Werk

Titel: 2 The cyclic bar construction.

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where d_r is the diagonal map (cf. 1.1). This uses the description of Λ_k^m from [J] (cf. 1.6). We leave the details for the reader. □

If we identify $\mathbb{R}/m\mathbb{Z}$ with S^1 in the standard fashion ($\theta \leftrightarrow e^{2\pi i\theta/m}$) then 1.11 simply says that

$$D_r\colon |sd_rX_\bullet| \to |X_\bullet|$$

is an S^1-homeomorphism.

Let us finally remark that the subdivision functor of course can be iterated and that

$$sd_rsd_sX_\bullet = sd_{rs}X_\bullet\,.$$

Moreover, the diagram

$$\begin{array}{ccc} |sd_{rs}X_\bullet| & \xrightarrow{D_r} & |sd_sX_\bullet| \\ & {\scriptstyle D_{rs}}\searrow \quad \swarrow{\scriptstyle D_s} & \\ & |X_\bullet| & \end{array} \tag{1.12}$$

is commutative.

2 The cyclic bar construction

Given a topological, group-like monoid G and a two-sided G-space E we can form the cyclic bar construction $N_\bullet^{cy}(E; G)$, cf. [W1]. It is the simplicial space:

$$\begin{aligned} N_n^{cy}(E; G) &= E \times G^n \\ d_0(e, g_1, \ldots, g_n) &= (eg_1, g_2, \ldots, g_n) \\ d_n(e, g_1, \ldots, g_n) &= (g_ne, g_1, \ldots, g_{n-1}) \\ d_i(e, g_1, \ldots, g_n) &= (e, g_1, \ldots, g_ig_{i+1}, \ldots, g_n), \quad 0 < i < n \\ s_i(e, g_1, \ldots, g_n) &= (e, g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)\,. \end{aligned}$$

If $E = G$, considered as a two-sided G-space via multiplication, we write $N_\bullet^{cy}(G)$ instead of $N_\bullet^{cy}(G, G)$. Setting

$$t_n(g_0, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1})$$

it becomes a cyclic space (with t_n corresponding to τ_n).

The r-fold edgewise subdivision $sd_rN_\bullet^{cy}(G)$ is again a cyclic bar construction, namely

$$sd_rN_\bullet^{cy}(G) \cong N_\bullet^{cy}(t(G^r), G^r) \tag{2.1}$$

where G^r is the r-fold Cartesian product of G and $t(G^r) = G^r$ but with a twisted two-sided G^r-structure:

$$\begin{aligned} (e_1, \ldots, e_r)\cdot(g_1, \ldots, g_r) &= (e_1g_1, \ldots, e_rg_r) \\ (g_1, \ldots, g_r)\cdot(e_1, \ldots, e_r) &= (g_re_1, g_1e_2, \ldots, g_{r-1}e_r)\,. \end{aligned}$$

Let π: $N^{cy}_\bullet(G) \to N_\bullet(G)$ be the simplicial map into the usual (one-sided) bar construction which in simplicial degree n maps $(g_0, \ldots, g_n)$ to $(g_1, \ldots, g_n)$. Combining with the S^1-action on $|N^{cy}_\bullet(G)|$ we have the map

$$S^1 \times |N^{cy}_\bullet(G)| \xrightarrow{\mu} |N^{cy}_\bullet(G)| \xrightarrow{\pi} |N_\bullet(G)|$$

whose adjoint is a map

$$f\colon |N^{cy}_\bullet(G)| \to \Lambda BG \tag{2.2}$$

into the free loop space of $BG = |N_\bullet(G)|$. Observe that f is an S^1-map when we give ΛBG the S^1-action from rotating the loops. It is well-known that f is a non-equivariant homotopy equivalence, [BF, G], since $\pi_0 G$ is assumed to be a group. We want to prove a corresponding equivariant statement. Let

$$\Delta_{r,\bullet}\colon N^{cy}_\bullet(G) \to sd_r N^{cy}_\bullet(G) \cong N^{cy}_\bullet(t(G^r), G^r)$$

be the diagonal map which sends a k-simplex $(g_0, \ldots, g_n)$ into $(\vec{g}_0, \ldots, \vec{g}_n)$ with $\vec{g}_i = (g_i, \ldots, g_i) \in G^r$.

