Invent. math. 108, 37–47 (1992)

Inventiones mathematicae

An isoperimetric comparison theorem

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Oblatum 13-II-1991 & 30-IX-1991

1 Introduction

The classical isoperimetric inequality says that if $D \subset E^n$ is a compact domain with smooth boundary ∂D, then

$$\operatorname{area}(\partial D) \geqq c_n (\operatorname{vol}(D))^{\frac{n-1}{n}}$$

where $\operatorname{area}(\partial D)$ denotes the $n-1$ dimensional volume of ∂D, $\operatorname{vol}(D)$ denotes the volume of D, and $c_n = \dfrac{\operatorname{area}(S^{n-1}(1))}{\operatorname{vol}(B^n(1))^{\frac{n-1}{n}}}$. The following appeared in [Aub, BZ, GLP]:

Conjecture 1 *If M^n is a complete, one-connected, Riemannian manifold with nonpositive sectional curvature, then any compact domain $D \subset M^n$ with smooth boundary ∂D satisfies the Euclidean isoperimetric inequality, i.e.*

$$(+) \qquad \operatorname{area}(\partial D) \geqq c_n (\operatorname{vol}(D))^{\frac{n-1}{n}} .$$

Here is some "evidence" in support of the conjecture:

1. If the domain $D \subset M^n$ is a geodesic ball, then the inequality (+) follows from standard comparison theorems.
2. If the sectional curvature of M^n satisfies $K_{M^n} \leqq k < 0$ then $\operatorname{area}(\partial D) \geqq (n-1)\sqrt{(-k)}\operatorname{vol}(D)$ (see [BZ, 34.2.6]) which implies $\operatorname{area}(\partial D) \geqq c_n(\operatorname{vol}(D))^{\frac{n-1}{n}}$ provided $\operatorname{vol}(D)$ is sufficiently large.
3. [HS, Cr2] show that for every n there are constants $\bar{c}_n < c_n$ such that $\operatorname{area}(\partial D) \geqq \bar{c}_n(\operatorname{vol}(D))^{\frac{n-1}{n}}$.

The two and four dimensional cases of Conjecture 1 were proved in [Weil] and [Cr1] respectively. The goal of this paper is to settle the three dimensional case of the conjecture with:

★ Supported by an NSF Postdoctoral Fellowship

Theorem 2 *Let M^3 be a complete, one-connected, three-dimensional Riemannian manifold with sectional curvature $K_{M^3} \leqq k \leqq 0$, and let N_k^3 be the model space with constant sectional curvature k. If $E \subset M^3$ is a compact domain with smooth boundary ∂E, and $\bar{E} \subset N_k^3$ is a geodesic ball with the same volume as E, then*

$$\text{area}(\partial E) \geqq \text{area}(\partial \bar{E}) .$$

Moreover, if $\text{area}(\partial E) = \text{area}(\partial \bar{E})$ *then E is isometric to $\bar{E}$.*

To indicate the idea of the proof of Theorem 2 we need

Definition 3 (Compare [BP, Gall, GLP]) The *isoperimetric profile* of a Riemannian manifold M^n is the function I_{M^n}: $[0, \text{vol}(M^n)) \to R$ defined by

$$I_{M^n}(V) = \inf\{\text{area}(\partial E) | E \subset M^n \text{ a compact domain with smooth boundary } \partial E, \text{vol}(E) = V\} .$$

Except for the last sentence, the conclusion of Theorem 2 can be restated as

$$(*) \qquad I_{M^3} \geqq I_{N_k^3}|_{[0, \text{vol}(M^3))} .$$

Observation 4 [Alm, BP, Gall] *Let $E_0 \subset M^n$ be a compact domain with smooth boundary ∂E_0, and let $V = \text{vol}(E_0)$. Suppose* $\text{area}(\partial E_0) = I_{M^n}(V)$, *i.e. E_0 has least boundary area among domains with volume V. The first variation formulas for volume and area imply that the mean curvature function of ∂E_0 (the trace of the second fundamental form) is everywhere equal to some constant H. We have*

$$(D_{-}I)(V) \overset{\text{def}}{=} \lim_{\Delta V \to 0^-} \frac{I_{M^n}(V + \Delta V) - I_{M^n}(V)}{\Delta V} \geqq H .$$

Proof. Embed E_0 in a smooth family of domains $\{E_t\}$ satisfying $\frac{d}{dt} \text{vol}(E_t)|_{t=0} \neq 0$. The curve $t \mapsto (\text{vol}(E_t), \text{area}(\partial E_t)) \in R^2$ lies above the graph of I_{M^n}, and by the first variation formulas for volume and area it has slope H at $(\text{vol}(E_0), \text{area}(\partial E_0))$. □

