Werk

Titel: 4 The topological Hochschild spectrum.

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such that the diagram

$$\begin{array}{ccccc} |sd_{sr}THH_\bullet(M_a(F))| & \leftarrow & \|X_{\bullet,\bullet}(sr)\| & \rightarrow & |sd_{sr}THH_\bullet(F)| \\ \downarrow D_s & & \downarrow \bar{D}_s & & \downarrow D_s \\ |sd_rTHH_\bullet(M_a(F))| & \leftarrow & \|X_{\bullet,\bullet}(r)\| & \rightarrow & |sd_rTHH_\bullet(F)| \end{array}$$

is homotopy commutative in the category of C_r-spaces.

In particular we obtain a homotopy equivalence

$$\operatorname{holim}_{\overleftarrow{D}} |sd_{p^n}THH_\bullet(M_a(F))|^{C_{p^n}} \to \operatorname{holim}_{\overleftarrow{D}} |sd_{p^n}THH_\bullet(F)|^{C_{p^n}} \tag{3.14}$$

cf. Sect. 2 and [BK, XI.5.6].

4 The topological Hochschild spectrum

The topological Hochschild space $THH(F)$, discussed in Sect. 3, turns out to be the the zero'th part of an Ω-spectrum $tHH(F)$. This is true even equivariantly, with respect to the group action of any finite cyclic group induced from the cyclic structure. We use the theory of equivariant Γ-spaces to construct the deloopings. In particular, we obtain deloopings of the fixed sets $THH(F)^{C_r}$; and this is what we are really after.

Let Γ_G^{op} be the category of finite based G-sets. We use the model where the objects are pairs $([n], \rho)$, with $[n] = \{0, \ldots, n\}$ and ρ is an action of G on $[n]$ which keeps 0 fixed. Following Segal (unpublished) and Shimakawa [Sh], a *special* G-equivariant Γ-space (or Γ_G-space) is a functor T_G from Γ_G^{op} to G-spaces with the property that $T_G([0])$ is G-contractible, and such that for each object $A = ([n], \rho)$, the natural map,

$$P_A\colon T_G(A) \to \mathrm{Map}_0(A, T_G([1]))\,, \tag{4.1}$$

is a G-homotopy equivalence. In (4.1) $P_A(t)(a) = P_a(t)$ and $P_a\colon [1] \to A$ is the based G-map with $P_a(1) = a$.

For $G = C_r$, the cyclic group of order r we will show that the r-fold subdivision $|sd_rTHH_\bullet(F)|$ is an equivariant Γ-space.

The basis of the construction is a certain map

$$\theta\colon E\Sigma_k \times THH(F_k) \to THH(F_k)$$

where $F_k = M_k(F)$ is the $(k \times k)$-matrix FSP associated with F.

Since the actual construction is rather technical, we first outline the main steps. The sum operation is produced by wedge $F_a \times F_b \to F_{a+b}$ corresponding to direct sum of matrices. It is not strictly commutative, but as usual it is commutative up to a permutation. The fact that this permutation is "irrelevant" is expressed by the existence of this θ, satisfying certain relations made precise in (4.4). We have to show that we can construct an equivariant Γ-space from these data.

We first rewrite Segals construction of categories with sum diagrams in a form which is uglier than his, but convenient for writing explicit formulas. This gives us an equivariant version of the Eilenberg-Maclane spectrum $H\mathbb{Z}$. The components of

this Γ-space are contractible, but carry free actions of various symmetric groups. The point of this is that these components of $E(\underline{k})$ can act as operations on spaces like $THH(F_k)$, using the map θ.

A Γ-space is a functor from the opposite of the category of finite, based sets. The functor which is going to give THH an equivariant Γ-space structure is first defined on objects. We take its values to be disjoint unions of products. One of the factors is a component of the simple Γ-space we just defined, and the other is of the form $THH(\prod F_{k_i})$. The main problem is to define the value of the functor on morphisms in such a way that we obtain a functor. For instance, given an equivariant, pointed map ϕ: $\underline{k} \to \underline{l}$, we have to produce maps

$$E(\underline{k})_{(a_1,\ldots,a_k)} \times THH(F_{a_1} \times F_{a_2} \times \ldots \times F_{a_k}) \to E(\underline{l})_{(b_1,\ldots,b_l)} \times THH(F_{b_1} \times \ldots \times F_{b_l}) .$$

After taking the union over all components of $E(\underline{k})$, this is to be a map over the already given map $E(\underline{k}) \to E(\underline{l})$. In a certain sense, this says that E acts on THH. The functoriality means, among other things, that if ϕ is invariant under some permutation of its source, this map is also invariant under the same permutation. The wedge sum defines maps

$$THH(F_{a_1} \times F_{a_2} \times \ldots \times F_{a_k}) \to THH(F_{b_1} \times \ldots \times F_{b_l})$$

which do not have this property. Composing with θ we obtain a space of such maps, that is a map

$$F(\phi, \underline{l}) \times THH(F_{a_1} \times F_{a_2} \times \ldots \times F_{a_k}) \to THH(F_{b_1} \times \ldots \times F_{b_l})$$

for some suitable space F. The permutations of the source of ϕ acts on F, so that at least this map is invariant with respect to the diagonal action. The main problem left is book keeping. We do this by specifying maps $E(\underline{k}) \to F(\phi)$, and declare that the action of E on THH is via these maps.