The simplicial action of the cyclic group C_r on $sd_r N^{cy}_\bullet(G)$, generated by $t^{n+1}_{(n+1)r-1}$ on the n-simplices, corresponds under the identification (2.1) to the permutation action on G^r. Thus

$$\Delta_{r,\bullet}\colon N^{cy}_\bullet(G) \to (sd_r N^{cy}_\bullet(G))^{C_r}\,; \tag{2.3}$$

this is a simplicial isomorphism, whose realization is denoted Δ_r. When G is a group there is the injection

$$i\colon N_\bullet(G) \to N^{cy}_\bullet(G)_{(1)}$$

$$i(g_1, \ldots, g_n) = ((\textstyle\prod g_i)^{-1}, g_1, \ldots, g_n)\,, \tag{2.4}$$

split by the map π used in (2.2). The realization of i corresponds to the inclusion of BG in ΛBG as the point loops.

Proposition 2.5 *For a topological group G,*

$$\begin{array}{ccccc} |N_\bullet(G)| & \xrightarrow{i} & |N^{cy}_\bullet(G)| & \xrightarrow{\Delta_{rs}} & |sd_{rs}N^{cy}_\bullet(G)|^{C_{rs}} \\ \downarrow i & & & & \wr \\ |N^{cy}_\bullet(G)| & \xrightarrow{\Delta_s} & |sd_s N^{cy}_\bullet(G)| & \xleftarrow{C_s D_r} & |sd_{rs}N^{cy}_\bullet(G)|^{C_s} \end{array}$$

is homotopy commutative by a homotopy $D^s_{rs,t}$ which is natural in G. Moreover, $D^s_{rs,t} = \Delta_s \circ D^1_{r,t}$

Proof. Suppose first that $s = 1$. Consider the homotopy

$$d_{r,t}\colon \Delta^n \to \Delta^n * \ldots * \Delta^n \quad (r \text{ factors})$$

$$d_{r,t}(u) = tu/r \oplus \ldots \oplus tu/r \oplus (tu/r + (1-t)u) \quad 0 \leqq t \leqq 1\,.$$

For $t = 1$ this is the map d_r used in 1.1. Let

$$D^1_{r,t}\colon |sd_r N^{cy}_\bullet(G)| \to |N^{cy}_\bullet(G)|$$

be the corresponding homotopy of the map D from 1.1. Since

$$d_{r,0}\colon \Delta^n \to \Delta^{(n+1)r-1}$$

is the $(n+1)(r-1)$-th iterate of the 0-th (co)face we have commutativity in

$$\begin{array}{ccc} \Delta^n \times [sd_r N^{cy}(G)]_n & \xrightarrow{1 \times d_0^{(n+1)(r-1)}} & \Delta^n \times N_n^{cy}(G) \\ \downarrow & & \downarrow \\ |sd_r N_\bullet^{cy}(G)| & \xrightarrow{D^1_{r,0}} & |N_\bullet^{cy}(G)| \,. \end{array}$$

A direct calculation shows that

$$d^{(n+1)(r-1)} \Delta_r(g_0, \ldots, g_n) = ((\prod g_i)^{r-1} g_0, g_1, \ldots, g_n) \,.$$

But $\prod g_i = 1$ when $(g_0, \ldots, g_n) \in \mathrm{Im}(i)$. Hence $D^1_{r,0} \circ \Delta_r \circ i = \mathrm{id}$ which proves the claim. For $s > 1$ one takes $D^r_{rs,t} = \Delta_s \circ D^1_{r,t}$. □

Proposition 2.6 *For a group-like topological monoid G and for each finite subgroup C of S^1,*

$$f\colon |N_\bullet^{cy}(G)|^C \to (\Lambda BG)^C$$

is a homotopy equivalence.

