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Titel: On the degree 5 L-function for Sp (2).

Autor: Kudla, Stephen S.; Rallis, S.; Soudry, D.

Jahr: 1992

PURL: https://resolver.sub.uni-goettingen.de/purl?356556735_0107|log28

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Invent. math. 107, 483–541 (1992)

Inventiones mathematicae
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On the degree 5 L-function for Sp(2)

Stephen S. Kudla [1,*], **Stephen Rallis** [2,**], **and David Soudry** [3,***]

[1] University of Maryland, College Park, MD 20742, USA
[2] Ohio State University, Columbus, Ohio 43210, USA
[3] School of Mathematical Sciences, The Raymond and Beverly Sackler, Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

Oblatum 19-IX-1990 & 23-V-1991

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Introduction

In this paper we study the poles and values of the degree 5 or standard Langlands L-function attached to an irreducible, cuspidal automorphic representation π for the rank 2 symplectic group $G=\mathrm{Sp}(2)$ over a (totally real) number field F. For $\mathrm{Sp}(n)$ with n even, an integral representation of the degree $2n+1$ Langlands L-function, twisted by a certain quadratic character, was discovered by Andrianov in the case of holomorphic Siegel modular forms [1] and was generalized and reformulated by Piatetski-Shapiro and Rallis [26]. The construction depends on the choice of a non-singular symmetric matrix T such that the cusp form in question admits a non-zero T-th Fourier coefficient, and the L-function $L_S(s, \chi_T, \pi)$ which naturally arises in the integral representation is

* Partially supported by NSF Grant DMS-87-04375
** Partially supported by NSF Grant DMS-84-01947
*** Partially supported by The Bat-Sheva de Rothschild Fund for the advancement of science and technology

twisted by the quadratic character χ_T associated to T. It is proved in [26] that the first possible pole of this L-function occurs at the point $s=\frac{n}{2}+1$, and that, if the residue at this point is non-zero, then the cusp form has a non-trivial theta lift to $O(V_T)$, the orthogonal group of the n dimensional quadratic space determined by T.

In the present paper we consider the case $n=2$ in more detail. The possible poles of the L-function now lie at $-1, 0, 1$, and 2, and, by the functional equation, it is sufficient to consider the points $s=1$ and $s=2$. The pole at $s=2$ is completely characterized in [33]. We then consider the pole of the L-function at $s=1$, which is at most simple [33]. We show (Theorem 7.1 (i)) that the non-vanishing of the residue of $L_S(s, \chi_T, \pi)$ at $s=1$ implies that the cusp form has a non-trivial theta lift to the orthogonal group of the 4-dimensional quadratic form $V_T+V_{1,1}$ obtained by adding a hyperbolic plane to V_T. We also show (Theorem 7.1 (ii)) that if the twist of the L-function by a quadratic character χ has a non-vanishing residue at $s=1$, then the cusp form has a non-trivial theta lift to the orthogonal group of the 4-dimensional quadratic form obtained by adding *some* binary space with character χ to V_T. These results, together with those of [33], thus give a complete interpretation of the poles of $L_S(s, \chi, \pi)$ for any quadratic twist χ, provided, of course, that some information about the existence of a non-singular Fourier coefficient is known. On the other hand, it $\chi^2 \neq 1$, then $L_S(s, \chi, \pi)$ is entire.

We also show, as that if π admits a non-singular binary quadratic form T as a nonvanishing Fourier coefficient, then π has a non-trivial theta lift to $O(V_T+V_{2,2})$, where V_T is the two dimensional inner product space associated to T and $V_{2,2}$ is a split inner product space of dimension 4, i.e., π has a theta lift to a certain $O(6)$.

Finally, as a further application of our methods, we answer a question raised by Blasius and Ramakrishnan [2, Conjecture 8.2, p. 60]. We say that an irreducible automorphic representation $\pi=\bigotimes_v \pi_v$ is locally generic if for each place v, the local component π_v of π admits a non-trivial Whittaker model for some non-degenerate character of the unipotent radical of our fixed minimal parabolic subgroup of $G=\mathrm{Sp}(2)$. Note that, in contrast to [2], we do not require that these characters are the restrictions of some global non-degenerate character. We then prove (Theorem 8.1) that if $\pi=\bigotimes_v \pi_v$ is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ which is locally generic and is such that the L-function $L_S(s, 1, \pi)$ does not have a zero at $s=1$, then π is globally generic, i.e., admits a standard Whittaker functional given by the usual Fourier coefficient integral on the space of automorphic forms. Note that we do not need a coherence condition in our definition of locally generic representations because this condition enters the proof only as a restriction on the local dual pairs for which π_v can occur in a theta correspondence (cf. § 8). It is not clear at the moment that the condition on the L-function is actually necessary, particularly in the case in which a stronger coherence condition is included in the locally generic hypothesis.

The main ingredient in the proofs of these results is a kind of generalized Siegel-Weil formula. Specifically, we study the Laurent expansion at $s=\frac{1}{2}$ of the family of Eisenstein series associated to a quadratic character χ of the Siegel

parabolic P of G (see § 1 for notation and definitions). Our Eisenstein series are constructed from the induced representation

$$I(s, \chi) = \bigotimes_v I_v(s, \chi_v)$$

with $I_v(s, \chi_v) = \mathrm{Ind}_{P_v}^{G_v} \chi |\ |_v^s$, a degenerate principal series associated to a maximal parabolic. At the point $s = \frac{1}{2}$, the local representation $I_v(\frac{1}{2}, \chi_v)$ is reducible and its constituents may be described in terms of certain representations $R_v(V_v)$ associated to 4-dimensional quadratic forms V_v of character χ_v. A standard section $\Psi(s) = \bigotimes_v \Psi_v(s) \in I(s, \chi)$ will be called homogeneous (at $s = \frac{1}{2}$) if all of the local components $\Psi_v(\frac{1}{2}) \in R_v(V_v)$ for some V_v. If the collection of quadratic spaces $\{V_v\}$ is the collection of localizations of a global quadratic space V, we will say that Ψ is *coherent* and, if no such global V exists, we will call Ψ *incoherent*.

If Ψ is a coherent section and if a corresponding global quadratic space V is anisotropic, then [18] $E(g, s, \Psi)$ is holomorphic at $s = 1$ and is equal to the integral over the compact space $O(V)(F) \backslash O(V)(\mathbb{A})$ of a certain theta function. On the other hand, if V cannot be chosen to be anisotropic, then $E(g, s, \Psi)$ can have a pole at $s = 1$ and a theta function associated to Ψ may not be integrable over $O(V)(F) \backslash O(V)(\mathbb{A})$.