To prove $(*)$, we control the left derivate $D_{-}I_{M^3}$ via Observation 4, i.e. by estimating the mean curvature of the boundaries of minimizing domains. In the case that $E_0 \subseteq M^3$ is a domain with smooth boundary ∂E_0, our estimate for the mean curvature H is:

$$\left[\left(\frac{H}{2}\right)^2 + k\right] \text{area}(\partial E_0) \geqq 4\pi .$$

This estimate appears (in slightly disguised form) in proposition 8, and is analogous to the mean curvature estimate in [Alm]. When ∂E_0 is smooth and homeomorphic to S^2, the proof of this mean curvature estimate simplifies considerably: using the Gauss-Bonnet formula for the induced Riemannian structure on ∂E_0, the Gauss equations for the surface $\partial E_0 \subset M^3$, and the inequality between arithmetic and geometric means, we have

$$\begin{aligned} 4\pi &= \int_{\partial E_0} (K_{int})\text{area}_{\partial E_0} \\ &= \int_{\partial E_0} (GK_{\partial E_0} + K_{amb})\text{area}_{\partial E_0} \\ &\leqq \int_{\partial E_0} \left[\left(\frac{H}{2}\right)^2 + k\right]\text{area}_{\partial E_0} \\ &= \left[\left(\frac{H}{2}\right)^2 + k\right]\text{area}_{\partial E_0} \end{aligned}$$

where $\text{area}_{\partial E_0}$ is the area form for ∂E_0, $GK_{\partial E_0}$ is the Gauss-Kronecker curvature of $\partial E_0 \subseteq M^3$ (the product of the principal curvatures), and H is the mean curvature of ∂E_0.

In order to follow through on the approach outlined above, we need to know that for every $V > 0$ there is a domain $E_0 \subseteq M^3$ with $\text{vol}(E_0) = V$ and $\text{area}(\partial E_0) = I_{M^3}(V)$. Unfortunately, since M^3 is noncompact such minimizing domains needn't exist. We circumvent this problem by replacing the noncompact manifold M^3 with a compact subset: we work with a geodesic ball $M_1^3 \subset M^3$ large enough to contain the domain E. Standard compactness and regularity theorems from Geometric Measure Theory guarantee, for any $V \in (0, \text{vol}(M_1^3))$, the existence of a domain $E_0 \subseteq M_1^3$ satisfying $\text{vol}(E_0) = V$, $\text{area}(\partial E_0) = I_{M_1^3}(V)$. There is a snag here: ∂E_0 is (a priori) only $C^{1,\alpha}$ (in fact $C^{1,1}$, see [Whi, Sect. 1]) at points where it touches ∂M_1^3. This necessitates the use of the weak notion of mean curvature in Definition 7.

Remarks. 1. Theorem 2 may be generalized to the case where M^3 has a smooth boundary ∂M^3 provided the second fundamental form of ∂M^3 with respect to the inward normal has at most one negative eigenvalue at every point (i.e. when ∂M^3 is next-to-convex).

2. The analytic framework for the proof of Theorem 2 works in higher dimensions as well. The only missing ingredient for a proof of Conjecture 1 for every n is an analog of the estimate in Lemma 5. This doesn't seem to follow from the generalized Gauss-Bonnet formula.

2 Preliminaries

Let M^3 be a three dimensional Riemannian manifold, and let $N^2 \subset M^3$ be a $C^{1,1}$ surface homeomorphic to S^2. By Rademacher's theorem N^2 is twice differentiable almost everywhere, so the Gauss-Kronecker curvature GK_{N^2} of N^2 is a well-defined element of $L^\infty(N^2)$.

Lemma 5 *If the sectional curvature of M^3 satisfies $K_{M^3} \leqq k \leqq 0$ then*

$$\int_{N^2} (GK_{N^2} + k)\text{area}_{N^2} \geqq 4\pi$$

where area_{N^2} *is the area form of* N^2. *Moreover equality holds here only if the sectional curvature of* M^3 *satisfies* $K_{M^3}(\sigma) = k$ *for every two-plane* σ *which is tangent to* N^2.

Proof. When N^2 is smooth the lemma follows immediately from the Gauss-Bonnet formula for the induced Riemannian structure on N^2 and the Gauss equations for the embedding $N^2 \subset M^3$. The general $C^{1,1}$ case follows by regularization. □

The next two definitions give a substitute for the mean curvature of the boundary of a domain when the boundary isn't twice differentiable.

Definition 6 Let M^n be a Riemannian manifold, and let $E \subset M^n$ be a closed set. A *smooth supporting hypersurface* for E at $p \in E$ is a smooth, normally oriented hypersurface $S \subset M^n$ such that $S \cap E = p$ and E lies on the same side of S as the oriented normal vector near p. The set of smooth supporting hypersurfaces for E at p will be denoted $\mathscr{S}(E, p)$.

