Werk

Titel: 4 The cojecture of Birch and Swinnerton-Dyer.

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that $H^3(S, \mathscr{I}_{\mathbb{Z}_p})$ vanishes. As the isomorphism $H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p}) \simeq H^2_{\text{cris}}(S/W)$ is compatible with Frobenius [F-M, II.1.3] the trace map $H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p}) \xrightarrow[\simeq]{\text{tr}} W$ induces the σ-linear endomorphism $F = p \cdot \sigma$ on W, where $\sigma(x_1, x_2, \ldots) = (x_1^p, x_2^p, \ldots)$.

From the exact sequence $0 \to \mu_{p^n} \to \mathscr{I}_n \xrightarrow{1-p^{-1}F} \mathcal{O}^{\text{cris}}_n \to 0$ (3.4) we now get the following commutative diagram which defines (4'):

$$
\begin{array}{ccccccc}
H^2(S, \mathscr{I}_{\mathbb{Z}_p}) & \xrightarrow{1-p^{-1}F} & H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p}) & \twoheadrightarrow & H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p})_{p^{-1}F} & \xrightarrow[\simeq]{\delta} & H^3(S, \mathbb{Z}_p(1)) \to 0 \\
\downarrow & & \| & & \downarrow \simeq & & \\
H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p}) & \xrightarrow{1-\sigma} & H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p}) & \twoheadrightarrow & H^2(S, \mathcal{O}^{\text{cris}}_{\mathbb{Z}_p})_{\sigma} & & \downarrow \simeq \\
\text{tr} \downarrow \simeq & & \text{tr} \downarrow \simeq & & \text{tr} \downarrow \simeq & & \\
W & \xrightarrow{1-\sigma} & W & \twoheadrightarrow & W_{\sigma} & \xrightarrow{\simeq} & \mathbb{Z}_p \, .
\end{array}
$$

Up to an isomorphism the composition $\mathbb{Z}_p = W^{\sigma} \hookrightarrow W \twoheadrightarrow W_{\sigma} \simeq \mathbb{Z}_p$ equals the multiplication by r. □

3.19. Proof of Theorem 3.2 This now follows immediately from (3.17) and (3.18). □

4 The Conjecture of Birch and Swinnerton-Dyer

In this last section we calculate the three factors in (3.2(ii)) and show that under the assumption made there, the conjecture of Birch and Swinnerton-Dyer as formulated by Tate in [TA, Conjecture B] is valid. We use the notations of the previous section.

At first we record the following observation:

4.1. Lemma. *Let ℓ be a prime ($\ell = p$ allowed). The following are equivalent:*

(i) $A(S) \otimes \mathbb{Z}_\ell \xrightarrow{\delta} H^1(S, T_\ell A)$ *is an isomorphism;*
(ii) $Ш(A_K/K)(\ell)$ *is finite;*
(iii) $rk_{\mathbb{Z}} A_K(K) = rk_{\mathbb{Z}_\ell} H^1(S, T_\ell A)$;
(iv) $Ш(\hat{A}_K/K)(\ell)$ *is finite.*

Proof. From $0 \to {}_{\ell^n}A \to A \xrightarrow{\ell^n} A \to 0$ one gets the exact sequence

$$0 \to A(S) \otimes \mathbb{Z}_\ell \xrightarrow{\delta} H^1(S, T_\ell A) \to T_\ell H^1(S, A) \to 0 \, .$$

As $T_\ell H^1(S, A)$ is torsionfree (i) is equivalent to (iii). By $H^1(S, A) \simeq Ш(A_K/K)$ [M1, §3, Lemma 1] we find $T_\ell H^1(S, A) = 0$ iff $Ш(A_K/K)(\ell)$ is finite, hence (i) $\Leftrightarrow$ (ii).To show the equivalence of (ii) and (iv) one uses a polarisation. □

4.2. Lemma. *If $Ш(A_K/K)(p)$ is finite, then*

$$\prod_{j=1}^{3} \# \operatorname{Tor} H^j(S, T_p \hat{A})^{(-1)^j} = \frac{\# Ш(A_K/K)(p)}{\# A_K(K)(p) \cdot \# \hat{A}_K(K)(p)} \, .$$

Proof. From the perfect pairing $H^j(S, {}_{p^n}A) \times H^{3-j}(S, {}_{p^n}\hat{A}) \to \mathbb{Q}/\mathbb{Z}$ [M5, III.9.1] one gains the perfect pairing

$$H^j(S, A(p))_{\mathrm{Div}} \times \mathrm{Tor}\, H^{3-j}(S, T_p\hat{A}) \to \mathbb{Q}/\mathbb{Z} \qquad \text{[M2, 2.3].}$$

Thus we recover $\# H^3(S, T_p\hat{A}) = \# A_K(K)(p)$.