Now we have to make this outline precise. We work simplicially (with cyclic sets). For each group Σ, $E_\bullet\Sigma$ is the cyclic set with

$$\begin{aligned} E_p\Sigma &= \Sigma^{p+1} \\ \partial_i(\sigma_0, \ldots, \sigma_p) &= (\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_p) \\ s_i(\sigma_0, \ldots, \sigma_p) &= (\sigma_0, \ldots, \sigma_i, \sigma_i, \ldots, \sigma_p) \\ t(\sigma_0, \ldots, \sigma_p) &= (\sigma_p, \sigma_0, \ldots, \sigma_{p-1}) . \end{aligned} \tag{4.2}$$

It is contractible and so is each of its fixed set $(E_\bullet\Sigma)^{C_r}$. Indeed,

$$|(E_\bullet\Sigma)^{C_r}| \underset{\simeq}{\overset{D_r}{\to}} |(sd_r E_\bullet\Sigma)^{C_r}| = |E_\bullet\Sigma| . \tag{4.3}$$

Hence $|E_\bullet\Sigma|$ with its induced C_r-action is a model for the equivariant $E_{C_r}\Sigma$, and its quotient (by the diagonal C_r-action) is model for the C_r-equivariant classifying space $B_{C_r}(\Sigma)$.

The symmetric group Σ_k acts on $F_k = M_k(F)$ by conjugation, hence on $THH_\bullet(F_k)$, and there is a simplicial map

$$\theta_\bullet\colon E_\bullet\Sigma_k \times THH_\bullet(F_k) \to THH_\bullet(F_k)$$

with the following properties:

$$\theta_\bullet(g_1 e g_2, t) = g_2^{-1}\theta_\bullet(e, g_1^{-1} t g_1) g_2$$

$$\theta_\bullet(e_1, \theta_\bullet(e_2, t)) = \theta_\bullet(e_1 e_2, t)\,. \tag{4.4}$$

Here and below the product of two simplicial spaces means (without further indication in notation) the diagonal simplicial space. The multiplication in $(E_p\Sigma_k) = \Sigma_k^{p+1}$ is component wise and the right and left action of Σ_k on Σ_k^{p+1} is multiplication on each factor Σ_k. The map $\theta_\bullet$ is defined as follows: Let $(\sigma_0, \ldots, \sigma_p) \in E_p\Sigma_k$ and let

$$f\colon S^{i_0} \wedge \ldots \wedge S^{i_p} \to F_k(S^{i_0}) \wedge \ldots \wedge F_k(S^{i_p})$$

represent an element of $THH_p(F_k)$. Then

$$\theta_p(\sigma_0, \ldots, \sigma_p, f)(u) = (\sigma_p^{-1} f_0(u)\sigma_0, \sigma_0^{-1} f_1(u)\sigma_1, \ldots, \sigma_{p-1}^{-1} f_p(u)\sigma_p)\,. \tag{4.5}$$

Let $\mathbb{N}_0$ denote the non-negative integers and write $P_0[n]$ for the subsets of $[n]$ which contain 0, the basepoint. The set of functions

$$\underline{k}\colon P_0[n] \to \mathbb{N}_0$$

which are additive,

$$\underline{k}(S \cup T) = \underline{k}(S) + \underline{k}(T) \text{ if } S \cap T = \{0\}\,,$$

will be denoted $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$. A G-action on $[n]$ implies a G-action on $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$, and a based G-map $\phi\colon [m] \to [n]$ induces a G-map

$$\phi_\#\colon \mathrm{Hom}(P_0[m], \mathbb{N}_0) \to \mathrm{Hom}(P_0[n], \mathbb{N}_0)$$

by the rule

$$\phi_\#(\underline{k})(S) = \underline{k}(\phi^\#(S))$$

where $\phi^\#(S) = \phi^{-1}(S - \{0\}) \cup \{0\}$.