Definition 7 Let M^n be a connected Riemannian manifold without boundary, and let E be a nonempty, compact, proper subset of M^n. Then $\mathscr{S}(E, q)$ is nonempty for some $q \in E$ and we define the mean curvature of the set E to be

$$H_E \doteq \sup\{H_S(p) \mid p \in E,\ S \in \mathscr{S}(E, p)\}$$

where $H_S(p)$ is the mean curvature of the smooth hypersurface $S \in \mathscr{S}(E, p)$ at p with respect to its oriented normal. Hence if E is a compact domain with C^2 boundary then H_E is just the maximum value of the mean curvature on ∂E.

The next proposition is the principal geometric ingredient in Theorem 2.

Proposition 8 *Let* M^3 *be a complete, one-connected, three dimensional Riemannian manifold without boundary and with sectional curvature satisfying* $K_{M^3} \leqq k \leqq 0$. *Let* $E_0 \subset M^3$ *be a compact set with nonempty interior. We define* $H_k\colon (0, \infty) \to (0, \infty)$ *by letting* $H_k(A)$ *be the mean curvature of a geodesic sphere with area* A *in the model space* N_k^3. *Then*

$$H_{E_0} \geqq H_k(\mathscr{H}^2(\partial E_0)) ,$$

where $\mathscr{H}^2$ *denotes two dimensional Hausdorff measure and* ∂E_0 *is the topological boundary of the set* E_0. *Moreover* $H_{E_0} = H_k(\mathscr{H}^2(\partial E_0))$ *only if* E_0 *is isometric to a geodesic ball* $\bar{E}_0 \subset N_k^3$ *with* $\text{area}(\partial \bar{E}_0) = \mathscr{H}^2(\partial E_0)$.

Proof. If E_0 is convex and ∂E_0 is C^2, then the inequality of the proposition follows directly from an application of Lemma 5 and the inequality between arithmetic and geometric means to both ∂E_0 and $\partial \bar{E}_0$, where $\bar{E}_0 \subset N_k^3$ is a geodesic ball with $\text{area}(\partial \bar{E}_0) = \text{area}(\partial E_0)$. We do the general case (following [Alm]) by taking the convex hull of E_0.

Let D_0 be the closure of the convex hull of E_0. D_0 is compact and convex since we may find a large geodesic ball containing D_0, and geodesic balls in M^3 are convex. Let $D_s = \{x \in M^3 \mid \text{dist}(x, D_0) \leqq s\}$, $C_s = \partial D_s$, for all $s \geqq 0$. D_s is convex and C_s is $C^{1,1}$ (see appendix B in [Alm2]) for each $s > 0$. The nearest point retraction $r\colon M^3 \setminus \text{Interior}(D_0) \to C_0$ is well defined and distance nonincreasing, and we set $r_s = r|_{C_s}$.

We claim that if C_s twice differentiable at $p \in C_s$, and $r_s(P) \in C_0 \cap \partial E_0$, then the mean curvature of C_s at p satisfies

$$H_{C_s}(p) \leqq H_{E_0} - (\text{Ricci}_-)s$$

where $\text{Ricci}_- = \inf\{\text{Ricci}(X, X) | X \in TM^3|_{D_s}, |X| = 1\}$. To see this, pick $S \in \mathscr{S}(D_s, p)$ with second fundamental form satisfying $II_{C_s}(p) > II_S(p)$, and let ν_S be the unit normal field for S which points into D_s at p. Then

$$\bar{S}_0 = \{\exp(s\nu_S(x)) | x \in S\}$$

will contain some $S_0 \in \mathscr{S}(E_0, r_s(p))$ whose mean curvature at $r_s(p)$ satisfies

$$H_{E_0} \geqq H_{S_0}(r_s(p)) \geqq H_S(p) + (\text{Ricci}_-)s$$

by the Riccati equation. Letting $H_S(p)$ tend to $H_{C_s}(p)$ we get $H_{C_s}(p) \leqq H_{E_0} - (\text{Ricci}_-)s$.

Now, using the fact that C_s is homeomorphic to S^2, we may apply Lemma 5 and the inequality between arithmetic and geometric means:

$$\begin{aligned}
4\pi &\leqq \int_{C_s} (k + GK_{C_s})\text{area}_{C_s} \\
&= \int_{r_s^{-1}(\partial E_0)} (k + GK_{C_s})\text{area}_{C_s} + \int_{C_s \backslash r_s^{-1}(\partial E_0)} (k + GK_{C_s})\text{area}_{C_s} \\
&\leqq \left(k + \left(\frac{H_{E_0} - (\text{Ricci}_-)s}{2}\right)^2\right)\text{area}(r_s^{-1}(\partial E_0)) \\
&\quad + k(\text{area}(C_s \backslash r_s^{-1}(\partial E_0))) + \int_{C_s \backslash r_s^{-1}(\partial E_0)} GK_{C_s}\text{area}_{C_s}.
\end{aligned}$$