If $Ш(A_K/K)(p) \simeq H^1(S, A)(p)$ [M1, §3, Lemma 1] is finite the exact sequence

$$0 \to A(S) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(S, A(p)) \to H^1(S, A)(p) \to 0$$

yields $\# Ш(A_K/K)(p) = \# H^1(S, A(p))_{\mathrm{Div}} = \# \mathrm{Tor}\, H^2(S, T_p\hat{A})$.

Finally $\# \mathrm{Tor}\, H^1(S, T_p\hat{A}) = \# \hat{A}(S)(p)$ is deduced from

$$0 \to \hat{A}(S) \otimes \mathbb{Z}_p \to H^1(S, T_p\hat{A}) \to T_p H^1(S, \hat{A}) \to 0. \qquad \square$$

4.3. Lemma. *Let $e: S \to A$ be the unit section and let $\omega_A := e^*\Omega^1_{A/S}$. Then*

$$\prod_{j=0}^{1} \# H^j(S, \mathbf{R}^1\pi_*\mathcal{O}_A)^{(-1)^j} = q^{-\deg \omega_A + d(1-g)},$$

where g denotes the genus of K and d the dimension of A_K.

Proof. Using $\mathbf{R}^1\pi_*\mathcal{O}_A \simeq \check{\omega}_{\hat{A}}$ [B-B-M, 5.5.1] this is an application of the Riemann-Roch theorem. $\square$

4.4. Lemma. *Let $\langle , \rangle : H^0(S, A) \times \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m) \to H^1(S, \mathbb{G}_m) \xrightarrow{\deg} \mathbb{Z}$ be the Yoneda pairing.*

If $Ш(A_K/K)(p)$ is finite one has an isomorphism of pairings:

$$\begin{array}{ccccc}
\langle , \rangle : & A(S) \otimes \mathbb{Z}_p & \times & \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m) \otimes \mathbb{Z}_p & \to & \mathbb{Z}_p \\
 & & & \simeq \downarrow & & \\
 & \simeq \downarrow \delta & & \hat{A}(S) \otimes \mathbb{Z}_p & = & \\
 & & & \simeq \downarrow \delta & & \\
\langle , \rangle_q : & H^1(S, T_pA) & \times & H^1(S, T_p\hat{A}) & \to & \mathbb{Z}_p .
\end{array}$$

Proof. By (4.1) we get the isomorphisms δ. We use the following explicit description of the Yoneda-pairing:

Let $e = (0 \to \mathbb{G}_m \to X \to A \to 0) \in \mathrm{Ext}(A, \mathbb{G}_m) \simeq \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m)$. For $a \in A(S)$ one has $\langle a, e \rangle = -\delta_e(a) \in H^1(S, \mathbb{G}_m)$.

We define $r_n : \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m) \to \mathrm{Ext}^1_{S_{fppf}}({}_{p^n}A, \mu_{p^n})$ by

$$(0 \to \mathbb{G}_m \to X \to A \to 0) \mapsto (0 \to \mu_{p^n} \to {}_{p^n}X \to {}_{p^n}A \to 0) .$$

Just as in [S1, p. 503] we get the following commutative diagram of pairings:

$$\begin{array}{ccccc}
H^0(S, A) & \times & \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m) & \to & H^1(S, \mathbb{G}_m) \\
\downarrow -\delta & & \downarrow r_n & (1) & \downarrow \delta \\
H^1(S, {}_{p^n}A) & \times & \mathrm{Ext}^1_{S_{fppf}}({}_{p^n}A, \mu_{p^n}) & \to & H^2(S, \mu_{p^n}) . \\
\| & & \uparrow \varepsilon & (2) & \| \\
H^1(S, {}_{p^n}A) & \times & H^1(S, {}_{p^n}\hat{A}) & \to & H^2(S, \mu_{p^n})
\end{array}$$

The commutativity of (1) follows from the above description of the Yoneda pairing. The morphism ε comes from the local to global spectral sequence and (2) commutes by [M3, V.1.20].

