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On the intersection of subgroups of a free group

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Summary. The Hanna Neumann Conjecture says that the intersection of subgroups of rank $n+1$ and $m+1$ of a free group has rank at most $nm+1$. This paper proves the conjecture for the case $m=1$. (See Theorem 1.) Our methods imply that the strengthened Hanna Neumann Conjecture is also true in this case (Theorem 2′).

1 Introduction

Let U and V be subgroups of a free group F. By the Nielsen-Schreier Theorem U, V and $U \cap V$ are all free groups. Suppose U and V are finitely generated, then so is $U \cap V$ by a result of Howson [4]. Hanna Neumann [6] improved Howson's upper bound on the rank of the intersection by showing that if U and V are nontrivial groups of rank $n+1$ and $m+1$ then rank $(U \cap V) \leqq 2nm+1$. She asked if the tighter bound rank $(U \cap V) \leqq nm+1$ is also true. This problem has come to be known as the "Hanna Neumann Conjecture". The conjecture is not yet proved in general and the best known upper bound for the rank of $U \cap V$ is $2nm - \min(n, m) + 1$ proved by Burns [1] (see also [8, 9] and for a modest improvement [2]).

The main result of this paper is proving the Hanna Neumann Conjecture for the case $m=1$, that is when one of the subgroups has rank 2:

Theorem 1 *If U and V are subgroups of the free group and V has rank two then* rank $(U \cap V) \leqq$ rank U.

Special attention has been paid to this case and a few subcases of it are solved in [3] and in Sect. 4 of [7].

We follow the graph-theoretical approach used by most authors on the subject, see [3, 5, 8]. Part of our notation is taken from [8]. In Sect. 2 we formulate a graph-theoretical version of the strengthened Hanna Neumann Conjecture. This conjecture implies the original Hanna Neumann Conjecture and was asked as a question in [7]. Theorem 2 solves a special case of the conjecture and implies

Theorem 1. In Sect. 3 we give the proof of Theorem 2. In Sect. 4 we briefly discuss possible generalizations. Theorem 6 is another result obtainable by our methods.

2 Graph-theoretical reformulation of Theorem 1

We may assume that U is finitely generated since otherwise Theorem 1 trivially holds. We may also suppose without loss of generality that U and V are subgroups of $F = F_{a,b}$ the group freely generated by elements a and b since the subgroup generated by U and V is a finitely generated free group and therefore it is isomorphic to a subgroup of $F_{a,b}$.

Definitions. For any subgroup W of F we define Γ_W to be a directed graph on the vertices $\{Wx | x \in F\}$ with a directed edge with label a from Wx to Wxa and an edge with label b from Wx to Wxb for any x in $F_{a,b}$.

The *core* of a connected graph is the minimal connected subgraph of it containing all circles of the graph. The *core* of a disconnected graph is the union of the cores of its components.

Let W be a subgroup of F. We define $\Gamma(W)$ to be the core of Γ_W.

We call a finite directed graph Γ with labels a and b on the edges *normal* if every vertex is the initial/terminal vertex of at most one edge of each label. We call a normal graph *basic* if it is connected and every vertex has degree at least 2. For a normal graph Γ let $Y(\Gamma)$ be the set of branching points, i.e. the vertices of the graph of degree at least 3. Let $X(\Gamma)$ be the set of vertices of degree 4 (of course, this is the maximal possible degree). We define the *branching number* of the graph to be $b(\Gamma) = |Y(\Gamma)| + |X(\Gamma)|$.

Let us first observe that Γ_W is a connected graph where every vertex is the initial/terminal vertex of exactly one edge of each label. It is not hard to prove that the rank of W is the cyclomatic number of Γ_W, that is the number of edges outside a spanning tree. It is easy to see that $\Gamma = \Gamma(W)$ is a basic normal graph whenever W is finitely generated, otherwise it is infinite. Suppose W has finite rank $n + 1$. Now $n + 1$ is the cyclomatic number of Γ_W, which equals to that of Γ which can be computed as $|E(\Gamma)| - |V(\Gamma)| + 1$, since the latter graph is finite. Therefore we have

$$n = |E(\Gamma)| - |V(\Gamma)| = \frac{1}{2} \sum_{v \in V(\Gamma)} (d_v - 2) = \frac{1}{2}(|Y(\Gamma)| + |X(\Gamma)|) = b(\Gamma)/2$$

and

$$\text{rank } W = b(\Gamma(W))/2 + 1 \,. \tag{1}$$

Definition. We define the product $\Gamma_1 \times \Gamma_2$ of the directed $\{a, b\}$ labelled graphs Γ_1 and Γ_2 the following way. The vertex set of the product is the Cartesian product of the vertex sets and we draw an edge with a certain label between two vertices if there is an edge of the same label with the same orientation in both projections.

