma \cdot \Gamma)^{b-r} = O(|x - \tilde{\beta}x|^{\max\{n-m, 0\}}) .$$

Equality occurs in (A.3.9) only if $2b - r + |F| = d_i^\beta + \hat{d}_i^\beta/2$.

When $i = 0$, this reduces to

$$\operatorname{tr} \tau\beta^* \tilde{h}^0_{\ell, s, s', A, 0} P_{s', S_2} = O(|x - \tilde{\beta}x|^{\max\{f - 2\ell + s + |A| + \varepsilon_s, 0\}}),$$

with $\varepsilon_0 = 0$ and $\varepsilon_s = 1$ for $s \neq 0$. Here we justify the insertion of the ε_s term with the fact that the second mass inequality in (A.3.2) is strict if $s > 0$.

Let E_1 and E_2 be as in (A.1.6). Applying (A.1.6), we obtain

$$\begin{aligned}
\text{(A.3.10)} \quad & \int_{E_1 \times E_2} F_{A, S_2}(x, \tilde{\beta}x) t^{2\ell + 2s - n + d_t - |A| + 2b - r - |B| - |K| - |S_2|)/2} \\
& \quad \times |f_{R \cup B \cup K}| \operatorname{tr} \tau\beta^* \tilde{h}^i_{\ell, s, s', A, B} P_{s', S_2, F, K} \Gamma^F G_R(\Gamma \cdot \Gamma)^{b-r} dx \\
& \leqq \operatorname{Vol}(E_1) O(t^{(2b - r - |B| - |K| + 2\ell + 2s + d_t^\beta - |A| - |S_2| - f + n - m)/2}) t^{-\sigma'(d')^2} \\
& \leqq \operatorname{Vol}(N \setminus E_1) O(t^{(|B| + s + d_t^\beta + \hat{d}_t^\beta/2 - \min\{2b - r + |F|, d_t^\beta + \hat{d}_t^\beta/2)/2}) t^{-2\sigma'(d')^2} ,
\end{aligned}$$

where we have used $|K| + |F| + |S_2| = 2s' \leqq 2s$.

The quantity in (A.3.10) is $\leqq O(t^{1 - 2\sigma'(d')^2})$ unless $s = |B| = |F| = 0$ and $2b - r \geqq d_i^\beta + \hat{d}_i^\beta/2$. Moreover, if $2b - r > d_i^\beta + \hat{d}_i^\beta/2$, then (A.3.9) is $O(t^{1 - 2\sigma'(d')^2})$, because the mass inequality preceding the definition of m is strict in this case. In particular, only those terms with the maximum possible mass can contribute to $\operatorname{tr} \beta^* \tau e^{-t\Delta}(x, \beta x)$, the remaining terms are rapidly decreasing. We collect these results into a Proposition. Let $\tilde{h}^i_{\ell, A} := \tilde{h}^i_{\ell, 0, 0, A, 0}$, and $F_A = F_{A, S_2}$. We continue to suppose $\beta = \beta_n \beta_m$ with $\beta_m \in K_m$.

(A.3.11) Proposition. *If* $a(x)^{-2\lambda'(1 + d' + 2)\sigma'} < t < a^{-2\lambda'_-/(1 + \sigma'/2)}$,

$$\sum_{\beta \in \tilde{\beta}} \int_{\tilde{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* e^{-t\Delta} d\mathrm{vol} = \sum_{\beta \in \tilde{\beta}} \int_{\tilde{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* (H_1^i(t) + H_0^i(t)) d\mathrm{vol} + O(t) .$$

If $i \neq 1$, or if P is not maximal then

$$\sum_{\beta \in \tilde{\beta}} \int_{\tilde{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* (H_1^i(t) + H_0^i(t)) d\mathrm{vol} = O(\operatorname{vol}(N \setminus E_1) t^{\sigma/4}).$$