We shall use $\theta_\bullet$ to exhibit an equivariant Γ-structure on $THH(F)$ for each finite cyclic group C. But first let us recall the general method for constructing Γ-spaces. To each $\underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)$ one associates a space $X(\underline{k})$, and to each morphism ϕ: $[m] \to [n]$ a map $\phi_*\colon X(\underline{k}) \to X(\phi_\#(\underline{k}))$ such that $(\phi\psi)_* = \phi_*\psi_*$. The n'th space in the associated Γ-structure is then

$$X_n = \coprod \{X(\underline{k}) \mid \underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)\}\,.$$

In the equivariant situation one has pairs $A = ([n], \rho)$ where ρ is a C-action on $[n]$ inducing a C-action $\underline{k} \mapsto \underline{k}^g$ on $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$, and one further needs maps from $X(\underline{k})$ to $X(\underline{k}^g)$ to define a C-action on

$$X_A = \coprod \{X(\underline{k}) \mid \underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)\}\,.$$

Let us first consider (non-equivariantly) a Γ-space which only involves permutation groups and whose underlying infinite loop space is homotopy equivalent to the integers. For two based sets of equal cardinality, let $\Sigma(S_1, S_2)$ denote the set of based bijections. It generates a contractible cyclic set (cf. (4.2)) with

$$E_p(S_1, S_2) = \Sigma(S_1, S_2)^{p+1}\,.$$

Given based ordered sets S and T we write $S \coprod T$ for the based concatenation of S and T; it starts with the elements of S and then lists the elements of $T' = T - \{0\}$. We are going to identify totally ordered sets of equal cardinality. In particular $[k] \coprod [l] = [k+l]$ and the bijection $[k] \coprod [l] \to [l] \coprod [k]$ induces the permutation of $[k+l]$ which fixes 0 and has $\sigma(i) = l + i$ for $0 < i \leqq k$ and $\sigma(i) = i - k$ for $k < i \leqq k + l$.

Consider an additive function $\underline{k}: P_0[m] \to \mathbb{N}_0$ as above, and let us write $k_i = \underline{k}\{0, i\}$. A subset $S \in P_0[m]$ has an ordering induced from the standard ordering of $[m] = \{0, 1, \ldots, m\}$, so we have an induced identification

$$\coprod_{i \in S} [k_i] = [\underline{k}(S)] .$$

We define the subspace

$$E_\bullet([m], \underline{k}) \subseteq \prod_{S \in P_0[m]} E_\bullet\left(\coprod_{i \in S} [k_i], [\underline{k}(S)]\right) \tag{4.6}$$

by the condition that $x = (x_S)$ belongs to $E_\bullet([m]; \underline{k})$ if

$$x_S = 1 \text{ when } \operatorname{card}(S_k) = 1,\ S_k = \{i \in S \mid k_i \neq 0\} . \tag{$*$}$$

We make $E_\bullet([\bullet], \underline{k})$ a functor on Γ^{op} as follows. Let $\phi: [m] \to [n]$ be a morphism in Γ^{op} with $\phi_\#(\underline{k}) = \underline{l}$. Then $l_j = \Sigma\{k_i \mid i \in \phi^{-1}(j)\}$ for $j > 0$. For $x \in E_\bullet([m], \underline{k})$ and $T \in P_0[n]$ we have elements

$$\begin{aligned} &x_{\phi^\#\{T\}} \in E_\bullet\left(\coprod_{i \in \phi^\#(T)} [k_i], \underline{l}(T)\right) \\ &x_{\phi^\#\{0,j\}} \in E_\bullet\left(\coprod_{i \in \phi^\#\{0,j\}} [k_i], [l_j]\right), \quad j > 0 \end{aligned} \tag{4.7}$$

where the orderings of $\phi^\#(T)$ and $\phi^\#\{0, j\}$ both are induced from $[m]$. Let

$$\mu: E_\bullet([k_1], [k_2]) \times E_\bullet([l_1], [l_2]) \to E_\bullet([k_1] \coprod [l_1], [k_2] \coprod [l_2])$$

be the obvious sum of permutations. We form

$$\mu((x^{-1}_{\phi^\#\{0,j\}})_{j \in T}) \in E_\bullet\left(\coprod_{j \in T} [l_j], \coprod_{j \in T} \coprod_{i \in \phi^\#\{0,j\}} [k_i]\right).$$

We can identify the indexing set of pairs

$$(j, i) \in T \times \phi^\#(T), \quad \phi(i) = j$$

with $\phi^\#(T)$ via projection onto the second factor. This gives rise to a bijection

$$\sigma_{\phi,T}: \coprod_{j \in T} \coprod_{i \in \phi^\#\{0,j\}} [k_i] \to \coprod_{i \in \phi^\#(T)} [k_i]$$

or in other words, a permutation of $[\underline{l}(T)]$. We define

$$E_\bullet(\phi)(x)_T = x_{\phi^\#(T)} \circ \sigma_{\phi,T} \circ \mu((x^{-1}_{\phi^\#\{0,j\}})_{j \in T}) \tag{4.8}$$

and note that $E_\bullet(\phi)(x)$ satisfies (*) above when x does, so that

$$E_\bullet(\phi)\colon E_\bullet([m], \underline{k}) \to E_\bullet([n], \phi_\#(\underline{k}))\ .$$