We will show in a moment that $\lim_{s\to 0}\text{area}(r_s^{-1}(\partial E_0)) = \mathscr{H}^2(\partial E_0 \cap C_0)$ and $\lim_{s\to 0}\int_{C_s\backslash r_s^{-1}(\partial E_0)} GK_{C_s}\text{area}_{C_s} = 0$, so by letting $s \to 0$ in the inequality above, we get

$$4\pi \leqq \left(k + \left(\frac{H_{E_0}}{2}\right)^2\right)\mathscr{H}^2(\partial E_0)\,.$$

On the other hand, if $\bar{E}_0 \subset N_k^3$ is a geodesic ball with $\text{area}(\partial\bar{E}_0) = \mathscr{H}^2(\partial E_0)$ the same reasoning applies, but this time giving the equation

$$4\pi = \left(k + \left(\frac{H_{\bar{E}_0}}{2}\right)^2\right)\text{area}(\partial\bar{E}_0) = \left(k + \left(\frac{H_k(\mathscr{H}^2(\partial E_0))}{2}\right)^2\right)\mathscr{H}^2(\partial E_0)\,.$$

Hence $H_{E_0} \geqq H_k(\mathscr{H}^2(\partial E_0))$.

Now suppose $H_{E_0} = H_k(\mathscr{H}^2(\partial E_0)) = H_{\bar{E}_0}$. Retracing steps we get $\mathscr{H}^2(\partial E_0 \cap C_0) = \mathscr{H}^2(\partial E_0)$. Hence $\mathscr{H}^2(\partial E_0 \cap \text{Interior}(D_0)) = 0$, which implies that either $\mathscr{L}^3(E_0) = \mathscr{L}^3(D_0)$, or $\mathscr{L}^3(E_0) = 0$. Since the latter case is impossible we have $E_0 = D_0$ because $E_0 \subseteq D_0$ is a closed set. Now $r_s^{-1}(\partial E_0) = r_s^{-1}(E_0) = C_s$ which implies that the mean curvatures of the C_s's remain uniformly bounded as $s \to 0$, and therefore their second fundamental forms remain uniformly bounded when $s \to 0$ as well. It follows that $C_0 = \partial D_0 = \partial E_0$ is $C^{1,1}$. Lemma 5 and the inequality between geometric and arithmetic means may be applied directly to ∂E_0, yielding:

1. The equality case in Lemma 5 holds, so $K_{M^3}(\sigma) = k$ for every two-plane σ tangent to ∂E_0.
2. ∂E_0 has mean curvature $H_{E_0} = H_{\bar{E}_0}$ almost everywhere, so it is a C^∞ surface by standard elliptic regularity theory.
3. ∂E_0 has the same second fundamental form as $\partial\bar{E}_0$ everywhere.

We may therefore cut $\bar{E}_0$ out of N_k^3 and glue in E_0 in its stead, getting a $C^{1,1}$ metric with sectional curvature $\leqq k$ almost everywhere. Applying (the $C^{1,1}$ modified version of) [SZ, Theorem 7] we conclude that E_0 is isometric to $\bar{E}_0$.

We now show that $\lim_{s \to 0} \text{area}(r_s^{-1}(\partial E_0)) = \mathscr{H}^2(\partial E_0 \cap C_0)$ and $\lim_{s \to 0} \int_{C_s \setminus r_s^{-1}(\partial E_0)} GK_{C_s} \text{area}_{C_s} = 0$.

Since $r_s|_{r_s^{-1}(\partial E_0)}$ is one-to-one, the area formula for Lipshitz maps [Fed] applied to $r_s : C_s \to C_0 \subset M^3$ gives

$$\int_{r_s^{-1}(\partial E_0)} \text{Jac}(r_s) \text{area}_{C_s} = \mathscr{H}^2(\partial E_0 \cap C_0)$$

where $\text{Jac}(r_s) = |\Lambda^2 r_{s*}|$ is the Jacobian of r_s. Now $\text{Jac}(r_s|_{r_s^{-1}(\partial E_0)}) \to 1$ uniformly as $s \to 0$ since $II_{C_s}|_{r_s^{-1}(\partial E_0)}$ is uniformly bounded in s, giving

$$\begin{aligned} \text{area}(r_s^{-1}(\partial E_0)) &= \int_{r_s^{-1}(\partial E_0)} (1) \text{area}_{C_s} \\ &= \int_{r_s^{-1}(\partial E_0)} \text{Jac}(r_s) \text{area}_{C_s} + \int_{r_s^{-1}(\partial E_0)} (1 - \text{Jac}(r_s)) \text{area}_{C_s} \\ &= \mathscr{H}^2(\partial E_0 \cap C_0) + \int_{r_s^{-1}(\partial E_0)} (1 - \text{Jac}(r_s)) \text{area}_{C_s} \to \mathscr{H}^2(\partial E_0 \cap C_0) \end{aligned}$$

as $s \to 0$.