In this case we must first regularize the theta integrals. This is done in § 3, where we introduce a certain central operator which eliminates divergent terms from the theta function. Introducing an auxiliary Eisenstein series on $O(V)(\mathbb{A})$, we obtain a meromorphic function of s whose Laurent expansion at $s = \frac{r}{2}$, where r is the Witt index of V, can be viewed as giving the regularization of the theta integral. If $B_{-2}(g, \varphi)$ and $B_{-1}(g, \varphi)$ are the first two coefficients in that expansion (a double pole occurs at this point), then our generalized Siegel-Weil formula expresses the first two coefficients $A_{-1}(g, \Psi)$ and $A_0(g, \Psi)$ of the Laurent expansion of $E(g, s, \Psi)$ at $s = \frac{1}{2}$ in terms of these "theta integrals". More precisely $A_{-1}(g, \Psi)$ is proportional to $B_{-2}(g, \varphi)$ (Theorem 4.1) while $A_0(g, \Psi)$ is a linear combination of $B_{-1}(g, \varphi)$ and $B_{-2}(g, \varphi')$ for some φ' (Theorem 6.1). This latter identity might be viewed as a kind of limit formula. Finally, if the section Ψ is incoherent we show that $A_{-1}(g, \Psi) = 0$, i.e., that $E(g, s, \Psi)$ has no pole at $s = \frac{1}{2}$, but we do not yet have a good expression for $A_0(g, \Psi)$ in this case.

Once these Siegel-Weil identities have been proved, our main results are obtained via the integral representation of the degree 5 L-function given in [26]. In fact, one of our main goals of this paper is to illustrate the form which the Siegel-Weil identities ought to take in the divergent case and the way in which those identities may be applied to the study of the poles and special values of certain Langlands L-functions. Another example may be found in [11] where the case $G = \mathrm{Sp}(3)$ and $s = 0$ is considered. Finally, it should be noted that the poles at $s = 1$ and 2 are also considered in [33], while the poles of the degree 4 or spin L-function are interpreted in [25, 27, 28, 33].

In Sect. 1 we fix notation and discuss the decomposition of the $I_v(\frac{1}{2})$'s in terms of $R_v(V)$'s. We also describe the behavior of the $R_v(V)$'s under the Jacquet functors, especially under those associated to a non-degenerate character of the unipotent radical N of P. In Sect. 2 we show (Theorem 2.2) that a certain

irreducible representation $\Pi(V_0)$ of $G(\mathbb{A})$ determined by a binary quadratic space V_0 occurs at most once as a submodule of the space $\mathscr{A}(G)$ of automorphic forms on G. We also show that the analogous representation determined by an incoherent family $\{V_{0,v}\}$ of local binary quadratic forms does not occur as a submodule of $\mathscr{A}(G)$, although, by [23], it must occur as a constituent. These results will be used to describe the residues of $E(g, s, \Psi)$ and the terms B_{-2}. In Sect. 3 we show how to regularize the various divergent theta integrals by using central operators. The case of $O(1, 1)$, i.e., of the split binary orthogonal group, provides motivation for the technique. Similar methods may be found in [23, 27] and in Sect. 5 of [18]. It follows that the representation $\Pi(V_0)$ mentioned above occurs with multiplicity precisely 1 in $\mathscr{A}(G)$. In Sect. 4 we show that the residue A_{-1} at $s=\frac{1}{2}$ of the family of Eisenstein series $E(g, s, \Psi)$ generates the (unique) copy of $\Pi(V_0)$ in $\mathscr{A}(G)$, and that $A_{-1}(g, \Psi)$ vanishes if Ψ is incoherent. In Sect. 5 we prove the analogous fact about the second term $B_{-2}(g, \varphi)$ in the Laurent expansion of the "regularized theta integral" $\mathscr{E}(g, s, \varphi)$ for $\varphi \in S(V(\mathbb{A})^2)$, when the Witt index r of V is 1 or 2. In Sect. 6 we prove that the second term $A_0(g, \Psi)$ in the Laurent expansion of $E(g, s, \Psi)$ at $s=\frac{1}{2}$ can be expressed in terms of B_{-2}'s and B_{-1}'s *provided* Ψ is coherent (Theorem 6.1). We have no such expression if Ψ is incoherent. The proof in the case $r=2$ is rather easy since in that case the series $\mathscr{E}(g, s, \varphi)$, which may be identified as an Eisenstein series on G, is very closely related to $E(g, s, \Psi)$. The case $r=1$ is more difficult and requires a comparison (Proposition 6.8) of the exponents of an automorphic representation with those of the Jacquet module of one of its non-archimedean components. This principle is rather general and is essentially a variant of a technique of Wallach [35]. Section 7 contains the applications to poles and values of L-functions described above, and Sect. 8 contains our result about locally generic forms.

Acknowledgements. While results of this paper have evolved over several years, the initial impetus was a question of Dinakar Ramakrishnan, elaborated in discussions with Michael Harris, about the possible significance of the pole of the degree 5 L-function at $s=1$. Also, the question about locally generic vs. globally generic forms was raised by Blasius and Ramakrishnan [2]; it arises in their work on Galois representation associated to Maass forms for GL(2). We have profited from discussion with D. Ramakrishnan, M. Harris, F. Shahidi, and Jian-Shu Li during the course of this work. We would also like to thank the referee for a number of useful comments. Finally, we would like to give our warm thanks to Ilya Piatetski-Shapiro and Steve Gelbart for their hospitality at their special year in automorphic forms at the Institute for Advanced Studies at Hebrew University of Jerusalem in 1987–88 and at the International Workshop on Automorphic Forms at Tel Aviv University in May of 1989.

1 Representations attached to quadratic forms

1.1

In this section we will set up the basic machinery which will be used throughout the paper. Our notation will be mostly that of [17, 18].

Let F be a totally real number field and let $G=\mathrm{Sp}(2)$ be the rank 2 symplectic group over F. Let $P \subset G$ be the maximal parabolic subgroup with Levi decomposition

$$P=MN$$

where

$$M=\left\{m(a)=\begin{pmatrix} a & \\ & {}^t a^{-1}\end{pmatrix} \mid a\in \mathrm{GL}(2)\right\}$$

and

$$N=\left\{n(b)=\begin{pmatrix} 1 & b \\ & 1\end{pmatrix} \mid b={}^t b\in \mathrm{Sym}(2)\right\}.$$

Here Sym(2) is the space of symmetric 2×2 matrices. We will sometimes refer to P as the Siegel parabolic.

For a place v of F we let $G_v=G(F_v)$, $P_v=P(F_v)$, etc., and we fix a maximal compact subgroup $K_v\subset G_v$:

$$K_v=\begin{cases}\mathrm{Sp}(2,O_v) & \text{v = non-archimedean}\\ \mathbf{U}(2) & \text{v = archimedean}\end{cases}$$

where

$$\mathbf{U}(2)=\left\{\begin{pmatrix} a & b\\ -b & a\end{pmatrix} \mid a,b\in M_2(\mathbb{R}),\ a+ib\in U(2)\right\}.$$

We then have a global Iwasawa decomposition

$$G(\mathbb{A})=P(\mathbb{A})\cdot K$$

where

$$K=\prod_v K_v$$

and local decompositions $G_v=P_v\cdot K_v$. If $g\in G_v$ (resp. $G(\mathbb{A})$),

$$g=nm(a)k,$$

we let

$$|a(g)|=|\det a|$$

where $|\ |=|\ |_v$ (resp. $|\ |_{\mathbb{A}}$).