We must still check that $E(\psi)\circ E(\phi) = E(\psi\circ\phi)$ when $\psi\colon [n]\to[r]$ is a further morphism in Γ^{op}. We leave the verification to the reader, but notice that it uses the commutative diagram (where $U\in P_0[r]$):

$$\begin{array}{ccc}
\coprod_{v\in U}\ \coprod_{\mu\in\psi^s\{0,v\}}\ \coprod_{i\in\phi^s\{0,\mu\}} [k_i] & \xrightarrow{\coprod \sigma_{\phi,\psi^s\{0,v\}}} & \coprod_{v\in U}\ \coprod_{i\in\phi^s\psi^s\{0,v\}} [k_i] \\
\downarrow \sigma_{\psi,U} & & \downarrow \sigma_{\psi\phi,U} \\
\coprod_{\mu\in\psi^s(U)}\ \coprod_{i\in\phi^s\{0,\mu\}} [k_i] & \xrightarrow{\sigma_{\phi,\psi^s(U)}} & \coprod_{i\in\phi^s\psi^s(U)} [k_i]\ .
\end{array}$$

We can now define a Γ-space by letting $X(\underline{k})$ be the geometric realization of $E_\bullet([m], \underline{k})$, i.e.

$$E_n = \coprod \{|E_\bullet([n]; \underline{k})| : \underline{k}\in \mathrm{Hom}(P_0[n], \mathbb{N}_0)\}\ .$$

This is precisely the classifying space of G. Segal's category of sum diagrams, [Se]; its associated spectrum is just the Eilenberg-Maclane spectrum $H\mathbb{Z}$. Let C be a finite cyclic group. Then there is a C-equivariant version of the above construction upon making use of the fact that $E_\bullet([m], \underline{k})$ is a cyclic set so that its subdivision $sd_{|C|}E_\bullet([m], \underline{k})$ is a simplicial C-set. More precisely, suppose $A = ([m], \rho)$ with ρ: $C\to Aut_0([m])$ and let $x = (x_S)$ with

$$x_S \in sd_{|C|}E_\bullet\left(\coprod_{i\in S}[k_i], [\underline{k}(S)]\right),\quad S\in P_0[m]\ .$$

For $g\in C$ and $\underline{k}$: $P_0[m]\to\mathbb{N}_0$, let $\underline{k}^g$: $P_0[m]\to\mathbb{N}_0$ be $\underline{k}^g(S) = \underline{k}(g^{-1}S)$, so that $k_i^g = k_{g^{-1}i}$. Then

$$x_{g^{-1}S}\in sd_{|C|}E_\bullet\left(\coprod_{i\in S}[k_i^g], \underline{k}^g(S)\right)$$

and we have

$$g_*\colon sd_{|C|}E_\bullet([m], \underline{k}) \to sd_{|C|}E_\bullet([m], \underline{k}^g)$$

by setting $g_*(x) = y$ with $y_S = gx_{g^{-1}S}$. We can then form the Γ_C-space,

$$E_C(A) = \coprod \{|sd_{|C|}E_\bullet([m], \underline{k})|\colon A = ([m], \rho),\ \underline{k}\in\mathrm{Hom}(A, \mathbb{N}_0)\}\ .$$

Its corresponding spectrum is the Eilenberg-MacLane spectrum $H\mathbb{Z}$ with trivial C-action. In order to get more complicated Γ (and Γ_C-spaces) we involve $THH_\bullet$ and the map $\theta_\bullet$ from (4.5). Let $\underline{k}$: $P_0[m]\to\mathbb{N}_0$ be an additive function, and F a functor with smash product. We define

$$\Sigma_{\underline{k}} = \prod_{i=1}^m \Sigma_{k_i},\ F_{\underline{k}} = \prod_{i=1}^m F_{k_i}\ .$$

The simplicial map $\theta_\bullet$ extends to a simplicial map

$$\theta_\bullet\colon E_\bullet\Sigma_{\underline{k}} \times THH_\bullet(F_{\underline{k}}) \to THH_\bullet(F_{\underline{k}})$$

upon using the old $\theta_\bullet$ on each of the coordinates, and we still have the relations (4.4) satisfied.