Let U and V be two subgroups of F. It is easy to see that $\Gamma_{U \cap V}$ is a connected component of $\Gamma_U \times \Gamma_V$. Since the core of the product is contained in the product of the cores this yields the following observation. If $U \cap V$ is nontrivial then $\Gamma(U \cap V)$

is the core of a connected component of $\Gamma(U) \times \Gamma(V)$. Applying this observation and (1) one can easily show that Theorem 1 is a consequence of the following graph theoretical result:

Theorem 2 *Let Γ_1 and Γ_2 be basic normal graphs. If $b(\Gamma_2) = 2$ then the branching number of the core of $\Gamma_1 \times \Gamma_2$ is not greater than $b(\Gamma_1)$.*

In fact this is a little bit stronger than Theorem 1, since for that it would be enough to prove the bound for a single component of the core of $\Gamma_1 \times \Gamma_2$. We conjecture that the stronger result holds in general too:

Conjecture. *If Γ_1 and Γ_2 are basic normal graphs then the branching number of the core of $\Gamma_1 \times \Gamma_2$ is not greater than $b(\Gamma_1)b(\Gamma_2)/2$.*

Note that this conjecture implies the Hanna Neumann Conjecture. To give group-theoretic meaning to this conjecture we note that the components of $\Gamma_U \times \Gamma_V$ are isomorphic to $\Gamma_{x-Ux\cap V}$ for $x \in F$ and all the graphs of the latter form correspond to components in $\Gamma_U \times \Gamma_V$. Note also that $\Gamma_{x-Ux\cap V}$ and $\Gamma_{y-Uy\cap V}$ correspond to the same component of $\Gamma_U \times \Gamma_V$ if and only if $UxV = UyV$. We note finally that this correspondence carries over to the components of the core of $\Gamma(U) \times \Gamma(V)$ and $\Gamma(x^- Ux \cap V)$ with the slight modification that for elements $x \in F$ for which $x^- Ux \cap V$ is trivial the graph $\Gamma(x^- Ux \cap V)$ is empty and therefore it does not correspond to any component of the core of $\Gamma(U) \times \Gamma(V)$. The above observations together with (1) and with noticing that the branching number of a disconnected graph is the sum of that of the components yields that the following group-theoretical statement is equivalent to the conjecture above.

Conjecture (Strengthened Hanna Neumann Conjecture, [7]) *If U and V are subgroups of a free group F and $X \subset F$ is a set of doublecoset representatives for those doublecosets UxV for which $x^- Ux \cap V$ is non-trivial then*

$$\sum_{x \in X} (\text{rank}(x^- Ux \cap V) - 1) \leqq (\text{rank } U - 1)(\text{rank } V - 1) .$$

The above line of reasoning also gives that the group-theoretic equivalent of Theorem 2 is the special case of this latter conjecture where V has rank 2 and thus we have:

Theorem 2' *The strengthened Hanna Neumann Conjecture holds for subgroups U and V if one of them has rank 2.*

3 Proof of Theorem 2

Definitions. We call $S = \{a, b, a^-, b^-\}$ the set of *literals*. For any $x \in F = F_{a,b}$ there is a unique way to write x in normal form $x = c_1 \ldots c_k$ with $k \geqq 0$ and $c_i \in S$ for $i = 1, \ldots, k$, where $c_i c_{i+1} \neq 1$ for $i = 1, \ldots, k-1$. We call k the *length* of x and denote it by $|x|$. We say that x *starts* with $y \in F$ if $y = c_1 \ldots c_i$ for some $0 \leqq i \leqq k$. Note that if $x \neq 1$ then it starts with exactly one literal c_1. We call $x \in F$ a *conjugate element* if the length of x is greater than that of some conjugate of x, i.e. if x and x^- start with the same literal.