For a character χ of $F_{\mathbb{A}}^{\times}/F^{\times}$ and for $s\in\mathbb{C}$, we let $I(s,\chi)$ denote the induced representation consisting of smooth functions Φ on $G(\mathbb{A})$ such that

$$\Phi(nm(a)g)=\chi(a)|a|^{s+(3/2)}\Phi(g).$$

Here we assume that $|\chi|=1$ and we require that Φ be right K_v-finite for each archimedean place v. Similarly, if v is any place of F, and for a character χ_v of $F_v^{\times}$ and for $s\in\mathbb{C}$, we let $I_v(s,\chi)$ denote the smooth induced representation

$$\mathrm{Ind}_{P_v}^{G_v}\chi_v|\ |_v^s,$$

where we once again require K_v finiteness in the archimedean case. Note that here and elsewhere we are using normalized induction. Finally a section $\Phi(s)\in I(s,\chi)$ will be called standard if its restriction to K is independent of s. Note that this restriction determines Φ.

As in [18, § 4], recall that we have an intertwining operator

$$M(s): I(s, \chi) \to I(-s, \chi^{-1})$$

defined, for $\mathrm{Re}(s) > \frac{3}{2}$, by the integral

$$M(s)\,\Phi(g) = \int_{N(\mathbb{A})} \Phi(wng, s)\,dn$$

where $w = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \in G(F)$. If v is a non-archimedean place at which the local component χ_v of χ is unramified, let $\Phi_v^0(s) \in I_v(s, \chi_v)$ be the section defined by

$$\Phi_v^0(k, s) = 1 \quad \forall k \in K_v.$$

Then

$$I(s, \chi) \simeq \bigotimes_v I_v(s, \chi_v),$$

the restricted tensor product with respect to the $\Phi_v^0(s)$, and

$$M(s) = \bigotimes_v M_v(s)$$

where, for $\mathrm{Re}(s) > \frac{3}{2}$, the local intertwining operator is defined by

$$M_v(s)\,\Phi_v(g) = \int_{N_v} \Phi_v(wng, s)\,dn.$$

Let $L_v(s, \chi_v)$ be the local Tate L-factor associated to χ_v. If $\chi_v = 1$, we will frequently write $L_v(s)$ in place of $L_v(s, 1)$. If we normalize $M_v(s)$ by setting

$$M_v^*(s) = \frac{L_v(s + (3/2), \chi_v) \cdot L_v(2s+1)}{L_v(s - (1/2), \chi_v) \cdot L_v(2s)} M_v(s),$$

then, at non-archimedean places at which χ_v is unramified,

$$M_v^*(s)\,\Phi_v^0 = \Phi_v^0(-s).$$

For $\Phi(s) \in I(s, \chi)$, we define the (Siegel) Eisenstein series:

$$E(g, s, \Phi) = \sum_{\gamma \in P(F) \backslash G(F)} \Phi(\gamma g, s).$$

This series is absolutely convergent for $\mathrm{Re}(s) > \frac{3}{2}$, and has a meromorphic analytic continuation and functional equation:

$$E(g, s, \Phi) = E(g, -s, M(s)\Phi).$$

1.2

In order to identify the various constituents of the $I_v(s, \chi_v)$'s at their points of reducibility, we need to recall a few facts about quadratic forms and the associated theta correspondence.

First suppose that F is a complete local field of characteristic 0.

For a non-degenerate inner product space V, $(,)$ over F of even dimension m, let

$$\Delta(V)=(-1)^{m/2}\det(V),$$

where $\det V=\det((x_i, x_j))$ for any basis $x_1, \ldots, x_m$ of V. For $x\in F^\times$, let

$$\chi_V(x)=(x, \Delta(V))_F,$$

where $(\cdot,\cdot)_F$ is the Hilbert symbol for F. The class of $\Delta(V)$ in $F^\times/F^{\times,2}$ is determined by χ_V.

If f is non-archimedean, then the isometry class of V is determined by m, χ_V, and the Hasse invariant ε_V, defined by

$$\varepsilon_V=\prod_{i<j}(a_i, a_j)_F$$

if we take any basis $\{x_i\}$ for V such that $(x_i, x_j)=\delta_{ij}a_i$.

If $F=\mathbb{R}$ the isometry class of V is determined by its signature (p, q), $p+q=m$, and, in this case, $\Delta(V)=(-1)^{(p-q)/2}$ and $\varepsilon_V=(-1)^{q(q-1)/2}$.

For our fixed additive character ψ of F, let $(\omega_V, \mathbf{S})$ denote the oscillator representation of $G=G(F)$ realized on $\mathbf{S}(V^2)$, the space of Schwartz-Bruhat functions on V^2. In the archimedean case we fix a decomposition $V=V_+\oplus V_-$ where $\dim V_+=p$, $\dim V_-=q$ and $(,)|_{V_+}$ (resp. $(,)|_{V_-}$) is positive (resp. negative) definite. Let $(,)_+$ be the majorant of $(,)$ determined by this decomposition:

$$(x, x)_+=\begin{cases}(x, x) & \text{if } x\in V_+\\ -(x, x) & \text{if } x\in V_-\end{cases},$$

and let $\varphi^0\in\mathbf{S}(V^2)$ be the Gaussian

$$\varphi^0(x)=e^{-\pi\operatorname{tr}((x, x)_+)},\quad x\in V^2.$$

Then let $S(V^2)\subset\mathbf{S}(V^2)$ be the space of functions of the form $\varphi(x)=P(x)\varphi^0(x)$, where P is a polynomial on V^2. This space is a $(\mathfrak{g}, K)$-module, where $\mathfrak{g}=\operatorname{Lie}(G)_{\mathbb{C}}$ and $K\subset G$ is the maximal compact subgroup, fixed as above. In the non-archimedean case we let $S(V^2)=\mathbf{S}(V^2)$.

Let $H=O(V)$ and let $R(V)$ denote the image of the map $S=S(V^2)\to I(s_0, \chi_V)$ $\varphi\mapsto\Phi$ where

$$\Phi(g)=\omega_V(g)\varphi(0)$$

and $s_0=\dfrac{m}{2}-\dfrac{3}{2}$. This map induces an isomorphism [19, 29],

$$S_H\xrightarrow{\sim}R(V),$$

where S_H is the space of H-coinvariants if F is non-archimedean and of $(\mathfrak{h}, L)$-coinvariants if F is archimedean. Here, in the archimedean case, $\mathfrak{h} = \mathrm{Lie}(H)_{\mathbb{C}}$ and L is the maximal compact subgroup of H which preserves the decomposition $V = V_+ \oplus V_-$ chosen above.

1.3

A full description of the constituents of the $I(s, \chi)$'s and their behavior under intertwining operators is given in [19, 20]. Since our goal in this paper will be to study the series $E(g, s, \Phi)$ at $s = \frac{1}{2}$, we record here only the relevant facts about the representation $I(\pm\frac{1}{2}, \chi)$ where $\chi^2 = 1$.

First assume that F is non-archimedean.

If $\chi = 1$, let V_0 be the split binary and let $V^{\pm}$ denote the quaternary space with $\chi_V = 1$ (i.e., with square determinant) and with $\varepsilon_V = \pm(-1, -1)_F$. Thus V^+ is the split quaternary space and V^- is anisotropic (quaternionic).

If $\chi \neq 1$, let V_0^+ denote the binary space given by the norm form from the quadratic extension of F determined by χ and let V_0^- be the binary space with the *multiple* of the norm form by a non-norm, i.e., by an element of $F^\times$ on which χ takes the value -1. Let $V^{\pm}$ be the orthogonal sum of $V_0^{\pm}$ with a hyperbolic plane.