If $i = 1$, then

$$\begin{aligned}
& \sum_{\beta \in \tilde{\beta}} \int_{\tilde{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* (H_1^1(t) + H_0^1(t)) d\mathrm{vol} \\
& \quad = \sum_{\beta \in \tilde{\beta}} \int_{\tilde{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \beta^* \tau F_A \tilde{h}^1_{\ell, A} \sum_{r=0} \sum_{|R| = r} f_R G_R (\Gamma \cdot \Gamma)^{(-r + d_t^\beta + \hat{d}_t^\beta/2)/2} \\
& \qquad \times t^{(2\ell - n + 2d_1^\beta - |A| + \hat{d}_1^\beta/2)/2} / r! [(-r + d_1^\beta + \hat{d}_1^\beta/2)/2]! d\mathrm{vol} .
\end{aligned}$$

If $t \leqq a(x)^{-2\lambda_-^1}(1 + (d_1^\beta + 2)\sigma)$,

$$\sum_{\beta \in \bar{\beta}} \int_{E_1 \times E_2} \operatorname{tr} \tau\beta^* e^{-t\Delta} d\mathrm{vol} = \sum_{\beta \in \bar{\beta}} \int_{E_1 \times E_2} \operatorname{tr} \tau\beta^* H_0^0(t) d\mathrm{vol} + O(t)$$

$$= \lim_{t \to 0} \sum_{\beta \in \bar{\beta}} \int_{E_1 \times E_2} \operatorname{tr} \tau\beta^* H_0^0(t) d\mathrm{vol} + O(t^{1/2})$$

$$\leqq O(\mathrm{Vol}(E_1)).$$

Proof. For $t < a^{-2\lambda_-^1/(1+\sigma^1/2)}$, the equality

$$\sum_{\beta \in \bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* e^{-t\Delta} d\mathrm{vol}$$

$$= \sum_{\beta \in \bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* \left(\sum_{b \leqq i} H_1^b(t) + H_0^i(t) \right) d\mathrm{vol} + O(t)$$

is merely a restatement of (A.1.4). The first assertion then follows from the simple estimate

$$\sum_{\beta \in \bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1 \times E_2)} \operatorname{tr} \tau\beta^* \sum_{b < i} H_1^b(t) d\mathrm{vol} = O(t),$$

for t in the given range. This is an immediate consequence of the estimate given for $|f_{B,\beta}^1|$ in case (A.2.1 ii).One merely substitutes this estimate into (A.3.8) and (A.3.10).