Let Γ be a normal graph. If there is an edge labelled a (respectively b) in Γ from v to w we also refer to the same edge as directed from w to v with the label a^-

(respectively b^-). For any literal c and $v \in V(\Gamma)$ let vc denote the terminal vertex of the edge labelled c starting at v. In case there is no such edge we say that vc is undefined. If the normal form of $x \in F$ is $x = c_1 \dots c_k$ and v is a vertex of Γ we define $vx = (\dots((vc_1)c_2)\dots)c_k$. If the right hand side is undefined we say that vx is undefined and denote this by $vx \notin \Gamma$. We denote the contrary case by $vx \in \Gamma$. Note that if $w = vx \in \Gamma$ then $v = wx^-$ and if x starts with y then vy is also defined. Notice that the identity $(vx)y = v(xy)$ holds whenever the left hand side is defined. But $v(xy) \in \Gamma$ implies that $(vx)y$ is defined only if the product xy starts with x, i.e. x^- and y don't start with the same literal. By vxy we mean $v(xy)$.

Let Γ be a normal graph $v \in V(\Gamma)$ and $x \in F$ with the normal form $x = c_1 \dots c_k$. Let i_0 be the greatest $i = 0, \dots, k$ for which $vc_1 \dots c_i$ is defined. By building vx we mean adding a path to the graph Γ at the vertex $vc_1 \dots c_{i_0}$, the length of the path is $k - i_0$ and the labels on the outward directed edges are $c_{i_0+1}, \dots, c_k$. Note that we modify the graph only if $vx \notin \Gamma$ and vx is always defined in the modified graph. Observe that building vx increases the branching number by at most 1.

Now we are ready to state and prove our main lemma.

Lemma 3 *Let Γ be a normal graph and x a conjugate element in F. Then we have*

$$|\{v \in Y(\Gamma) | vx \in \Gamma\}| + |\{v \in Y(\Gamma) | vx^- \in \Gamma\}| \leqq b(\Gamma) . \tag{2}$$

Proof. We proceed by induction on $|x|$. Let us suppose that the lemma is true for any conjugate element x' with $|x'| < |x|$. We may also suppose without loss of generality that x starts with the literal a. Let us call a branching point v in Γ *good* if both vx and vx^- are defined, and we call v *bad* if neither of them is defined. We denote the set of good branching points by $G(\Gamma)$ and the set of the bad ones by $B(\Gamma)$. With this notation (2) is equivalent to

$$|G(\Gamma)| \leqq |B(\Gamma)| + |X(\Gamma)| . \tag{3}$$

Now we write $x = yzy^-$ where $y \neq 1 \neq z$ and z, z^- and y^- start with different literals, say c, d and e, all from S. The elements y and z are uniquely determined by the conjugate element x. Note that y starts with a. For any good branching point v vy is defined and it is also a branching point since edges labelled c, d and e go out of it. Note also that the function f defined by $f(v) = vy$ is 1-1 where defined. We distinguish cases according to the values of c, d and e.

Suppose first that $a \notin \{c, d, e\}$. In this case for any good branching point v vy is either a bad branching point since there is no edge labelled a going out from it or it is a degree 4 vertex. This directly proves inequality (3) since the 1-1 map f maps the set on the left hand side into the union of the sets on the right hand side.

For the second case we take $a = e$, that is when y^- starts with a. In this case y is a conjugate element, with $|y| < |x|$ so we may use the inductive hypothesis for y. Let v be a not bad branching point then $w = vy$ is defined since both x and x^- start with y. Therefore we have $Y(\Gamma) \backslash B(\Gamma) \subset \{v \in Y(\Gamma) | vy \in \Gamma\}$. If v is good then w is a branching point and $v = wy^-$ is defined. So we have $|G(\Gamma)| \leqq |\{w \in Y(\Gamma) | wy^- \in \Gamma\}|$ since f maps the former set into the latter. The left hand side of (2) can be rewritten as $|Y(\Gamma) \backslash B(\Gamma)| + |G(\Gamma)|$ which is not greater than $|\{v \in Y(\Gamma) | vy \in \Gamma\}| + |\{v \in Y(\Gamma) | vy^- \in \Gamma\}|$ by the observations above. Since this is just the left hand side of (2) for y the inductive hypothesis completes the proof for this case.