Proposition 1.1 *If $\chi = 1$, then*

(i) *$R(V_0)$ is the unique irreducible subrepresentation of $I(-\frac{1}{2})$.*

(ii) *$R(V^+) = I(\frac{1}{2})$ and $R(V^-)$ is its unique irreducible submodule.*

(iii) *$R(V^+)/R(V^-) \xrightarrow{\sim} R(V_0)$ via the normalized intertwining operator $M^*(\frac{1}{2})$.*

(iv) *$I(-\frac{1}{2})/R(V_0) \xrightarrow{\sim} R(V^-)$ via the normalized intertwining operator $M^*(-\frac{1}{2})$.*

If $\chi \neq 1$, then

(i) *$R(V_0^{\pm})$ is irreducible and $R(V_0^+) \oplus R(V_0^-)$ is the maximal submodule of $I(-\frac{1}{2})$.*

(ii) *$R(V^+)$ and $R(V^-)$ are maximal submodules of $I(\frac{1}{2})$, $I(\frac{1}{2}) = R(V^+) + R(V^-)$, and $R(V^+) \cap R(V^-)$ is the unique irreducible submodule of $I(\frac{1}{2})$.*

(iii) *$R(V^{\pm})/(R(V^+) \cap R(V^-)) \xrightarrow{\sim} R(V_0^{\pm})$ via the normalized intertwining operator $M^*(\frac{1}{2})$.*

(iv) *$I(-\frac{1}{2})/(R(V_0^+) \oplus R(V_0^-)) \xrightarrow{\sim} R(V^+) \cap R(V^-)$ via the normalized intertwining operator $M^*(-\frac{1}{2})$.*

When χ is unramified, these facts may be found in [29] and are essentially due to Gustafson [7]. In the general case, they are proved in [20].

Next suppose that F is real archimedean.

A quadratic space $V, (,)$ is then determined by its signature (p, q), $p + q = \dim V$, and so we will write $R(p, q)$ for the image in $I(s_0, \chi)$, $s_0 = \frac{1}{2}\dim V - \frac{3}{2}$, $\chi = \chi_V$, of the space $S(V^2)$ defined above. The analogue of Proposition 1.1 is then

Proposition 1.2 *If $\chi = 1$, then*

(i) *$R(1, 1)$ is the unique irreducible submodule of $I(-\frac{1}{2})$.*

(ii) *$R(2, 2) = I(\frac{1}{2})$ and $R(4, 0) \oplus R(0, 4)$ is its unique submodule. Here $R(4, 0)$ and $R(0, 4)$ are irreducible.*

(iii) *$R(2, 2)/(R(4, 0) \oplus R(0, 4)) \xrightarrow{\sim} R(1, 1)$ via the normalized intertwining operator $M^*(\frac{1}{2})$.*

(iv) $I(-\frac{1}{2})/R(1,1) \xrightarrow{\sim} R(4,0) \oplus R(0,4)$ *via the normalized intertwining operator* $M^*(-\frac{1}{2})$.

If $\chi \neq 1$, *then*

(i) $R(2,0)$ *and* $R(0,2)$ *are irreducible, and* $R(2,0) \oplus R(0,2)$ *is the maximal submodule of* $I(-\frac{1}{2})$.

(ii) $R(3,1)$ *and* $R(1,3)$ *are maximal submodules of* $I(\frac{1}{2})$, *and* $I(\frac{1}{2}) = R(3,1) + R(1,3)$. *Moreover,* $R(3,1) \cap R(1,3)$ *is the unique irreducible submodule of* $I(\frac{1}{2})$.

(iii) $R(3,1)/(R(3,1) \cap R(1,3)) \xrightarrow{\sim} R(2,0)$ *and* $R(1,3)/(R(3,1) \cap R(1,3)) \xrightarrow{\sim} R(0,2)$ *via the normalized intertwining operator* $M^*(\frac{1}{2})$.

(iv) $I(-\frac{1}{2})/(R(2,0) \oplus R(0,2)) \xrightarrow{\sim} R(3,1) \cap R(1,3)$ *via the normalized intertwining operator* $M^*(-\frac{1}{2})$.

These facts are easily checked using the results of [19]. We omit the details.

1.4

We will need some information about the behavior of the $R(V)$'s under various Jacquet functors.

Assume that F is a non-archimedean local field.

Let $S = S(V^2)$ and $R(V) = S_H$ be as in 1.2 above. For $\beta = {}^t\beta \in M_2(F)$, let ψ_β denote the character of N given by

$$\psi_\beta(n(b)) = \psi(\mathrm{tr}(b\beta)),$$

and let

$$S_{N,\beta} = S_{N,\psi_\beta} = S/\langle \omega(n)\varphi - \psi_\beta(n)\varphi \rangle$$

be the Jacquet module of S associated to ψ_β and N.

Lemma 1.3 [30] *Let* $\Omega_\beta = \{x \in V^2 \mid (x,x) = 2\beta\}$. *Then the quotient map* $S \to S_{N,\psi_\beta}$ *factors through the restriction* $S \to S(\Omega_\beta)$, $\varphi \mapsto \varphi|_{\Omega_\beta}$. *In particular, if* $\det \beta \neq 0$, *then* Ω_β *is a single* H*-orbit and the quotient map* $S \to (S_{N,\psi_\beta})_H$ *is given by*

$$\varphi \mapsto \int_{\Omega_\beta} \varphi \, d\mu_\beta$$

where $d\mu_\beta$ *is the unique (up to a scalar)* H*-invariant measure on* Ω_β.

Remark. If β is not represented by V, then, of course, $\Omega_\beta = \phi$ and $S_{N,\psi_\beta} = 0$.

We now specialize to the case in which $\dim V = 4$. Let $\chi = \chi_V$. Suppose that V is isotropic and let V_0 be the (unique) binary space such that V is the direct sum of V_0 with a hyperbolic plane. Also let V' be the quaternary space with $\chi_{V'} = \chi_V$ but with the opposite Hasse invariant: $\varepsilon_{V'} = -\varepsilon_V$. Let $\mathcal{O}_{V_0}$ be the set of all $\beta \in \mathrm{Sym}_2(F)$ such that β is non-singular and β is represented by V_0.

Lemma 1.4 *If* $\beta \in \mathcal{O}_{V_0}$, *then* V' *does not represent* β. *Thus*

$$S(V'^2)_{N,\beta} = 0$$

and hence $R(V')_{N,\beta} = 0$ *and* $(R(V) \cap R(V'))_{N,\beta} = 0$.