We are left to justify the claim that the trace is rapidly decreasing unless $i = 1$ and P is maximal. This is proved in [26, (4.4.9), (4.4.11), (5.1.17)] for $\beta =$ identity. The general case is the same, but for the convenience of the reader, we will summarize the arguments involved. Suppose that $i > 1$. The space $\mathbf{n}_1$ and hence $\mathbf{n}_1^\beta$ contains a vector e of weight λ_+^1 on which K_M acts trivially (see [26, (4.4.11)]). The factor $C(e)$ can only enter the maximal mass terms in $H_1^i + H_0^i$ from terms of the form $\Gamma_W \Gamma_W$, where $W \in \mathbf{n}_1$. Let v denote the weight of W. We may write $\Gamma_W = \Gamma_W^2 + \Gamma_W^1$, where Γ_W^2 only contains Clifford factors (two) from $\mathbf{n}_1^\perp$, and Γ_W^1 contains one Clifford factor from $\mathbf{n}_1$. According to [26, (4.4.9)], $\Gamma_W^1 \Gamma_W^1$ cannot contribute a factor of $C(e)$. Hence, $C(e)$ may only enter through $\Gamma_W^1 \Gamma_W^2$. Write $\Gamma_W^2 = \sum a_{kj} C(Z_k) C(Z_j)$, for weight vectors Z_1. The sum of the weights of any pair (Z_k, Z_j) for which $a_{kj} \neq 0$, is v. If neither element of the pair lies in $\mathbf{n}_i$, but the sum of their weights is $v > (1 + \sigma)\lambda_+^2$, then we see that the function $a^{-\hat{d}_i \lambda^1}$ which we used as an upper bound on the volume of $\mathrm{Vol}(\Gamma_N N_i \backslash N)(x)$ can be improved to $\mathrm{Vol}(\Gamma_N N_i \backslash N)(x) \leqq a^{-(\hat{d}_i + \sigma)\lambda_-^1}$. The extra factor of $a^{-\sigma\lambda_-^1}$ is sufficient to swamp the $O(t^{-2\sigma^1(d^1)^2})$ upper bound (A.3.8) for $\mathrm{Vol}(\Gamma_N N_i \backslash N)(x) |f_B|$ in this case. If one element of the pair lies in $\mathbf{n}_i$ and the other lies in $\mathbf{n}_i$ or in $\hat{\mathbf{n}}_i$, then its contribution to $H_1^i + H_0^i$ via $\Gamma_W^1 \Gamma_W^2$ is not maximal mass. If one element lies $\mathbf{n}_i$ and the other lies in the complement of $\mathbf{n}_i \oplus \hat{\mathbf{n}}_i$, then our volume estimate can be improved as before. Thus we are reduced to the case $i = 1$. Suppose that P is not maximal. Recall that we may pick a basis for $\mathbf{a}_P^*$ so that the roots of $\mathbf{a}_P$ on $\mathbf{n}$ are of the form $\gamma_i \pm \gamma_j$, and γ_i (with possible zero multiplicity of some roots). Suppose that P is not maximal. Then there exists a maximal parabolic subgroup Q such that $\mathbf{a}_P$ is contained in its $\mathbf{g}_l$ summand. Using the K_Q equivariant isomorphism between $\mathbf{p}_l(Q)$ and the center

of the unipotent radical of Q, we may find $r_P := \dim A_P\ \mathbf{a}_P$ weight vectors $e_1, \ldots, e_{r_P}$ which are fixed by K_P. Moreover their weights are strongly orthogonal (see [20, 11.6]), and therefore span $\mathbf{a}_P^*$. We denote them $2\gamma_i$.

Because these vectors are fixed by K_P, we may argue as above to deduce that each $e_i \in \mathbf{n}_1$ if $C(e_i)$ can appear in a maximal mass term. This implies that $2\gamma_i \in \phi_\sigma^1$, contradicting assumption (A.1.1) applied to the roots $\gamma_i - \gamma_{i+1}$. (Recall that the real and rational root system of G is of type C or BC and therefore $\gamma_i - \gamma_j$ occurs as a root.) □

A.4 We now modify the above discussion to consider

$$\lim_{R\to\infty} \int_{e^{-4R/(1-\varepsilon)}}^{\infty} \operatorname{tr} C(e_n)\tau D_\rho \beta^* e^{-t\Delta}\, dt$$

instead of $\operatorname{tr} \tau\beta^* e^{-t\Delta}$, in a neighborhood of a point x where $a(x)^{-\alpha_1} \sim e^{-2R}$. We have the following analogue of (A.3.11).

(A.4.1) Proposition. *If* $a(x)^{-2\lambda_-^1(1+d^1+2)\sigma^1)} < t < a^{-2\lambda_-^1/(1+\sigma^1/2)}$,

$$\sum_{\beta\in\bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1\times E_2)} \operatorname{tr} C(e_n)\tau\beta^* D_\rho e^{-t\Delta} d\mathrm{vol}$$
$$= \sum_{\beta\in\bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1\times E_2)} \operatorname{tr} C(e_n)\tau\beta^* D_\rho (H_1^i(t) + H_0^i(t)) d\mathrm{vol} + O(t)\,.$$

If $i \neq 1$ *or* P *is not maximal, then*

$$\sum_{\beta\in\bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1\times E_2)} \operatorname{tr} C(e_n)\tau\beta^* D_\rho (H_1^i(t) + H_0^i(t)) d\mathrm{vol} = \mathrm{Vol}(N\setminus E_1) O(t^{-2\sigma^1(d^1)^2})\,.$$