Given $\phi\colon [m] \to [n]$ there is a cyclic map

$$THH(\phi)\colon THH_\bullet(F_{\underline{k}}) \to THH_\bullet(F_{\phi^*(\underline{k})})\,. \tag{4.10}$$

Indeed, if $\underline{l} = \phi_*(\underline{k})$ then $l_j = \sum_{i\in\phi^{-1}(j)} k_i$.

The ordering of $\phi^{-1}(j)$ as a subset of $[m]$ and the wedge sum

$$\mu\colon F_a \times F_b \to F_{a+b}$$

defines a map from $\prod_{j=1}^n \prod_{i\in\phi^*\{0,j\}} F_{k_i}$ to $F_{\underline{l}}$. Finally we have a projection from $F_{\underline{k}}$ onto $\prod_{j=1}^n \prod_{i\in\phi^*\{0,j\}} F_{k_i}$, and we can apply the functor THH to get $THH(\phi)$. Similarly, using the sum (concatenation) map from $\Sigma_a \times \Sigma_b$ to Σ_{a+b} there is a cyclic map

$$\mu(\phi)\colon E_\bullet\Sigma_{\underline{k}} \to E_\bullet\Sigma_{\underline{l}}\,. \tag{4.11}$$

We must examine functoriality. In the special case where $\phi\colon [m] \to [n]$ and $\psi\colon [n] \to [r]$ are *order preserving*, then it is easily seen that

$$\begin{aligned} THH(\psi)\circ THH(\phi) &= THH(\psi\circ\phi) \\ \mu(\psi)\circ\mu(\phi) &= \mu(\psi\circ\phi)\,. \end{aligned} \tag{4.12}$$

Moreover, for order preserving ϕ, $\theta_\bullet$ is functorial in the sense that we have a commutative diagram

$$\begin{array}{ccc} E_\bullet\Sigma_{\underline{k}} \times THH_\bullet(F_{\underline{k}}) & \xrightarrow{\theta_\bullet(\underline{k})} & THH_\bullet(F_{\underline{k}}) \\ \downarrow \mu(\phi)\times THH(\phi) & & \downarrow THH(\phi) \\ E_\bullet\Sigma_{\underline{l}} \times THH_\bullet(F_{\underline{l}}) & \xrightarrow{\theta_\bullet(\underline{l})} & THH_\bullet(F_{\underline{l}})\,. \end{array} \tag{4.13}$$

In order to handle more general set maps we introduce the cyclic set

$$F_\bullet(\phi, \underline{k}) \subseteq \prod_{j=1}^n E_\bullet\Big(\coprod_{i\in\phi^*\{0,j\}} [k_i], [l_j]\Big), \quad \underline{l} = \phi_*(\underline{k})\,,$$

defined by the condition

$$f_j = 0 \text{ if card } \{i \mid \phi(i) = j,\ k_i \neq 0\} = 1\,. \tag{**}$$

Here $\phi\colon [m] \to [n]$ is arbitrary. There is a natural map of cyclic sets

$$\lambda(\phi, \underline{k})\colon E_\bullet([m], k) \to F_\bullet(\phi, \underline{k})$$

defined by projection onto the relevant components.

As for functoriality, if ϕ and ψ are composable there is a product map induced by composition

$$\mu_F\colon F_\bullet(\psi, \phi_*(\underline{k})) \times F_\bullet(\phi, \underline{k}) \to F_\bullet(\psi\phi, \underline{k})\,.$$

To be precise, if $\underline{l} = \phi_\#(\underline{k})$, $\underline{p} = \psi_\#(\underline{l})$ then the component h_v is defined to make the following diagram commutative

$$\begin{array}{ccc} \coprod_{i \in (\psi\phi)^{\#}\{0, v\}} [k_i] & \xrightarrow{h_v} & [p_v] \\ \uparrow \sigma_{\phi, \psi^{\#}\{0, v\}} & & \uparrow g_v \\ \coprod_{j \in \psi^{\#}\{0, v\}} \coprod_{i \in \phi^{\#}\{0, j\}} [k_i] & \xrightarrow{\coprod f_j} & \coprod_{j \in \psi^{\#}\{0, v\}} [l_j] \end{array}$$

It is direct from the definitions involved that the following diagram is commutative

$$\begin{array}{ccc} E_\bullet([m], k) & \xrightarrow{(\lambda, E(\phi))} & E_\bullet([n], l) \times F_\bullet(\phi, \underline{k}) \\ \downarrow (E(\psi\phi), \lambda) & & \downarrow (E(\psi), \lambda, \mathrm{id}) \\ E_\bullet([r], p) \times F_\bullet(\psi\phi, \underline{k}) & \xrightarrow{(id, \mu_F)} & E_\bullet([r], p) \times F_\bullet(\psi, \underline{l}) \times F_\bullet(\phi, \underline{k}). \end{array} \tag{4.14}$$