Exactly as in [S2, §6, Lemma 3] one shows that the diagram

$$\begin{array}{ccc} H^1(S, {}_{p^n}\hat{A}) & \xleftarrow{-\delta} & H^0(S, \hat{A}) \\ \downarrow \varepsilon & & \downarrow \simeq \\ \mathrm{Ext}^1_{S_{fppf}}({}_{p^n}A, \mu_{p^n}) & \xleftarrow{r_n} & \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m) \end{array}$$

is commutative which implies the lemma. □

4.5. Lemma. *Let $h: A_K(K) \times \hat{A}_K(K) \to \mathbb{R}$ be the Néron-Tate height pairing* [N]. *If $Ш(A_K/K)(p)$ is finite, then $\langle\,,\rangle_q$ is non-degenerate and*

$$|\det \langle\,,\rangle_q|_p = |(\log q)^{-rk_{\mathbb{Z}}A_K(K)} \cdot \det h|_p\,.$$

Proof. By [S1, p. 509] the diagram of pairings

$$\begin{array}{ccccc} A_K(K) & \times & \hat{A}_K(K) & \xrightarrow{h} & \mathbb{R} \\ \| & & \| & & \uparrow \cdot(-\log q) \\ A(S) & \times & \mathrm{Ext}^1_{S_{fppf}}(A, \mathbb{G}_m) & \xrightarrow{\langle\,,\rangle} & \mathbb{Z} \end{array}$$

is commutative. As h is non-degenerate the pairing $\langle\,,\rangle$ is too, and

$$\det h = \pm (\log q)^{rk_{\mathbb{Z}}A_K(K)} \cdot \det \langle\,,\rangle\,.$$

Hence by (4.4) we get the lemma. □

Now we obtain our main result:

4.6. Theorem. *Let A_K/K be an abelian variety of dimension d possessing good reduction everywhere. Let ρ be the order of the zero of the Hasse-Weil L-function $L(s)$ of A_K/K at $s = 1$ and $\alpha := (\log q)^{-\rho}$. Let $S/\mathbb{F}_q$ be the model of K, g be the genus of K and A/S be an abelian scheme with generic fibre A_K. Further let $e: S \to A$ be the unit section, $\omega_A := e^*\Omega^1_{A/S}$ and $h: A_K(K) \times \hat{A}_K(K) \to \mathbb{R}$ denote the height pairing. Then it holds:*

(i) $\rho \geqq rk_{\mathbb{Z}}A_K(K)$;

(ii) $\rho = rk_{\mathbb{Z}}A_K(K) \Leftrightarrow Ш(A_K/K)(p)$ *is finite and in this case:*

$$\left|\alpha \cdot \lim_{s\to 1} L(s)\cdot(s-1)^{-\rho}\right|_p^{-1} = \frac{\#\,Ш(A_K/K)(p)\cdot|\alpha\cdot\det h|_p^{-1}}{\#\,A_K(K)(p)\cdot\#\,\hat{A}_K(K)(p)} \cdot q^{-\deg\omega_A + d(1-g)}\,.$$

Proof. (i) By (3.2) we have $\rho \geqq rk_{\mathbb{Z}_p}H^1(S, T_p\hat{A})$. From the exact sequence

$$0 \to \hat{A}(S) \otimes \mathbb{Z}_p \to H^1(S, T_p\hat{A}) \to T_pH^1(S, \hat{A}) \to 0 \tag{4.1}$$

we deduce:

$$\rho \geqq rk_{\mathbb{Z}_p}H^1(S, T_p\hat{A}) \geqq rk_{\mathbb{Z}}\hat{A}_K(K) = rk_{\mathbb{Z}}A_K(K)\,.$$

(ii) If $\rho = rk_{\mathbb{Z}}A_K(K)$ then $rk_{\mathbb{Z}_p}H^1(S, T_p\hat{A})$ equals $rk_{\mathbb{Z}}\hat{A}_K(K)$ and $Ш(A_K/K)(p)$ is finite (4.1).

If $Ш(A_K/K)(p)$ is finite then the pairing $\langle\,,\rangle_q$ is non-degenerate (4.5) and thus $\rho = rk_{\mathbb{Z}_p}H^1(S, T_p\hat{A}) = rk_{\mathbb{Z}}A_K(K)$ by (3.2) and (4.1).