We now show that $(++)$ $\lim_{s \to 0} \int_{C_s \setminus r_s^{-1}(\partial E_0)} GK_{C_s} \text{area}_{C_s} = 0$. If M^3 is Euclidean space it is easy to check that $GK_{C_s}(p) = 0$ for every $p \in C_s \setminus r_s^{-1}(\partial E_0)$ at which GK_{C_s} is defined, so $(++)$ is immediate. For general M^3, if $C_0 = \partial D_0$ is twice differentiable at $q \in C_0 \setminus \partial E_0$, then $GK_{C_0}(q) = 0$ for otherwise D_0 could be pushed in near q to produce a smaller convex set contaning E_0; therefore $\lim_{s \to 0} GK_{C_s}(r_s^{-1}(q)) = GK_{C_0}(q) = 0$. But in general, it needn't be true that $\lim_{s \to 0} GK_{C_s}(r_s^{-1}(q)) = 0$ when $q \in C_0 \setminus \partial E_0$, so we give a more convoluted argument.

To establish $(++)$ we fix $s_0 > 0$, and show that

1. $|(r_{s_0 s}^* GK_{C_s} \text{area}_{C_s})(p)| \leqq F(s_0, |K|)$
2. $\lim_{s \to 0} (r_{s_0 s}^* GK_{C_s} \text{area}_{C_s})(p) = 0$

where $0 < s < s_0$, $r_{s_0 s} : C_{s_0} \to C_s$ is the Lipshitz closest point map, $p \in C_{s_0} \setminus r_{s_0}^{-1}(\partial E_0)$, C_s is twice differentiable at p, and $|K| = \sup\{|K_{M^3}(\sigma)|\}$ where σ runs over all two planes in D_{s_0}. From 1 and 2 we have

$$\begin{aligned} \int_{C_s \setminus r_s^{-1}(\partial E_0)} GK_{C_s} \text{area}_{C_s} &= \int_{C_{s_0} \setminus r_{s_0}^{-1}(\partial E_0)} (r_{s_0 s}^* GK_{C_s} \text{area}_{C_s}) \\ &\to 0 \text{ as } s \to 0 . \end{aligned}$$

Let $v : C_{s_0} \to TM^3$ be the inward unit normal field for C_{s_0}, and pick $p \in C_{s_0} \setminus r_{s_0}^{-1}(\partial E_0)$ at which v is differentiable. Let $\gamma : [0, s_0] \to M^3$ be the geodesic segment $\gamma(t) = \exp tv(p)$, and for every $e \in T_p C_{s_0}$ let $\bar{e}$ be the Jacobi field along γ given by $\bar{e}(\gamma(t)) = (\exp \circ (t \cdot v))_* e$.

For $s \in (0, s_0]$ consider the maps $W_{s_0 s} : T_p C_{s_0} \to T_{r_{s_0 s}(p)} C_s$ given by $e \to \nabla_{\dot{\gamma}(s_0 - s)} \bar{e}$. We claim that the maps $W_{s_0 s}$ are bounded above uniformly in terms of s_0 and the geometry of M^3, while the lower bound on $W_{s_0 s}$, i.e. $\inf\{|W_{s_0 s} e| \,|\, e \in T_p C_{s_0}, |e| = 1\}$, goes to zero as $s \to 0$. To see the former, note that the second fundamental form of C_{s_0} is bounded above uniformly in terms of s_0 and the geometry of M^3 since C_{s_0} is convex and supported from the inside by a ball of radius s_0; consequently the maps $W_{s_0 s}$ are bounded above uniformly because they are obtained by solving the Jacobi equation with initial conditions determined by the second fundamental form of C_{s_0}. The latter follows from the factorization $W_{s_0 s} = W_s \circ r_{s_0 s *}$, where W_s: $T_{\gamma(s_0 - s)} C_s \to T_{\gamma(s_0 - s)} C_s$ is the Weingarten map for the inward normal to C_s, and the fact that the lower bound on W_s goes to zero with s since $r_{s_0}(p) = r_s(r_{s_0 s}(p)) \in C_0 \setminus \partial E_0$. Now $(r^*_{s_0 s} GK_{C_s} \text{area}_{C_s})(p) = (-W_{s_0 s})^* \text{area}_{C_s}(r_{s_0 s}(p))$ so 1 and 2 follow immediately from the bounds on $W_{s_0 s}$. □

We will need the following from Geometric Measure Theory:

Fact 9 (see [Sim]) (Existence, compactness, and regularity of minimizing domains) *Let M^3 be a compact Riemannian manifold with smooth boundary ∂M^3. If $V \in (0, \mathrm{vol}(M^3)]$, then there is a domain $E_0 \subset M^3$ with C^1 boundary ∂E_0 such that* $\mathrm{vol}(E_0) = V$ *and* $\mathrm{area}(\partial E_0) = I_{M_3}(V) = \inf\{\mathrm{area}(\partial E) \mid E \subseteq M^3, \mathrm{vol}(E) = V\}$.