Proof. If $\chi = 1$, then V and hence V_0 is split, while V' is the quaternion norm form. Thus V' does not represent $\beta \in \mathcal{O}_{V_0}$. If $\chi \neq 1$, let V_0' be the "other" binary

form with character $\chi=\chi_V=\chi_{V_0}$. Then V' is the direct sum of V_0' and a hyperbolic plane. If $\beta\in\mathcal{O}_{V_0}$ were represented by V', then we could write V' as a direct sum of V_0 and some other binary space, say V_0''. But since V_0 and V_0' have the same determinant (modulo squares) the space V_0'' must have square determinant and thus is split. Then Witt cancellation would force $V_0\simeq V_0'$ and so, again, β cannot be represented by V'. □

Combining these facts we obtain

Proposition 1.5 *Suppose that* dim $V=4$ *and that* V *is isotropic. Let* V_0 *be the associated binary space as above. Then for any* $\beta\in\mathcal{O}_{V_0}$,

$$\dim R(V)_{N,\psi_\beta}=1.$$

Proof. By the result of 1.3 above, we have an exact sequence

$$0\to R(V)\cap R(V')\to R(V)\to R(V_0)\to 0$$

and hence, by the exactness of the Jacquet functor,

$$0\to(R(V)\cap R(V'))_{N,\psi_\beta}\to R(V)_{N,\psi_\beta}\to R(V_0)_{N,\psi_\beta}\to 0.$$

Since $(R(V)\cap R(V'))_{N,\psi_\beta}=0$, we obtain $R(V)_{N,\psi_\beta}\xrightarrow{\sim} R(V_0)_{N,\psi_\beta}$ and this last space has dimension 1. □

Corollary 1.6 *For any* $\beta\in\mathrm{Sym}_2(F)$ *with* $\det\beta\neq 0$ *which is represented by* V, *the space of linear functionals* $\mu: S\to\mathbb{C}$ *which satisfy*

$$\mu(\omega(n)\varphi)=\psi_\beta(n)\,\mu(\varphi)\quad\forall n\in N,$$

and

$$\mu(\omega(h)\varphi)=\mu(\varphi)\quad\forall h\in H=O(V),$$

has dimension 1.

Now suppose that F is real archimedean.

In the archimedean case we must pay more attention to questions of continuity. First, for V, (,) a non-degenerate inner product space, let $\mathbf{S}=\mathbf{S}(V^2)$ be the Schwartz space of V with its usual topology and let $\mathbf{S}'$ be the space of tempered distributions. The map

$$i: S\to R(V)\subset I(s_0).\qquad s_0=\frac{m}{2}-\frac{3}{2}$$

extends to a continuous map

$$i:\mathbf{S}\to R^\infty(V)\subset I^\infty(s_0)$$
$$\varphi\to\Phi,\qquad \Phi(g)=\omega(g)\,\varphi(0),$$

where $I^\infty(s_0)$ is the smooth induced representation with its Frechet topology, and $R^\infty(V)$ is the image $i(\mathbf{S})$. Moreover, by a result of Casselman, [6], $R^\infty(V)$ is closed in $I^\infty(s_0)$ and every $H=O(V)$ invariant tempered distribution $\lambda\in(\mathbf{S}')^H$ has the form $\lambda=\mu\circ i$ for some continuous linear functional $\mu\in I^\infty(s_0)'$. Note that

every continuous linear functional on $R^\infty(V)$ is the restriction of a continous linear functional on $I^\infty(s_0)$.

The analogue of Lemma 1.3 is then proved in [30, Lemma 4.2]:

Proposition 1.7 *Suppose that* $\beta \in \mathrm{Sym}_2(\mathbb{R})$ *with* $\det \beta \neq 0$ *and that* $m = \dim V \geqq 2$. *Let* $(\mathbf{S}'_\beta)^H$ *be the space of* H*-invariant tempered distributions* μ *such that*

$$\mu(\omega(\chi)\varphi) = d\psi_\beta(X) \cdot \mu(\varphi)$$

for all $X \in \mathfrak{n} = \mathrm{Lie}\, N$ *and for all* $\varphi \in \mathbf{S}$. *Here* $d\psi_\beta : \mathfrak{n} \to \mathbb{C}$ *is the differential of the character* $\psi_\beta : N \to \mathbb{C}$ *given by*

$$\psi_\beta(n(b)) = \psi(\mathrm{tr}(b\beta)).$$

Let $\Omega_\beta = \{x \in V^2 | (x, x) = 2\beta\}$ *be the hyperboloid, as before.*
(i) *If* $\Omega_\beta = \phi$, *then* $(\mathbf{S}'_\beta)^H = 0$.
(ii) *If* $\Omega_\beta \neq \phi$, *then* $\dim(\mathbf{S}'_\beta)^H = 1$ *and this space is spanned by the orbital integral*

$$\varphi \mapsto \int_{\Omega_\beta} \varphi \, d\mu_\beta$$

for the unique (up to scalar) H*-invariant measure on* Ω_β.

Now let $R^\infty(V)'_\beta$ denote the space of all continuous linear functionals μ on $R^\infty(V)$ which satisfy

$$\mu(r(X)\Phi) = d\psi_\beta(X) \cdot \mu(\Phi) \quad \forall X \in \mathfrak{n} = \mathrm{Lie}\, N, \qquad \forall f \in R^\infty(V).$$

Then the previous lemma asserts that

Corollary 1.8 *If* β *is not represented by* V, *i.e., if* $\Omega_\beta = \phi$, *then* $R^\infty(V)'_\beta = 0$. *If* β *is represented by* V, *then* $\dim R^\infty(V)'_\beta = 1$.

Now suppose that $m = \dim V = 4$, and that V is isotropic, and let V_0 be the binary space such that $V = V_0 + V_{1,1}$, where $V_{1,1}$ is a hyperbolic plane. If V is split, then so is V_0; while if V has signature (3, 1) (resp. (1, 3)), then V_0 is positive (resp. negative) definite.

Proposition 1.9 *Suppose that* $\beta \in \mathcal{O}_{V_0}$.
(i) *If* V *has signature* (2, 2), *so that* V_0 *is split, then* $(R^\infty(4, 0) \oplus R^\infty(0, 4))'_\beta = 0$.
(ii) *If* V *has signature* (3, 1), *then* $R^\infty(1, 3)'_\beta = 0$, $\dim R^\infty(3, 1)'_\beta = 1$ *and the restriction map*

$$R^\infty(3, 1)'_\beta \to (R^\infty(3, 1) \cap R^\infty(1, 3))'_\beta$$

is zero. Moreover, the transpose of $M^*(\frac{1}{2})$: $R^\infty(3, 1) \to R^\infty(2, 0)$ *induces an isomorphism*

$$(*) \qquad R^\infty(2, 0)'_\beta \xrightarrow{\sim} R^\infty(3, 1)'_\beta.$$

The analogous statements hold if V *has signature* (1, 3).

Proof. Case (i) follows immediately from Corollary 1.8 above, as do the first statements in (ii). Finally note that we have an isomorphism

$$M^*(\tfrac{1}{2}): R^\infty(3,1)/(R^\infty(3,1)\cap R^\infty(1,3)) \xrightarrow{\sim} R^\infty(2,0)$$

and thus an exact sequence

$$0 \to R^\infty(2,0)'_\beta \to R^\infty(3,1)'_\beta \xrightarrow{res} (R^\infty(3,1)\cap R^\infty(1,3))'_\beta.$$

Since both spaces on the left are one-dimensional, we obtain the remaining assertions of (ii). Note that the intertwining operator $M^*(\frac{1}{2})$: $I(\frac{1}{2}) \to I(-\frac{1}{2})$ has a unique continuous extention to $M^*(\frac{1}{2})$: $I^\infty(\frac{1}{2}) \to I^\infty(-\frac{1}{2})$ via [6]. The image of this extension is closed and hence we obtain the isomorphism (*) above from Proposition 1.2. □

2 Uniqueness of binary theta series

In this section we will prove that the global representation associated to a binary quatratic form can occur at most once in the space of automorphic forms on G.