Proof. First observe that

$$\Delta_{r,\bullet}\colon P_r N_\bullet^{cy}(G) \to sd_r N_\bullet^{cy}(G)$$

is a map of Λ_r^{op}-spaces with P_r from (1.9). It follows from 1.10 and 1.11 that the composition

$$\bar{\Delta}_r\colon |N_\bullet^{cy}(G)| \xrightarrow{\Delta_r} |sd_r N_\bullet^{cy}(G)| \xrightarrow{D_r} |N_\bullet^{cy}(G)|$$

has the following equivariance property for the S^1-action:

$$\Delta_r(z^r \cdot x) = z \cdot \Delta_r(x) \tag{2.7}$$

for $z \in S^1$, $x \in |N_\bullet^{cy}(G)|$. Let $g\colon |N_\bullet^{cy}(G)| \to \Lambda |N_\bullet^{cy}(G)|$ be the adjoint of the S^1-action. We have the diagram

$$\begin{array}{ccccc} |N_\bullet^{cy}(G)|^{C_r} & \xrightarrow{g^{C_r}} & (\Lambda |N_\bullet^{cy}(G)|)^{C_r} & \xrightarrow{\Lambda\pi} & (\Lambda |N_\bullet(G)|)^{C_r} \\ \uparrow \bar{\Delta}_r & & \uparrow P_r & & \uparrow P_r \\ |N_\bullet^{cy}(G)| & \xrightarrow{g} & \Lambda |N_\bullet^{cy}(G)| & \xrightarrow{\Lambda\pi} & \Lambda |N_\bullet(G)| \end{array} \tag{2.8}$$

with P_r the power map, $P_r(\sigma)(z) = \sigma(z^r)$. We claim that the outer diagram in (2.8) is homotopy commutative. For $x \in |N_\bullet^{cy}(G)|$, $(P_r \circ g)(x)$ is the loop $\sigma(z) = z^r \cdot x$ and $g \circ \bar{\Delta}_r(x)$ is the loop $\bar{\sigma}(z) = z \cdot \bar{\Delta}_r(x) = \bar{\Delta}_r(z^r \cdot x)$. So we have left to show that the maps

$$\pi,\ \pi \circ \bar{\Delta}_r\colon |N_\bullet^{cy}(G)| \to |N_\bullet G|$$

are homotopic. The homotopy is $\pi \circ D^1_{r,t} \circ \Delta_r$ with $D^1_{r,t}$ the homotopy from 2.5. Clearly

$$P_r\colon \Lambda BG \to (\Lambda BG)^{C_r}$$

is a homeomorphism. By (2.3) the same is true for Δ_r, so f^{C_r} is conjugate to f. Since f is a homotopy equivalence, so is f^{C_r}. □

Remark 2.9 The equivariant Whitehead theorem asserts that an equivariant map is an equivariant homotopy equivalence if and only if the induced maps on all fixed sets are homotopy equivalences, at least if the transformation group in question is compact. Thus $f\colon |N^{cy}_\bullet(G)| \to \Lambda BG$ is a C-homotopy equivalence for each finite $C \subset S^1$. However, f is not an S^1-equivalence, since

$$|N^{cy}_\bullet(G)|^{S^1} = \{g \in G \mid s_0(g) = t_1 s_0(g)\} = \{1\}\,,$$
$$(\Lambda BG)^{S^1} = BG\,.$$

We need a blown-up version of $N^{cy}_\bullet(G)$, e.g. the bi-simplicial set used in [B1], to get an S^1-equivalence.

The homotopies specified in Proposition 2.5 gives a well-defined map

$$I\colon |N_\bullet(G)| \to \operatorname{holim}_{\overleftarrow{D}} |sd_r N^{cy}_\bullet(G)|$$

with the limits running over the compositions

$$D\colon |sd_{rs}N^{cy}_\bullet(G)|^{C_{rs}} \to |sd_{rs}N^{cy}_\bullet(G)|^{C_r} \xrightarrow{D_r} |sd_r N^{cy}_\bullet(G)|^{C_r}\,.$$

The reader is referred to [BK, Chap. XI] for the definition and general properties of homotopy inverse limits. For the purpose of this paper it suffices to replace the limit system above by the simpler system where r runs over the powers of a fixed prime number p. In this case there is a more well-known description of homotopy inverse limits which we now recall. Given a string of spaces

$$\ldots \to S_n \xrightarrow{\sigma_n} S_{n-1} \to \ldots \to S_0$$

we can replace it by a string of fibrations

$$\ldots \to f(S_n) \xrightarrow{f(\sigma_n)} f(S_{n-1}) \to \ldots \to f(S_0)$$

by iterating the usual mapping path space construction which converts a map into a fibration, and one has

$$\operatorname{holim}_{\longleftarrow} S_n \simeq \lim_{\longleftarrow} f(S_n)\,.$$

This follows from [BK, XI, 4.1 and 5.6].