Moreover, if $E_1, E_2, \ldots \subseteq M^3$ is a sequence of domains with C^1 boundary with $\mathrm{vol}(E_i) \to V > 0$ *and* $\mathrm{area}(\partial E_i) = I_{M^3}(\mathrm{vol}(E_i))$ *then there is a domain E_0 with C^1 boundary and a subsequence $\{E_{i_k}\}$ such that*

1. *$\partial E_{i_k} \to \partial E_0$ in the C^1 topology.*
2. *The characteristic functions $\chi_{E_{i_k}}$ converge to χ_{E_0} in $L^1(M^3)$.*

Lemma 10 *Let M^n be a Riemannian manifold. Suppose $E_0 \subset M^n$ is a compact domain with C^1 boundary ∂E_0, $p \in \partial E_0$, $S \in \mathscr{S}(E_0, p)$ (see Definition 6), and assume the mean curvature of S at p with respect to the inward normal of ∂E_0, $H_S(p)$, satisfies $H_S(p) > H_0$. Then there is a family of domains with C^1 boundary $\{E_t\}$ such that*

1 $E_t \subseteq E_0$ *for* $t \geqq 0$
2 $\mathrm{vol}(E_t)$ *and* $\mathrm{area}(\partial E_t)$ *are smooth functions of* t

3. $\dfrac{d}{dt}\mathrm{vol}(E_t)|_{t=0} < 0$

4. $\dfrac{d}{dt}\mathrm{area}(\partial E_t)|_{t=0} < H_0 \dfrac{d}{dt}\mathrm{vol}(E_t)|_{t=0}$.

Proof. If ∂E_0 is smooth near p, then the mean curvature of ∂E_0 is well defined and satisfies $H_{\partial E_0}(p) \geqq H_S(p) > H_0$. In this case the lemma follows by pushing ∂E_0 inward near p and applying the first variation formulas for area and volume. We now turn to the general case.

Fix $H \in (H_0, H_S(p))$. Choose $\bar{S} \subset S \subset M^n$ a compact, connected, hypersurface with smooth boundary, and $\varepsilon > 0$ such that

(i) $p \in \bar{S}$
(ii) The normal exponential map $v\bar{S} \to M^n$ is well defined and one-to-one on $v_\varepsilon \bar{S} = \{\xi \in v\bar{S} \mid |\xi| < \varepsilon\} \subset v\bar{S}$. For the rest of the proof this map $v_\varepsilon \bar{S} \to M^n$ will be denoted simply exp.
(iii) $\exp(v_\varepsilon \bar{S}|_{\partial \bar{S}}) \cap E_0 = \varnothing$
(iv) Let $v: \bar{S} \to v\bar{S}$ be the unit normal vector field restricted from S, and let $N^n = \exp(v_\varepsilon \bar{S}) \subset M^n$. Define $s \in C^\infty(N^n)$ by $(s \circ \exp)(\xi) = \langle \xi, v \rangle$ for $\xi \in v_\varepsilon \bar{S}$, in other words s is the signed distance function from $\bar{S}$. We want $\varepsilon > 0$ small enough that ∇s points inward (i.e. toward E_0) along $\partial E_0 \cap N^n$.

(v) The mean curvature of the hypersurface with boundary $s^{-1}(\lambda)$ with respect to the unit normal field ∇s is $\geqq H$ for every $\lambda \in (-\varepsilon, \varepsilon)$.

The family of domains $\{E_t\}$ will be produced by "squeezing" the box N^n, i.e. by applying the flow Φ_t of a vector field $X = (f \circ s)\nabla s$, for suitably chosen f, to $E_0 \cap N^n$. By taking $f \in C^\infty(-\varepsilon, \varepsilon)$ with $f \geqq 0$ and support$(f) \subset (-\varepsilon, \delta]$, $0 < \delta < \varepsilon$, we will get a family of domains with C^1 boundary $\{E_t\}$ such that conditions 1 and 2 are satisfied.