Let F be a totally real number field, and for a binary quadratic space V_0 over F define a global representation

$$\Pi(V_0) = \otimes_v R_v(V_0) \tag{2.1}$$

associated to the binary quadratic space V_0. As a representation of $G(\mathbb{A})$ – strictly speaking it is only a $(\mathfrak{g}_v, K_v)$-module at the archimedean places – $\Pi(V_0)$ is irreducible and has a very strong uniqueness property.

First we fix $\beta \in \mathrm{Sym}_2(F)$ with $\det \beta \neq 0$ and define a character ψ_β of $N(\mathbb{A})$ by

$$\psi_\beta(n(b)) = \psi(\mathrm{tr}(b\beta)) \tag{2.2}$$

where ψ is our fixed global additive character. Consider the space $\mathscr{W}_\beta(\Pi(V_0))$ of linear functionals μ on $\Pi(V_0)$ which satisfy

$$\mu(\pi(n)f) = \psi_\beta(n)\,\mu(f) \quad \forall n \in N(\mathbb{A}_f) \tag{2.3}$$

and

$$\mu(\pi(X)f) = d\psi_\beta(X)\,\mu(f) \quad \forall X \in \mathfrak{n} = \mathrm{Lie}(N_\infty) \tag{2.4}$$

where $d\psi_\beta : \mathfrak{n} \to \mathbb{C}$ is the differential of the restriction of the character ψ_β to N_∞. Here $N_\infty = \prod_w N_w$ as w runs over the archimedean places of F. For each archimedean place v, let $R_v^\infty(V_0)$ be the closure of $R_v(V_0)$ in $I_v^\infty(\frac{1}{2})$ as in Sect. 1. Also, let

$$\Pi^\infty(V_0) = \Big(\bigotimes_{\substack{v \\ \text{arch}}} R_v^\infty(V_0)\Big) \otimes \Big(\bigotimes_{\substack{v \\ \text{finite}}} R_v(V_0)\Big) \tag{2.5}$$

so that $\Pi^\infty(V_0)$ is, in fact, a $G(\mathbb{A})$-module. Let $\mathcal{W}_\beta^\infty(\Pi^\infty(V_0))$ be the space of all linear functionals μ on $\Pi^\infty(V_0)$ which satisfy (2.3) and (2.4) and whose restrictions to $R_v^\infty(V_0)$ lie in $R_v^\infty(V_0)'_\beta$, the space of continuous functionals. Let $\mathcal{W}_\beta^\infty(\Pi(V_0))$ be the subspace of $\mathcal{W}_\beta(\Pi(V_0))$ spanned by the restrictions of functionals in $\mathcal{W}_\beta^\infty(\Pi^\infty(V_0))$.

Proposition 2.1 *Let*

$\mathcal{O}_{V_0}=\{\beta\in\mathrm{Sym}_2(F)|\det\beta\neq 0$ *and* $2\beta=(x,x)$ *for some* $x\in V_0(F)^2\}$.

(i) *If* $\beta\notin\mathcal{O}_{V_0}$ *and* $\det\beta\neq 0$, *then* $\mathcal{W}_\beta^\infty(\Pi(V_0))=0$.

(ii) *If* $\beta\in\mathcal{O}_{V_0}$ *then* $\dim\mathcal{W}_\beta^\infty(\Pi(V_0))=1$.

Proof. Suppose that $\mu\in\mathcal{W}_\beta^\infty(\Pi(V_0))$. *If* $f^0=\bigotimes_v f_v^0\in\Pi(V_0)$, we choose a place v and define a linear functional μ_v on $\Pi_v(V_0)$ by

$$\mu_v(f_v)=\mu(f_v\otimes f^{0,v})$$

where $f^{0,v}=\bigotimes_{w\neq v} f_w^0$. If v is non-archimedean, this linear functional is a ψ_β eigenfunctional for N_v and thus factors through the space $R_v(V_0)_{N_v,\beta}$. If v is archimedean, then μ_v is the restriction to $R_v(V_0)$ of an element of $R_v^\infty(V_0)'_\beta$. Now if $\beta\notin\mathcal{O}_{V_0}$, then there must exist a place v at which β is not represented by $V_{0,v}$. At this place $R_v(V_0)_{N_v,\beta}=0$ by Lemma 1.3, or $R_v^\infty(V_0)'_\beta=0$ by Corollary 1.8, and thus $\mu=0$. This proves (i). Next we observe that if $\beta\in\mathcal{O}_{V_0}$, then the spaces $\mathcal{W}_\beta^\infty(\Pi(V_0))$ are all one-dimensional and so we obtain (ii) by the usual local to global argument [32]. □

Let $\mathcal{A}(G)$ denote the space of automorphic forms on $G(\mathbb{A})$ and note that, for any $\beta\in\mathrm{Sym}_2(F)$ the β-th Fourier coefficient

$$W_\beta(f)(g)=\int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\,\psi(-\mathrm{tr}(\beta b))\,db, \tag{2.6}$$

for $f\in\mathcal{A}(G)$, defines a linear functional

$$\begin{aligned} W_\beta &: \mathcal{A}(G)\to\mathbb{C} \\ f &\mapsto W_\beta(f)(e) \end{aligned} \tag{2.7}$$

such that

$$W_\beta(r(n)f)=\psi_\beta(n)\,W_\beta(f)$$

for all $n\in N(\mathbb{A})$.

Let $\mathrm{Hom}(\Pi(V_0),\mathcal{A}(G))$ be the space of equivariant embeddings i: $\Pi(V_0)\to\mathcal{A}(G)$.

Theorem 2.2 $\dim\mathrm{Hom}(\Pi(V_0),\mathcal{A}(G))\leqq 1$.

Remark. In fact, the dimension of this space is exactly one, as we will show in the next section.

Proof. Suppose that A, B: $\Pi(V_0)\to\mathcal{A}(G)$ are two embeddings, and consider the linear functionals $A_\beta=W_\beta\circ A$ and $B_\beta=W_\beta\circ B$ on $\Pi(V_0)$. We now want to apply a result of Casselman [6, Corollary 10.5], to show that the embeddings A and B both extend to continuous $G(\mathbb{A})$ intertwining operators

$$A^\infty, B^\infty: \Pi^\infty(V_0)\to\mathcal{A}^\infty(G),$$

where $\mathscr{A}^\infty(G)$ is the space of smooth automorphic forms without the K-finiteness condition. To do so we must check that the hypotheses of his result are satisfied in our situation.