The last case is when $a = c$ or d. By symmetry we may assume that $a = d$, i.e. z^- starts with a. Now $t = yz$ is a conjugate element shorter than $|x|$ for which we can use the inductive hypothesis. We modify Γ. Let us build vx for every $v \in Y(\Gamma) = Y$. We call the constructed graph Γ'. Since building a new vertex increases the branching number by at most 1 we have

$$|\{v \in Y | vx \notin \Gamma\}| \geqq b(\Gamma') - b(\Gamma) . \tag{4}$$

For any $v \in Y$ we have $vx \in \Gamma'$ and therefore $vt \in \Gamma'$. If furthermore $vx^- \in \Gamma$ then $w = vy$ is a branching point in Γ' and wt^- is defined. Therefore $|\{v \in Y | vx^- \in \Gamma\}| \leqq |\{w \in Y(\Gamma') | wt^- \in \Gamma'\}|$ since f maps the former set into the latter. This yields

$$\begin{aligned}
&|\{v \in Y(\Gamma) | vx \in \Gamma\}| + |\{v \in Y(\Gamma) | vx^- \in \Gamma\}| \\
&= |Y| - |\{v \in Y | vx \notin \Gamma\}| + |\{v \in Y | vx^- \in \Gamma\}| \\
&\leqq |\{v \in Y(\Gamma') | vt \in \Gamma'\}| - (b(\Gamma') - b(\Gamma)) + |\{w \in Y(\Gamma') | wt^- \in \Gamma'\}| \\
&\leqq b(\Gamma') - (b(\Gamma') - b(\Gamma)) = b(\Gamma)
\end{aligned}$$

by (4) and the induction hypothesis. □

First we give a trivial reformulation of this lemma then we strengthen it a little bit.

Lemma 4 *Let Γ be a normal graph and $x \in F$ a conjugate element. We have*

$$|\{v \in Y(\Gamma) | vx \in \Gamma \text{ or } vx^- \in \Gamma\}| + |\{v \in V(\Gamma) | vx \in \Gamma \text{ and } vx^- \in \Gamma\}| \leqq b(\Gamma) .$$

Proof. The only difference from the previous lemma is that in the second set on the left hand side we allow any vertex not just branching points. To solve this difficulty we add one or two new edges to the graph at every vertex of that set that is not a branching point so that the degree becomes 3. Each such modification increases both $b(\Gamma)$ and the first summand on the left by 1, so applying Lemma 3 to the new graph gives the result. □

Lemma 5 *Let Γ be a normal graph and $x \in F$ a conjugate element. We have*

$$|\{v \in Y(\Gamma) | \exists n \neq 0 \; vx^n \in \Gamma\}| + |\{v \in V(\Gamma) | \exists m < 0 < n \; vx^m \in \Gamma \text{ and } vx^n \in \Gamma\}| \leqq b(\Gamma) .$$

Proof. We extend Γ again. Let $v \in V(\Gamma)$ such that $\exists n > 0$ with $vx^n \in \Gamma$. Let n_0 be the first such n. Let us build vx^i for $i = 1, \ldots, n_0 - 1$. For any of these vertices w both wx and wx^- will be defined in the new graph. Let Γ' be the graph that we construct by making this extension for all vertices v of Γ with the given property. Then by the above observation we have:

$$|\{v \in V(\Gamma') \setminus V(\Gamma) | vx \in \Gamma' \text{ and } vx^- \in \Gamma'\}| \geqq b(\Gamma') - b(\Gamma) .$$

Note that if $v \in V(\Gamma)$ and $vx^n \in \Gamma$ with $n > 0$ then $vx \in \Gamma'$. If $vx^n \in \Gamma$ with $n < 0$ then we extended the graph at $w = vx^n$ since $wx^{-n} \in \Gamma$ and therefore we have $vx^- \in \Gamma'$. Applying Lemma 4 for Γ' together with the last two observations we get the statement of Lemma 5. □

Proof of Theorem 2 First note that the degree of $v \times w \in V(\Gamma_1 \times \Gamma_2)$ is not bigger than the smaller of the degrees of v and w. Furthermore, if $(v \times w)x$ is defined then its projections are vx and wx. Let Γ be the core of $\Gamma_1 \times \Gamma_2$. We distinguish cases according to the topological form of Γ_2. A basic normal graph with branching number 2 can have three different forms.

The first case is when Γ_2 consists of two circles with a common vertex of degree 4. This case is trivial since here the branching number of the whole product $\Gamma_1 \times \Gamma_2$ is at most $b(\Gamma_1)$ since the branching points of the product must have the form $v \times w$ where v is a branching point of Γ_1 and w is the degree 4 vertex in Γ_2.