We have $\coprod_{i \in \phi^{\#}\{0, j\}} [k_i] = [l_j]$ so that

$$E_\bullet\left(\coprod_{i \in \phi^{\#}\{0, j\}} [k_i], [l_j] \right) = E_\bullet([l_j], [l_j]) = E_\bullet \Sigma_{l_j}$$

and there is a corresponding cyclic map

$$\phi_1 \colon F_\bullet(\phi, \underline{k}) \to E_\bullet \Sigma_{\underline{l}}$$

which is equivariant with respect to the right action over

$$\mu_1(\phi) \colon \Sigma_{\underline{k}} \to \Sigma_{\underline{l}}$$

(compare (4.11): $\mu(\phi) = E_\bullet(\mu_1(\phi))$). We can now define the cyclic map

$$\pi(\phi) \colon F_\bullet(\phi, \underline{k}) \times THH_\bullet(F_{\underline{k}}) \xrightarrow{(\phi_1, THH(\phi))} E_\bullet \Sigma_{\underline{l}} \times THH_\bullet(F_{\underline{l}}) \xrightarrow{\theta(\underline{l})} THH_\bullet(F_{\underline{l}}) \tag{4.15}$$

where $\theta_\bullet(\underline{l}) = (\theta_\bullet(l_1), \ldots, \theta_\bullet(l_n))$, and $\theta_\bullet(l_j)$ is the map from (4.5). It follows from (4.4) that $\pi(\phi)$ has the following equivariance property

$$\pi(\phi)(g_1 f g_2, t) = g_2^{-1} \pi(\phi)(x, g_1^{-1} t g_1) g_2 \tag{4.16}$$

for $g_1 \in \Sigma_{\underline{l}}$, $g_2 \in \Sigma_{\underline{k}}$ and where the action of g_2 on the right hand side of (4.15) is via $\mu_1(\phi) \colon \Sigma_{\underline{k}} \to \Sigma_{\underline{l}}$.

We finally let

$$X(\phi) \colon E_\bullet([m], \underline{k}) \times THH_\bullet(F_{\underline{k}}) \to E_\bullet([n], \underline{l}) \times THH_\bullet(F_{\underline{l}})$$

be the cyclic map defined by

$$X(\phi)(e, t) = (E(\phi)(e), \pi(\phi)(\lambda(\phi, \underline{k})(e), t)$$

Lemma 4.17 *For based maps $\phi \colon [m] \to [n]$, $\psi \colon [n] \to [r]$, $X(\psi\phi) = X(\psi) \circ X(\phi)$.*

Proof. We have already showed that $E(\psi \circ \phi) = E(\psi) \circ E(\phi)$ so we have left to examine the functoriality of $\pi(\phi)$.

Let $\underline{l} = \phi_*(\underline{k})$, $p = \psi_\#(\underline{l})$. Assuming first that ϕ and ψ are order preserving. We can then combine (4.13), applied to ψ, with the diagram

$$\begin{array}{ccc} E_\bullet\Sigma_{\underline{p}} \times E_\bullet\Sigma_{\underline{p}} \times THH_\bullet(F_{\underline{p}}) & \xrightarrow{\mu \times id} & E_\bullet\Sigma_{\underline{p}} \times THH_\bullet(F_{\underline{p}}) \\ \downarrow (\mathrm{id}, \theta(\underline{p})) & & \downarrow \theta(\underline{p}) \\ E_\bullet\Sigma_{\underline{p}} \times THH_\bullet(F_{\underline{p}}) & \xrightarrow{\theta(\underline{p})} & THH_\bullet(F_{\underline{p}}) \end{array}$$

which is commutative by the second formula in (4.4), to show that

$$\begin{array}{ccc} F_\bullet(\psi, \underline{l}) \times F_\bullet(\phi, \underline{k}) \times THH_\bullet(F_{\underline{k}}) & \xrightarrow{(\mu_F, \mathrm{id})} & F_\bullet(\psi\phi, \underline{k}) \times THH_\bullet(F_{\underline{k}}) \\ \downarrow (\mathrm{id}, \pi(\phi)) & & \downarrow \pi(\psi\phi) \\ F_\bullet(\psi, \underline{l}) \times THH_\bullet(F_{\underline{l}}) & \xrightarrow{\pi(\psi)} & THH_\bullet(F_{\underline{p}}) \end{array} \tag{4.18}$$

is commutative.