Let C be the maximum value of $|k|$ where k runs over all the principal curvatures of the hypersurfaces $s^{-1}(\lambda)$, $\lambda \in (-\varepsilon, \delta)$. Find $\lambda_0 \in (0, \delta)$ such that area$(\partial E_0 \cap s^{-1}((\lambda_0 - \sigma, \lambda_0 + \sigma))) \to 0$ as $\sigma \to 0$. Fix $\sigma > 0$ and choose $f \in C^\infty(-\varepsilon, \varepsilon)$ such that $f \geqq 0$, support$(f) \subset (-\varepsilon, \lambda_0 + \sigma)$, $f(x) \geqq 1$ for $x \in (-\varepsilon, \lambda_0]$, $f(x) \leqq 1$ for $x \in [\lambda_0, \lambda_0 + \sigma]$, $f' \leqq 0$, and finally $f'(x) < -(\max(H, 0) + (n-2)C)$ when $x \in (-\varepsilon, \lambda_0]$. We have

$$\frac{d}{dt}\operatorname{area}(\partial E_t)\Big|_{t=0} = \frac{d}{dt}\operatorname{area}(\Phi_t(\partial E_0 \cap N^n))\Big|_{t=0}$$

$$= \int_{\partial E_0 \cap N^n} \frac{d}{dt}(\operatorname{Jac}_{\partial E_0}\Phi_t)\Big|_{t=0}$$

where $\operatorname{Jac}_{\partial E_0}\Phi_t = |\Lambda^{n-1}(\Phi_t|_{\partial E_0})_*|$, and

$$\frac{d}{dt}\operatorname{vol}(E_t)\Big|_{t=0} = \int_{\partial E_0 \cap N^n} \langle X, \nu_{\partial E_0}\rangle < 0$$

so 3 holds.

Pick $q \in s^{-1}(\lambda) \cap \partial E_0$. Let $e_1, \ldots, e_{n-1} \in T_q(s^{-1}(\lambda))$ be an orthonormal basis of principal directions for the hypersurfaces $s^{-1}(\lambda)$, and let $k_1, \ldots, k_{n-1}$ be the corresponding principal curvatures with respect to the normal direction ∇s. Set $e_n = \nabla s$, and $k_n = -f'(s(q))$. Calculation shows that if $\nu_{\partial E_0}(q) = \sum_{i=1}^n \alpha_i e_i$, then

$$\frac{d}{dt}(\operatorname{Jac}_{\partial E_0}\Phi_t)(q)\Big|_{t=0} = -f(s(q))\left(\sum_{i=1}^n \alpha_i^2\left(\sum_{j \neq i} k_j\right)\right)$$

which is

$$\leqq f(s(q))(n-1)C\,.$$

If $q \in \partial E_0 \cap s^{-1}((-\varepsilon, \lambda_0))$, then

$$\frac{d}{dt}(\operatorname{Jac}_{\partial E_0}\Phi_t)(q)\Big|_{t=0} = -f(s(q))\left(H_{s^{-1}(\lambda)}(q)\alpha_n^2 + \sum_{i=1}^{n-1}\alpha_i^2\left(\sum_{j \neq i} k_j\right)\right)$$

$$\leqq -f(s(q))\left(H\alpha_n^2 + \sum_{i=1}^{n-1}\alpha_i^2\left(\sum_{j \neq i} k_j\right)\right)$$

$$\leqq -f(s(q))\left(H\alpha_n^2 + \sum_{i=1}^{n-1}\alpha_i^2(\max(H, 0))\right)$$

$$= -f(s(q))(H\alpha_n^2 + (1-\alpha_n^2)(\max(H, 0)))$$

$$\leqq f(s(q))H\alpha_n = H\langle X(q), \nu_{\partial E_0}\rangle$$

since $-1 \leqq \alpha_n = \langle e_n, \nu_{\partial E_0} \rangle = \langle \nabla s, \nu_{\partial E_0} \rangle < 0$.

Now

$$\frac{d}{dt}\operatorname{area}(\partial E_t)\big|_{t=0} = \int_{\partial E_0 \cap N^n} \frac{d}{dt}(\operatorname{Jac}_{\partial E_0}\Phi_t)(q)\big|_{t=0}$$

$$= \int_{\partial E_0 \cap s^{-1}(-\varepsilon, \lambda_0)} \frac{d}{dt}(\operatorname{Jac}_{\partial E_0}\Phi_t)(q)\big|_{t=0} + \int_{\partial E_0 \cap s^{-1}(\lambda_0, \lambda_0+\sigma)} \frac{d}{dt}(\operatorname{Jac}_{\partial E_0}\Phi_t)(q)\big|_{t=0}$$

$$\leqq H \int_{\partial E_0 \cap s^{-1}(-\varepsilon, \lambda_0)} \langle X, \nu_{\partial E_0} \rangle + (n-1)C\operatorname{area}(\partial E_0 \cap s^{-1}(\lambda_0, \lambda_0 + \sigma))$$

$$= H\frac{d}{dt}\operatorname{vol}(E_t)\big|_{t=0} - \int_{\partial E_0 \cap s^{-1}(\lambda_0, \lambda_0+\sigma)} \langle X, \nu_{\partial E_0} \rangle$$

$$+ (n-1)C\operatorname{area}(\partial E_0 \cap s^{-1}(\lambda_0, \lambda_0 + \sigma))$$

$$\leqq H\frac{d}{dt}\operatorname{vol}(E_t)\big|_{t=0} + ((n-1)C + 1)\operatorname{area}(\partial E_0 \cap s^{-1}(\lambda_0, \lambda_0 + \sigma)) .$$