The second case is when Γ_2 consists of two circles connected by a path. Let s and t be the two degree 3 vertices, y the product of the labels on the directed path from s to t, and z the product of labels on the directed circle starting at t. We may consider the circle to be directed in any direction. So we have $sy = tz = t$. The word $x = yzy^-$ is a conjugate element of F. By symmetry we may assume that y starts with a. By the observation at the beginning of the proof there are no degree 4 vertices in Γ and any branching point in Γ is the product of a branching point of Γ_1 and one of s and t.

First we claim that if $w = u \times s \in Y(\Gamma)$ for a vertex $u \in Y(\Gamma_1)$ then there exists an $n \neq 0$ such that $ux^n \in \Gamma_1$. The projection to Γ_2 of the three edges starting at w are those starting at s one of which is labelled a so $wa \in \Gamma$. It follows from this and the fact that Γ is the core of a graph that there is an $x' \in F$ starting with a such that $wx' = w$. Projecting this to Γ_2 yields $sx' = s$. Using the simple structure of Γ_2 and that x' starts with a this means that x' must start with $yz^n y^- = x^n$ for some $n \neq 0$. Since $wx' \in \Gamma$ we have also $wx^n \in \Gamma$ and therefore $ux^n \in \Gamma_1$ as claimed.

Now we claim that if $v \times t \in Y(\Gamma)$ for a vertex $v \in Y(\Gamma_1)$ then $u = vy^- \in \Gamma_1$ and there exist numbers $m < 0 < n$ such that $ux^m \in \Gamma_1$ and $ux^n \in \Gamma_1$. The proof is as follows: The labels on the three edges starting at $w = v \times t$ are identical to the labels on the edges starting at t. These labels are the literals with which z, z^- and y^- start, let us call them c, d and e. First we take $x' \in F$ starting with e such that $wx' = w$. Projecting to Γ_2 yields $tx' = t$ and therefore x' starts with y^-. So $wx' \in \Gamma$ implies that $w' = wy^- \in \Gamma$. Projecting this to Γ_1 proves the first part of the claim, that $u = vy^- \in \Gamma_1$. Using again that Γ is the core of a graph we can find an element $y' \in F$ starting with d and $z' \in F$ starting with c such that $wy' = wz' = w'$. Projecting to Γ_2 we have $ty' = tz' = s$. Using the simple structure of Γ_2 again and that we know what literals they start with we conclude that y' starts with $z^m y^-$ for some $m < 0$ while z' starts with $z^n y^-$ for some $n > 0$. Projecting to Γ_1 we find that both vy' and vz' are defined and therefore so are $vz^m y^-$ and $vz^n y^-$. Since $v = uy$ both $uyz^m y^- = ux^m$ and $uyz^n y^- = ux^n$ are defined as claimed.

The statement of Theorem 2 for this case follows from the last two claims after applying Lemma 5 to the graph Γ_1 and the conjugate element x.

The last case is when Γ_2 consists of two vertices connected by three disjoint paths. The proof in this case is more complicated than that of the previous case. Applying Lemma 5 to Γ_1 is not enough, we have to extend Γ_1.

Let the two degree 3 vertices in Γ_2 be s and t again and let the product of the labels on the directed paths from s to t be y, z_1 and z_2. So we have $sy = sz_1 = sz_2 = t$. The word $x = yz_1^- z_2 y^-$ is a conjugate element of F. As in the previous case there are no degree 4 vertices in Γ and the branching points project to s or t in Γ_2. Let $V = V(\Gamma_1)$. Let V_s, the set of s-vertices consist of all $u \in V$ such that $u \times s \in V(\Gamma)$ and let V_t, the set of t-vertices consist of all $v \in V$ such that $v \times t \in V(\Gamma)$.

Let Y_s (respectively Y_t) be the set of all $u \in V_s$ ($v \in V_t$) for which $u \times s \in Y(\Gamma)$ (respectively $v \times t \in Y(\Gamma)$). We have

$$b(\Gamma) = |Y_s| + |Y_t| . \tag{5}$$

It is easy to see that for any branching point w of Γ that projects to s (respectively t) each of wy, wz_1 and wz_2 (respectively wy^-, wz_1^- and wz_2^-) are defined in Γ and if w is not a branching point but still maps to s (respectively t) then two of the three products are defined. All the defined products project to t (respectively s) in Γ_2 therefore projecting to Γ_1 yields t-vertices (respectively s-vertices). So if u is an s-vertex (respectively v is a t-vertex) then at least two out of the three products uy, uz_1 and uz_2 (respectively vy^-, vz_1^- and vz_2^-) are defined t-vertices (s-vertices) in Γ_1, while if $u \in Y_s$ (respectively $v \in Y_t$) then this is true for all three products.