We claim that (4.18) commutes for all based set maps. Since a set map is the composition of an order preserving map and a permutation (of the non-zero elements) there is really only two cases to consider, namely the cases where either ϕ or ψ is a based permutation. Then $F(\phi, \underline{k})$ or $F(\psi, \underline{l})$ is a one-point space, and the commutativity of (4.18) follows from the equivariance property (4.16). Indeed, if ψ is a permutation then the two maps around in (4.18) are

$$(e, t) \to \pi(\phi)(e\psi, t), \quad (e, t) \to \psi^{-1}\pi(\phi)(e, t)\psi$$

and if ϕ is a permutation the two compositions are

$$(e, t) \to \pi(\phi e, t), \quad (e, t) \to \pi(e, \phi^{-1}t\phi)$$

In either case the two compositions are identical by (4.16). □

We can define the wanted equivariant Γ-space structure on $THH(F)$. Let C be a cyclic group of order r, $A = ([m], \rho)$ with $\rho: C \to \mathrm{Aut}_0([m])$, and $\phi: ([m], \rho) \to ([n], \rho)$ a morphism in Γ_C^{op}. We define

$$T_C F(A) = \coprod \{|sd_r(E_\bullet([m], \underline{k}) \times THH_\bullet(F_{\underline{k}}))|: \underline{k} \in Hom(P_0[m], \mathbb{N}_0)\} ,$$

$$T_C F(\phi) = |sd_C X(\phi)| . \tag{4.19}$$

Here the C-action on $T_C F(A)$ is the conjugation action when we view (4.19) as the space of mappings with domain $\mathrm{Hom}(P_0[n], \mathbb{N}_0)$ (with its C-action induced from ρ) and the range as the C-space induced from the simplicial C-action on the subdivision. It is clear from (4.17) that we have defined a functor

$$T_C(F): \Gamma_C^{op} \to \{C - \text{spaces}\} .$$

We have left to prove that $T_C F$ is a special Γ_C-space, i.e. that it satisfies (4.1).

Proposition 4.20 *For any two functors with smash product, the product of projections define a C-equivariant homotopy equivalence*

$$|\phi_\bullet|: |sd_r THH_\bullet(F' \times F'')| \to |sd_r THH_\bullet(F')| \times |sd_r THH_\bullet(F'')| .$$

Proof. A map between equivariant CW-complexes is an equivariant homotopy equivalence if it induces ordinary homotopy equivalences on all fixed sets, [A, Sect. 2]. It thus suffices to show that

$$|\phi_\bullet|: |sd_r THH_\bullet(F' \times F'')^{C_r}| \to |sd_r THH_\bullet(F')^{C_r}| \times |sd_r THH_\bullet(F'')^{C_r}|$$

is a homotopy equivalence.

The domain and range for $|\phi_\bullet|$ are H-spaces, hence simple, so we can check on integral homology whether $|\phi_\bullet|$ is a homotopy equivalence or not. Moreover, for each simplicial space the skeleton filtration of $|X_\bullet|$ defines a spectral sequence with abutment $H_*(|X_\bullet|)$ and with $E^1_{p,q} = H_q(X_p)$.

Given any FSP, define a simplicial group $H^r_q(F)_\bullet$ by

$$[p] \mapsto H_q(sd_r THH_p(F)^{C_r}) .$$

We show the projections define a homotopy equivalence of simplicial groups

$$H^r_q(\phi)_\bullet: H^r_q(F' \times F'')_\bullet \to H^r_q(F')_\bullet \times H^r_q(F'')_\bullet$$

This in turn will imply that the $|\phi_\bullet|$ induces an isomorphism on the E^1-term of the spectral sequences, and hence is an integral homology isomorphism.

There are (not unit preserving) inclusions ψ_i: $F^{(i)} \to F' \times F''$ of FSP's, giving simplicial maps $H^r_q(\psi_i)_\bullet$ for $i = 1, 2$. We consider the sum

$$\psi_\bullet = H^r_q(\psi_1)_\bullet + H^r_q(\psi_2)_\bullet .$$

One composition is the identity, $H^r_q(\phi)_\bullet \circ \psi_\bullet = \mathrm{id}$. We shall construct a simplicial homotopy between the identity and the other composition.

$$K: \Delta[1]_\bullet \times H_q(sd_r THH_\bullet(F' \times F'')^{C_r}) \to H_q(sd_r THH_\bullet(\mathrm{F}' \times \mathrm{F}'')^{C_r}) .$$

A p-simplex of $\Delta[1]_\bullet$ is a weakly increasing map σ: $[p] \to [1]$, so is determined by the number k for which $\sigma(k-1) = 0$ and $\sigma(k) = 1$. Let $F = F' \times F''$ and consider

$$f: S^{|i|R} \to F(S^{i_0})^{(r)} \wedge \ldots \wedge F(S^{i_p})^{(r)} ,$$

representing an element of $sd_r THH_p(F)$.