Hence by letting $\sigma \to 0$ we will get condition 4 of the lemma satisfied since $\frac{d}{dt}\operatorname{vol}(E_t)$ stays bounded away from zero. □

3 The proof of Theorem 2

Pick a geodesic ball which contains the domain $E \subset M^3$, and call it M_1^3. M_1^3 is a compact domain with smooth boundary ∂M_1^3. We will show that (see Definition 3) $I_{M_1^3} \geqq I_{N_k^3}^{ball}|_{[0, \operatorname{vol}(M_1^3)]}$ where $I_{N_k^3}^{ball}: [0, \infty) \to [0, \infty)$ is the geodesic ball profile of the model space N_k^3 with constant sectional curvature k, i.e.

$$I_{N_k^3}^{ball}(V) = \operatorname{area}(\partial(Ball))$$

where $Ball \subset N_k^3$ is a geodesic ball with volume V.

First note that by Fact 9, $I_{M_1^3}: [0, \operatorname{vol}(M_1^3)] \to R$ is continuous.

Fix $V \in (0, \operatorname{vol}(M_1^3))$ and let $E_0 \subset M_1^3$ be a domain with C^1 boundary ∂E_0 satisfying $\operatorname{vol}(E_0) = V$, $\operatorname{area}(\partial E_0) = I_{M_1^3}(V)$; the existence of such a domain is guaranteed by Fact 9. By Lemma 8

$$H_{E_0} = \sup\{H_S(p) \mid p \in E_0, S \in \mathscr{S}(E_0, p)\}$$

$$\geqq H_k(\operatorname{area}(\partial E_0))$$

$$= H_k(I_{M_1^3}(V))$$

where, as before, $H_k(A)$ is the mean curvature of a geodesic sphere in N_k^3 with surface area A. By Lemma 10, for every $H < H_{E_0}$ there is a family of domains with

C^1 boundary $\{E_t\}$ such that

1. $E_t \subset E_0, t \geqq 0$.
2. $\text{vol}(E_t)$ and $\text{area}(\partial E_0)$ are smooth functions of t.
3. $\frac{d}{dt}\text{vol}(E_t)|_{t=0} < 0$.
4. $\frac{d}{dt}\text{area}(\partial E_t)|_{t=0} < H\frac{d}{dt}\text{vol}(E_t)|_{t=0}$.

From the fact that the curve $t \mapsto (\text{vol}(E_t), \text{area}(\partial E_t))$ lies above the graph of $I_{M_1^3}$ we may conclude that

$$(D_- I_{M_1^3})(V) \overset{\text{def}}{=} \liminf_{\Delta V \to 0^-} \frac{I_{M_1^3}(V+\Delta V) - I_{M_1^3}(V)}{\Delta V}$$

$$\geqq H_{E_0} \geqq H_k(I_{M_1^3}(V)) > 0 \ .$$

We will use this to deduce that $I_{M_1^3} \geqq I_{N_k^3}^{ball}|_{[0,\text{vol}(M_1^3)]}$. Foliate the upper half plane using the graph of $I_{N_k^3}^{ball}$ and all its translates $\{(\bar{V}, I_{N_k^3}^{ball}(\bar{V} - V_0)) | \bar{V} \in [V_0, \infty)\}$. Since $(I_{N_k^3}^{ball})'(\bar{V}) = H_k(I_{N_k^3}^{ball}(\bar{V}))$ for all $\bar{V} \in (0, \infty)$, the graph of $I_{M_1^3}$ crosses this foliation monotonically. It follows that $I_{M_1^3} \geqq I_{N_k^3}^{ball}|_{[0,\text{vol}(M_1^3)]}$ because $I_{M_1^3}(0) = I_{N_k^3}(0) = 0$.

Now suppose $\text{area}(\partial E) = I_{N_k^3}^{ball}(\text{vol}(E))$. Then we get $\text{area}(\partial E) = I_{M_1^3}(\text{vol}(E)) = I_{N_k^3}^{ball}(\text{vol}(E))$ which forces $I_{M_1^3} = I_{N_k^3}^{ball}|_{[0,\text{vol}(E)]}$. In particular, $H_E \leqq (D_- I_{M_1^3})(\text{vol}(E)) = (I_{N_k^3}^{ball})'(\text{vol}(E)) = H_k(\text{area}(\partial E))$. By proposition 8, this implies that E is isometric to a geodesic ball in N_k^3. □

Acknowledgements. I would like to thank Chris Croke, Lisa Haner, and Hsueh-ling Huynh for their influence on the evolution of this paper. I would also like to thank the referee for bringing [Whi] to my attention.

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