The homotopy groups of $\operatorname{holim}_{\longleftarrow} S_n$ can be calculated from the exact sequence

$$0 \to \lim^{(1)}_{\longleftarrow} \pi_{k+1} S_n \to \pi_k \operatorname{holim}_{\longleftarrow} S_n \to \lim_{\longleftarrow} \pi_k S_n \to 0$$

cf. [BK, XI, 7.4] or [Mi]. If we assume that each S_n is an infinite loop space and σ_n an infinite loop map, and this will always be the case for us, then one has an exact sequence

$$0 \to \lim^{(1)}_{\longleftarrow} [\Sigma X, S_n] \to [X, \operatorname{holim}_{\longleftarrow} S_n] \to \lim_{\longleftarrow} [X, S_n] \to 0$$

for every X. The $\lim^{(1)}_{\longleftarrow}$-term is in general non-zero.

We shall use at several places below that $\varprojlim^{(1)} A_n = 0$ if A_n is a string of compact abelian groups. Indeed, $\varprojlim^{(1)} A_n$ is the cokernel of the map $\prod A_n \to \prod A_n$ which sends (a_n) to $(\sigma_n(a_n) - a_{n-1})$. This has dense image and $\prod A_n$ is compact; thus it is onto.

Lemma 2.10 *Suppose (S_n, σ_n) is a string of infinite loop spaces, and that $f_n\colon X \to S_n$ are maps so that $\sigma_n \circ f_n \simeq f_{n-1}$. There is a homotopy class $[f]$, $f\colon X \to \operatorname{holim} S_n$, inducing $[f_n]$. Its p-adic completion $[\hat{f}_p]$ is well-defined, when each $[\Sigma X, S_n]$ is finitely generated.*

Let us return to the inverse limit system at hand. It is direct from the definitions to check that there are commutative diagrams

$$\begin{array}{ccc} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} & \xrightarrow{\Delta_p} & |sd_{p^{n+1}} N_\bullet^{cy}(G)|^{C_{p^{n+1}}} \\ \downarrow D & & \downarrow D \\ |sd_{p^{n-1}} N_\bullet^{cy}(G)|^{C_{p^{n-1}}} & \xrightarrow{\Delta_p} & |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} \end{array} \tag{2.11}$$

where Δ_p is the homeomorphism induced by (2.3). There is an induced homeomorphism

$$\Delta_p\colon \operatorname{holim} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} \to \operatorname{holim} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}}$$

and it is clear from (2.5) that $\Delta_p \circ I = I$. Let $\Phi_p = \Delta_p^{-1}$ be the inverse homeomorphism. We have

$$I\colon |N_\bullet(G)| \to (\operatorname{holim} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}})^{\Phi_p} \tag{2.12}$$

Our preference of Φ_p over Δ_p in (2.12) will become apparent in Sect. 5 below.

In our definition of the cyclotomic trace (Sect. 5) we need to apply the above in a situation where G is a group-like monoid (a topological monoid with $\pi_0 G$ a group). The map I is not a priori defined for monoids, since it uses strict inverses. However, there is a well-known trick to get around this difficulty.

There is a functor $G \to G^\wedge$ which replaces a topological monoid by a group, and another functor $G \to G^\vee$ (the free group) together with natural transformations

$$G \leftarrow G^\vee \to G^\wedge$$

cf. [BF, p. 311] or [G, Sect. I, 1.8].

When G is group-like the induced maps

$$|N_\bullet(G)| \leftarrow |N_\bullet(G^\vee)| \to |N_\bullet(G^\wedge)|$$

$$|sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}} \leftarrow |sd_{p^n} N_\bullet^{cy}(G^\vee)|^{C_{p^n}} \to |sd_{p^n} N_\bullet^{cy}(G^\wedge)|^{C_{p^n}}$$

are all homotopy equivalences. Since homotopy inverse limits, and in particular homotopy fixed points are homotopy invariant notions, we get a well-defined homotopy class

$$I\colon |N_\bullet(G)| \to (\operatorname{holim} |sd_{p^n} N_\bullet^{cy}(G)|^{C_{p^n}})^{h\Phi_p} \tag{2.13}$$

for every group-like topological monoid. (Here $h\Phi_p$ indicates homotopy fixed set, i.e. the homotopy equalizer of the self-maps Φ_p and id.)