If $i = 1$,

$$\sum_{\beta\in\bar{\beta}} \int_{\bar{\pi}_i^{-1}(E_1\times E_2)} \operatorname{tr} C(e_n)\tau\beta^* D_\rho (H_1^1(t) + H_0^1(t)) d\mathrm{vol} = \mathrm{Vol}(N\setminus E_1) O(t^{-1-2\sigma(d^1)^2})\,.$$

Proof. The proof, which we omit, is essentially the same as (A.3.12). One merely counts the extra mass coming from $C(e_n)D_\rho$. This can have mass as high as four arising from terms of the form $C(e_n)C(e_i)\Gamma_{W_i}$. This term is similar to $\Gamma\cdot\Gamma$ and therefore enters (A.3.9) and the definition of m by replacing $2b$ by $2b+2$. This raises $n-m$ by 2 at most, and lowers the power of t by at most -1. The remaining terms are similar.

Observe that any term in the approximation to $\operatorname{tr} C(e_n)\tau\beta^* D_\rho e^{-t\Delta}$ which is $O(t^a)$ for $a > -1$ makes an exponentially decreasing contribution to the integral $\int_{e^{-4R/(1-\varepsilon)}}^{e^{-\varepsilon R}} \operatorname{tr} C(e_n)\tau\beta^* D_\rho e^{-t\Delta}$. Hence, (A.4.1) reduces the computation of this integral (in the limit as $R\uparrow\infty$) to the evaluation of $\int_{e^{-4R/(1-\varepsilon)}}^{e^{-3R}} \operatorname{tr} C(e_n)\tau\beta^* D_\rho (H_1^1 + H_0^1) dt$, for some $\varepsilon > 0$; in fact, we clearly need only include the maximal mass terms in the integrand. In the notation of §5, these maximal mass terms are given by the maximal mass terms in

$$\sum_{\mu\in\Lambda^*} \operatorname{tr} C(e_n)\tau D_\rho \beta^* e^{-t\Delta_h} e^{-t\Delta_E} e^{-t\Delta_u} T_\mu$$

(see [26, (5.2.2)]).

Theorem (A.4.1) may be viewed as a reduction of the trace computation to maximal rational parabolic subgroups. The integral $\int_{e^{-rR}}^{\infty} \operatorname{tr} \tau C(e_n)\beta * D_\rho e^{-t\Delta}\, dt$ can now be evaluated in several different ways. We use the construction $H_{P'}^0$ discussed in §3. This allows us to frame our constructions entirely in terms of rational maximal parabolic subgroups.Alternatively, we may continue as before performing Fourier expansions and obtaining the associated estimates showing that various terms are decreasing like $a(x)^{-\sigma' \alpha_i}$, for some i. The equality of these two expressions then implies the convergence as $j \to \infty$ of $\sum_{b \in C_P(\beta)} \int_{\partial B_j} \int_{e^{-\nu}}^{\infty} \operatorname{tr} C(e_n)\tau b * H_{P'}^0(t)\, dt\, dx$. This may also be deduced from the computation of this integral given in §7.