For any t-vertex v we build vy^- and then for any s-vertex u we build both uz_1y^- and uz_2y^-. Let us call the constructed graph Γ_1'. First we need an estimate on the increase of the branching number. By building $u = vy^-$ for a t-vertex v we increase the branching number by 1 only if $vy^- \notin \Gamma_1$. In this case both $uyz_1^- = vz_1^-$ and $uyz_2^- = vz_2^-$ are s-vertices and therefore for the new vertex u we have $uyz_1^- z_2 y^- = ux \in \Gamma_1'$ and $uyz_2^- z_1 y^- = ux^- \in \Gamma_1'$. Now if building uz_1y^- (uz_2y^-) for an s-vertex u increased the branching number then uz_1 (uz_2) was not a t-vertex (remember we started with building vy^- for all t-vertices v). So building only one of uz_1y^- and uz_2y^- could increase the branching number and only if $u \notin Y_s$ but $uy \in \Gamma_1$ and it is a t-vertex. Therefore with $V' = V(\Gamma_1')$ we have

$$b(\Gamma_1') - b(\Gamma_1) \leqq |\{u \in V' \setminus V | ux \in \Gamma_1' \text{ and } ux^- \in \Gamma_1'\}| + |\{u \in V_s \setminus Y_s | uy \in V_t\}| . \tag{6}$$

Take any vertex $v \in Y_t$. Then all $u = vy^-$, vz_1^- and vz_2^- are s-vertices and therefore both $ux = vz_1^- z_2 y^-$ and $ux^- = vz_2^- z_1 y^-$ are defined in Γ_1'. So we conclude that

$$|Y_t| \leqq |\{u \in V | ux \in \Gamma_1' \text{ and } ux^- \in \Gamma_1'\}| . \tag{7}$$

Let $u \in V_s$ with $uy \in V_t$. First note that all $u \in Y_s$ satisfy this condition. Now u must be branching point in Γ_1' since all uy, uz_1 and uz_2 are defined there and y, z_1 and z_2 start with different literals. Since uy is a t-vertex, uyz_i^- is an s-vertex for either $i = 1$ or 2. Therefore we have $uyz_i^- z_{3-i} y^- \in \Gamma_1'$. Since this is either ux or ux^- we have

$$|\{u \in V_s \setminus Y_s | uy \in V_t\}| + |Y_s| \leqq |\{u \in Y(\Gamma_1') | ux \in \Gamma_1' \text{ or } ux^- \in \Gamma_1'\}| . \tag{8}$$

Now if we add up (5), (6), (7), (8) and the statement of Lemma 4 for the graph Γ_1' and x, that says

$$|\{u \in Y(\Gamma_1') | ux \in \Gamma_1' \text{ or } ux^- \in \Gamma_1'\}| + |\{u \in V' | ux \in \Gamma_1' \text{ and } ux^- \in \Gamma_1'\}| \leqq b(\Gamma_1') ,$$

then we get $b(\Gamma) \leqq b(\Gamma_1)$. This completes the proof of Theorem 2. □

4 Concluding remarks

Let us remark that in this proof of Theorem 2 the second case could have been handled the way the third case was, building vy^- for all t-vertices v and applying

Lemma 4 for the constructed graph. We phrased Lemma 5 instead in the belief that it may be interesting in itself.

Although we could not extend our result for subgroups of rank bigger than 2, or what is the same, for graphs of higher cyclomatic number, there are certain topological forms of graphs with higher cyclomatic number where this method can be used. For example the following is not hard to prove.

Theorem 6 *Let V be nontrivial subgroup of the free group F and suppose that $\Gamma(V)$ consists of several disjoint circles, with one of them being connected to all others by disjoint paths. Then the strengthened Hanna Neumann Conjecture is true for V and any nontrivial subgroup U of F.*

Here we suppose that $F = F_{a,b}$, or we have to extend the definition for $\Gamma(V)$ for the general case (this can be done easily). We omit the proof.

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