Fix a norm $\|\cdot\|$ on $G(\mathbb{A})$ as in [4, 4.3 (4)], and let

$$\|g\|_F = \inf_{\gamma \in G(F)} \|\gamma \cdot g\|.$$

Recall that the space $\mathscr{A}^\infty(G)$ contains a subspace $\mathscr{A}^\infty_{ung,n}(G)$ consisting of the smooth, $\mathfrak{z}(\mathfrak{g}_\infty)$-finite functions f on $G(F)\backslash G(\mathbb{A})$ such that, for all $X \in U(\mathfrak{g}_\infty)$ the quantity

$$\|g\|_F^{-n} \cdot |r(X)f(g)|$$

is bounded on $G(\mathbb{A})$. Note that n here is fixed independent of X. Casselman shows that $\mathscr{A}^\infty_{ung,n}(G)$ is a Frechet space with seminorms

$$v_X(f) = \sup_{g \in G(\mathbb{A})} \|g\|_F^{-n} \cdot |r(X)f(g)|.$$

Our goal is to prove

Lemma 2.3 *Suppose that Π is a smooth irreducible representation of $G(\mathbb{A})$ on a Frechet space T and let*

$$C: T_{K_\infty} \to \mathscr{A}(G)$$

be a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$ intertwining map, where T_{K_∞} is the space of K_∞-finite vectors in T. Then

(i) *There exists a positive integer n such that the image $C(T_{K_\infty})$ lies in $\mathscr{A}^\infty_{umg,n}(G)$.*
(ii) *The closure $\overline{C(T_{K_\infty})}$ in $\mathscr{A}^\infty_{umg,n}(G)$ is an irreducible $G(\mathbb{A})$ invariant subspace.*
(iii) *There exists a unique continous extension C^∞ of C*

$$C^\infty: T \to \mathscr{A}^\infty_{umg,n}(G) \subset \mathscr{A}^\infty(G).$$

Proof. Let $f \in C(T_{K_\infty})$ be any non-zero vector, and note that f is cyclic for the action of $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A})$. Since f is both $\mathfrak{z}(\mathfrak{g}_\infty)$ finite and K-finite, Lemma 14 of Harish-Chandra [9] implies that there is an element $\xi \in C_c^\infty(G(\mathbb{A}))$ such that $f * \xi = f$ and hence that f lies in $\mathscr{A}^\infty_{umg,n}(G)$ for some n (cf. Sect. 2.2 of [4]). Since $\mathscr{A}^\infty_{umg,n}(G)$ is a Frechet space with a smooth action of $G(\mathbb{A})$, the cyclicity of f implies that $C(T_{K_\infty}) \subset \mathscr{A}^\infty_{umg,n}(G)$, as claimed. A result of Borel [3, p. 28] implies that the closure $\overline{C(T_{K_\infty})}$ is stable for the action of $G(\mathbb{A})$. On the other hand, we have

Lemma 2.4 *The action of $G(\mathbb{A})$ on $\mathscr{A}^\infty_{umg,n}(G)$ is of moderate growth in the sense of* [5].

Proof. For g and $g' \in G(\mathbb{A})$ and for $\gamma \in G(F)$ observe that

$$\|\gamma g' g\| \leqq \|\gamma g'\| \cdot \|g\|,$$

so that

$$\|g' g\|_F \leqq \|g\| \cdot \|g'\|_F,$$

and, for any non-negative real number n,

$$\|g\|^n \cdot \|g'g\|_F^{-n} \geqq \|g'\|_F^{-n}.$$

For $X \in U(\mathfrak{g}_\infty)^{(d)}$, the subspace of $U(\mathfrak{g}_\infty)$ of degree less than or equal to d, and for $f \in \mathscr{A}^\infty_{umg,n}(G)$ and $g \in G(\mathbb{A})$,

$$\begin{aligned} \nu_X(r(g)f) &= \sup_{g' \in G(\mathbb{A})} \|g'\|_F^{-n} |r(X)\, r(g) f(g')| \\ &= \sup_{g'} \|g'\|_F^{-n} |r(\mathrm{Ad}\, g^{-1} \cdot X) f(g'g)| \\ &\leqq \sum_i |\alpha_i(g)| \sup_{g'} (\|g\|^n \|g'g\|_F^{-n} |r(X_i) f(g'g)|) \\ &\leqq c \cdot \|g\|^{n'} \sum_i \nu_{X_i}(f), \end{aligned}$$

where the X_i's are a basis for the finite dimensional space $U(\mathfrak{g}_\infty)^{(d)}$, and the constant c and exponent n' may depend on d. □

Thus the closure $\overline{C(T_{K_\infty})}$ is also of moderate growth, and since its space $\overline{C(T_{K_\infty})}_{K_\infty} = C(T_{K_\infty})$ of K_∞ finite functions is admissible and irreducible as a representation of $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$, Corollary 10.5 of [6] yields (iii). This completes the proof of Lemma 2.3. □

Since W_β defines a continuous linear functional on $\mathscr{A}^\infty(G)$, we conclude that both A_β and B_β lie in $W_\beta^\infty(\Pi(V_0))$. If rk $\beta = 2$ and $\beta \notin \mathcal{O}_{V_0}$, then by (i) of Proposition 2.1, $A_\beta = B_\beta = 0$. If $\beta \in \mathcal{O}_{V_0}$, then these functionals both satisfy the hypotheses of (ii) Proposition 2.1 above and hence, assuming that one of them, say A_β, is non-zero, there exists a constant c_β such that $B_\beta = c_\beta A_\beta$. But $\mathcal{O}_{V_0}$ forms a single orbit under GL(2, F), and, for $a \in \mathrm{GL}(2, F)$, we have

$$\begin{aligned} A_\beta^\infty(r(m(a))\, \Phi) &= W_\beta(A^\infty(r(m(a))\, \Phi)) \\ &= W_\beta(r(m(a))\, A^\infty(\Phi)) \\ &= W_{\beta[a]}(A^\infty(\Phi)) \\ &= A^\infty_{\beta[a]}(\Phi) \end{aligned}$$

and similarly for B_β^∞. Here $\beta[a] = {}^t a \beta a$. Thus c_β is independent of β. Thus $D = B - cA$: $\Pi(V_0) \to \mathscr{A}(G)$ is an intertwining map such that $D_\beta = W_\beta \circ D = 0$ for all β of rank 2. Note that if we let $D^\infty = B^\infty - cA^\infty$, then for all $G(\mathbb{A})$

$$D^\infty(r(g)\, \Phi) = r(g)\, D^\infty(\Phi).$$

Thus, setting $D_\beta^\infty = W_\beta \circ D^\infty$, we have

$$D_\beta^\infty(r(g)\, \Phi) = D^\infty(\Phi)_\beta(g),$$

where $D^\infty(\Phi)_\beta(g)$ is the β-th Fourier coefficient of $D^\infty(\Phi)$, as a function on $G(\mathbb{A})$. Therefore, the vanishing of D_β^∞ as a linear functional implies that for all $\Phi \in \Pi^\infty(V_0)$, $D^\infty(\Phi)_\beta(g) = 0$ for all $g \in G(\mathbb{A})$. Now we apply

Lemma 2.5 *If D: $\Pi(V_0) \to \mathscr{A}(G)$ is intertwining with $D_\beta = W_\beta \circ D = 0$ for all β of rank 2, then $D = 0$.*