References

1. Andrianov, A.: Quadratic Forms and Hecke Operators. Berlin Heidelberg New York: Springer 1987
2. Ash, A., Mumford, D., Rapoport, M., Tai, Y.: Smooth Compactifications of Locally Symmetric Varieties. Brookline: Math. Sci. Press 1975
3. Arthur, J.: Characters, harmonic analysis, and an L^2-Lefschetz Formula. Proc. Sympos. Pure Math., vol 48, Gmer. Math. Soc., Providence, R.I. (1987), 167–179
4. Arthur, J.: The L^2-Lefschetz numbers of Hecke operators. Invent. Math. **97**, 257–290 (1981)
5. Atiyah, M.F., Bott, R.: A Lefschetz fixed point formula for elliptic complexes: II, Applications.Ann. Math. **88**, 451–491 (1968)
6. Atiyah, M.F., Bott, R., Patodi, V.K.: On the heat equation and the index theorem. Invent. Math. **19**, 279–330 (1973)
7. Atiyah, M.F., Bott, R., Shapiro, A.: Clifford modules. Topology **3** (Suppl. 1), 3–38 (1964)
8. Borel, A.: Density and maximality of arithmetic subgroups, J. Reine Angew. Math. **224**, 78–89 (1966)
9. Borel, A.: Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. J. Differ. Geom. **6**, 543–560 (1972)
10. Borel, A., Casselman, W.: L^2-cohomology of locally symmetric manifolds of finite volume. Duke Math. J. **50**, 625–647 (1983)
11. Borel, A., Serre, J.-P.: Corners and arithmetic groups. Comment.Math. Helv. **48**, 244–297 (1974)
12. Faraut, J., Satake, I.: The functional equation of zeta distributions associated with formally real Jordan algebras, Tôhoku Math. J. **36**, 469–482 (1984)
13. Goresky, M., MacPherson, R.: (a) Weighted cohomology of Satake compactifications. (Preprint) (b) Lefchetz Numbers of Hecke correspondences. (Preprint, August 1990)
14. Harder, G.: A Gauss-Bonnet formula for discrete arithmetically defined groups.Ann. Sci. Ec. Norm. Supér. **4**, 409–455 (1971)
15. Hitchin, N.: Harmonic Spinors, Adv. Math. **14**, 1–55 (1974)
16. Matsushima, Y., Murakami, S.: On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds.Ann. Math. **78**, 417–449 (1963)
17. Moscovici, H.: (a) The signature with local coefficients of locally symmetric spaces. Tôhoku Math. J. 37, 513–522 (1985). (b) Lefschetz formulae for Hecke operators. In: Herb, R. et al. (eds.) Lie Group Representations, III. (Lect. Notes Math., vol. 1077, pp. 321–358) Berlin Heidelberg New York: Springer 1984
18. Muller, W.: L^2-Index Theory, Eta Invariants and Values of L-functions. Contemp. Math. **105**, 145–189 (1990)
19. O'Meara, O.: Introduction to Quadratic Forms.Berlin Heidelberg New York: Springer 1963
20. Saper, L., Stern, M.: L_2-cohomology of arithmetic varieties. Ann. Math. **132**, 1–69 (1990)
21. Satake, I.: Algebraic Structures of Symmetric Domains, Princeton, NJ: Princeton University Press 1980
22. Satake, I.: Special values of zeta functions associated with self-dual homogeneous cones. In: Hano, J. et al. (eds.) Manifolds and Lie Groups, Papers in Honor of Yozo Matsushima, pp. 359–384. Boston: Birkhäuser 1981

23. Sato, F.: Zeta functions in several variables associated to prehomogeneous vector spaces I: functional equations. Tôhoku Math. J. **34**, 437–483 (1982)
24. Sato, M., Shintani, T.: On zeta functions associated to prehomogeneous vector spaces. Ann. Math., Il. Ser. **100**, 131–170 (1974)
25. Shintani, T.: On the zeta functions associated to the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo **22**, 25–66 (1975)
26. Stern, M.: L^2-index theorems on locally symmetric spaces. Invent. Math. **96**, 231–282 (1989)
27a. Stern, M.: Eta invariants and hermitian locally symmetric spaces. J.Differ. Geom. **31**, 771–789 (1990)
27b. Stern, M.: Index theory for certain complete Kahler manifolds. J. Differ. Geom. **37**, 467–503 (1993)
[BV] Berline, N., Vergne, M.: A computation of the equivariant index of the Dirac operator, Bull. Soc. Math. France, **113**, 305–345 (1985)
[F] Fedosov, B.V.: Analytic formulas for the index of elliptic operators, Trans. Moscow Math. Soc. **30**, 159–240 (1974)
[H] Harder, G.: A Gauss-Bonnet formula for discrete arithmetically defined groups, Ann. Sci. E.N.S. **4**, 409–455 (1971)