Suppose first $r = 1$. Write $\pi_{\sigma,1}$ and $\pi_{\sigma,2}$ for the following compositions where $v = (')$ or $('')$

$$F(S^{i_0}) \wedge \ldots \wedge F(S^{i_p}) \to F^{(v)}(S^{i_0}) \wedge \ldots \wedge F^{(v)}(S^{i_{k-1}}) \wedge F(S^{i_k}) \wedge \ldots \wedge F(S^{i_p})$$
$$\to F(S^{i_0}) \wedge \ldots \wedge F(S^{i_p}) .$$

Here the first map is projection onto $F^{(v)}(S^{i_t})$, $0 \leqq t \leqq k-1$, and the second is induced from the inclusion of $F^{(v)}$ into $F = F' \times F''$.

Assigning $(\pi_{\sigma,1} \circ f, \pi_{\sigma,2} \circ f)$ to f induces two maps

$$\Delta[1]_p \times H_q(THH_p(F)) \to H_q(THH_p(F))$$

which we can add to get K_p. In the special case where $\sigma(i) = 0$ for all i, we let $K_p = \mathrm{id}$. If $\sigma(i) = 1$ for all i then $K_p = \psi_p \circ H^r_q(\phi_p)$. One can easily check that $K_\bullet$ is a simplicial map, thus defines the required simplicial homotopy.

Finally, if $r > 1$, we use $\pi^{(r)}_{\sigma,1}$ and $\pi^{(r)}_{\sigma,2}$ to get equivariant maps, and define $K^{(r)}_p$ using the sum of these two maps to obtain the required homotopy. □

Remark 4.21 The above map K_p can be defined directly on the space level,

$$\Delta[1]_p \times sd_r THH_p(F)^{C_r} \to sd_r THH_p(F)^{C_r} .$$

However, this will not be a simplicial map; one will only have the simplicial identities satisfied up to homotopy. This is the reason that we apply the integral homology functor.

Corollary 4.22 *The functor $T_{C_r}F$ defined in* (4.19) *is a special Γ_G-space.*

In [Sh], Shimakawa constructs to each special Γ_G-space T_G a G-equivariant spectrum $\mathbb{B}T_G$. It is an "almost" $G-\Omega$ spectrum in the sense that the structure maps

$$S^W \wedge \mathbb{B}_V T_G \to \mathbb{B}_{W \oplus V} TG$$

adjoin to become G-homotopy equivalences for each pair of $\mathbb{R}G$-modules with $V^G \neq 0$. In particular, $V \mapsto \Omega\mathbb{B}_{V \oplus \mathbb{R}} T_G$ is a $G-\Omega$ spectrum. Moreover, the natural map (adjoined to the inclusion of $T_G([1])$ in the 1-skeleton)

$$T_G([1]) \to \Omega\mathbb{B}_{\mathbb{R}} T_G$$

is an equivariant group completion.

Recall the terminology that a G-infinite loop space is the zero'th space in a $G-\Omega$ spectrum, and that a G-infinite loop map is the zero'th level of map between $G-\Omega$ spectra.

The above applies to the Γ_C-space $T_C F$ defined in (4.19).

Proposition 4.23 *For each finite cyclic group C and each functor with smash product $THH(F)$ is a C-infinite loop space in such a way that the product $H(\mathbb{Z}) \times THH(F)$ is the C-infinite loop space associated with the Γ_C-space of* (4.19).

Proof. Since the cyclic space $|E\Sigma_\bullet(\underline{k})|$ is equivariantly contractible and since THH is a Morita-invariant by (3.9),

$$|sd_r(E\Sigma_\bullet(\underline{k}) \times THH_\bullet(F_k))| \simeq_{C_r} |sd_r THH_\bullet(F)| .$$

The resulting map

$$T_C F([1]) \to \mathbb{Z} \times THH(F) ,$$

which maps the k'th term into $\{k\} \times THH(F)$, is an equivariant group completion. We have left to see that $\mathbb{Z}$, with its standard infinite loop space structure arising from the C-trivial Eilenberg-Maclane spectrum, splits off. To this end we can project $T_C F$ to the Γ_C-space E_C given by

$$[n] \to \coprod |sd_r E\Sigma_\bullet(\underline{k})|, \quad \underline{k} \in \mathrm{Hom}(P_0[n], \mathbb{N}_0)$$

and observe that the projection is split (as Γ_C-spaces) by the inclusion of the base point in $THH(F_{\underline{k}})$ (FSP's take value in pointed spaces).

Since $|sd_r E\Sigma_\bullet(\underline{k})|$ is C-contractible, the equivariant Γ-space E_C is homotopy equivalent to

$$([n], \rho) \to \mathrm{Hom}(P_0[n], \mathbb{N}_0) .$$

The spectrum of E_C is the Eilenberg-Maclane spectrum $H(\mathbb{Z})$, with trivial action of C. □