Proof. Choose a non-archimedean place v and let $f \in S(N_v)$ be the function such that $\hat{f}$, its Fourier transform with respect to ψ_v, is the characteristic function of the set $\{n(b) \in N_v | b \in \mathcal{O}_{V_{0,v}} \cap \mathrm{Sym}_2(\mathcal{O}_{F_v})\}$. First consider the action of f in $R_v(V_0)$. Here we evaluate at $w = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \in G_v$. Let $\mu(x) = \frac{1}{2}(x, x)$. Then

$$\pi(f)\Phi_v(w) = \int_{\mathrm{Sym}_2(F_v)} \Phi_v(w\,n(b)) f(b)\,db$$

$$= \int_{\mathrm{Sym}_2(F_v)} \int_{V_0(F_v)^2} \psi(\mathrm{tr}(b\mu(x)))\,\varphi(x)\,dx\,f(b)\,db$$

$$= \int_{V_0(F_v)^2} \varphi(x)\hat{f}(\mu(x))\,dx$$

$$= \int_{\mu^{-1}(\mathcal{O}_{V_{0,v}} \cap \mathrm{Sym}_2 \mathcal{O}_{F_v})} \varphi(x)\,dx \tag{2.8}$$

and this integral does not vanish for suitable choice of $\varphi \in S(V_0(F_v)^2)$. Thus f does not act by zero in $R_v(V_0)$ or in $\Pi(V_0)$. On the other hand, if we take $g \in G(\mathbb{A})$ with $g_v = 1$ for our chosen place v, then

$$r(f)D(\Phi)(g) = \int_{N_v} D(\Phi)(g\,n(b)) f(b)\,db$$

$$= \int_{N_v} \sum_{\alpha \in \mathrm{Sym}_2(F)} D(\Phi)_\alpha(g\,n(b)) f(b)\,db$$

$$= \sum_\alpha \int_{N_v} \psi_v(\mathrm{tr}(\alpha b)) f(b)\,db \cdot D(\Phi)_\alpha(g)$$

$$= \sum_\alpha \hat{f}(\alpha)\,D(\Phi)_\alpha(g)$$

$$= 0, \tag{2.9}$$

since $D(\Phi)_\alpha$ is zero for $\mathrm{rk}\,\alpha = 2$ and $\hat{f}(\alpha) = 0$ for $\mathrm{rk}\,\alpha < 2$. This shows that the restriction of $D(\pi(f)\Phi)$ to $G(F) \cdot \prod_{w \neq v}' G_w$ is zero. Since this set is dense in $G(\mathbb{A})$ and since $D(\pi(f)\Phi)$ is smooth, we conclude that $D(\pi(f)\Phi) = 0$.

Thus the kernel of D on $\Pi(V_0)$ is non-zero, so that D is zero by the irreducibility of $\Pi(V_0)$. □

This completes the proof of Theorem 2.2 □

Note that, if we fix χ and choose a collection $\{V_{0,v}\}$ of binary quadratic spaces $V_{0,v}$ over F_v with $\chi_{V_{0,v}} = \chi_v$ and with Hasse invariant $\varepsilon_{V_{0,v}} = 1$ for almost all v, then the only obstruction to the existence of a global binary space V_0 with these localizations is the product formula for the Hasse invariants. Suppose that

$$\prod_v \varepsilon_{V_{0,v}} = -1 \tag{2.10}$$

so that no global V_0 exists, and let

$$\Pi = \Pi(\{V_{0,v}\}) = \bigotimes_v R_v(V_{0,v}). \tag{2.11}$$

Also define Π^∞ as in (2.5) above, so that Π^∞ is an irreducible representation of $G(\mathbb{A})$.

Proposition 2.6 $\mathrm{Hom}(\Pi, \mathscr{A}(G)) = 0$.

Proof. Suppose that $D\colon \Pi \to \mathscr{A}(G)$ is an intertwining map, and for $\alpha \in \mathrm{Sym}_2(F)$ of rank 2, consider the linear functional $D_\alpha = W_\alpha \circ D$ on Π. For the restriction of D_α to the local component $\Pi_v = R_v(V_{0,v})$ to be non-zero, we must have $\alpha \in \mathcal{O}_{V_{0,v}}$ and hence $\varepsilon_{\alpha,v} = \varepsilon_{V_{0,v}}$ for the Hasse invariant. The product formula for the Hasse invariants of α then contradicts (2.10) and we conclude that $D_\alpha = 0$ for all α of rank 2. But Lemma 2.5, whose proof is "local at v", applies to D and Π and so $D = 0$. □

3 Regularization of theta integrals

We now return to our global situation.

Let $V, (,)$ be a non-degenerate inner product space over F with $\dim_F V = m = 2$ or 4, and let $H = O(V)$. For $\varphi \in S(V(\mathbb{A})^2)$, $g \in G(\mathbb{A})$ and $h \in H(\mathbb{A})$ we have the usual theta kernel

$$\theta(g, h; \varphi) = \sum_{x \in V(F)^2} \omega(g)\, \varphi(h^{-1} x).$$

If V is anisotropic over F, then the integral

$$I(g; \varphi) = \int_{H(F)\backslash H(\mathbb{A})} \theta(g, h; \varphi)\, dh \tag{3.1}$$

is well defined. Our goal in this section is to define an analogue of $I(g, \varphi)$ when V is isotropic by a regularization technique involving differential operators. Since this technique will be discussed more fully in a forthcoming paper [21], we will restrict ourselves to the special cases of interest for our present applications. Let r be the Witt index of V over F, i.e., the dimension of a maximal isotropic subspace of V. We assume for the remainder of this section that $r > 0$.

3.1 The case $m = 2$

We begin with the case of a split binary form V_0.

Choose a Witt basis for V_0 so that $V_0 \simeq F^2$ (column vectors), and $V_0^2 \simeq M_2(F)$. Note that for each place v we have $H_v \simeq F_v^\times \rtimes \langle \sigma_v \rangle$ where σ_v has order 2 and acts on $F_v^\times$ by inversion. Let

$$\Xi = \prod_v \langle \sigma_v \rangle.$$

Then for $H^1 = \mathrm{SO}(V_0)$, $H^1(\mathbb{A}) \simeq F_{\mathbb{A}}^\times$, and

$$H(\mathbb{A}) = H^1(\mathbb{A}) \rtimes \Xi \simeq F_{\mathbb{A}}^\times \rtimes \Xi.$$

The oscillator representation for the dual pair (G, H) and associated to the fixed additive character ψ may be realized in either the standard model on

$S(V(\mathbb{A})^2)$ or on the space $S(W(\mathbb{A}))$ where $W=F^4$ (row vectors) is the symplectic space with right G action. The two models are intertwined by

$$S(M_2(\mathbb{A}))\to S(\mathbb{A}^4)$$
$$\varphi\mapsto\hat{\varphi}$$

where

$$\hat{\varphi}(x,y)=\int_{\mathbb{A}^2}\psi(\mathrm{tr}({}^t u y))\,\varphi\left(\begin{pmatrix}x\\u\end{pmatrix}\right)du \tag{3.2}$$

for x, y and u row vectors of length 2. For $a\in F_{\mathbb{A}}^{\times}$, the corresponding element $h(a)\in H^1(\mathbb{A})$ acts via

$$\omega(h(a))\,\hat{\varphi}(x,y)=|a|^2\,\hat{\varphi}(ax,ay) \tag{3.3}$$

and the action of

$$\sigma=\begin{pmatrix}&1\\1&\end{pmatrix}$$

is given by

$$\omega(\sigma)\,\hat{\varphi}(x,y)=\tilde{\hat{\varphi}}(x,y)=\int_{u,v\in\mathbb{A}^2}\hat{\varphi}(u,v)\,\psi(x^t v-y^t u)\,du\,dv. \tag{3.4}$$

Also, if $g\in G(\mathbb{A})$

$$\omega(g)\,\hat{\varphi}(x,y)=\hat{\varphi}((x,y)\cdot g).$$

Then, by Poisson summation, we have

$$\begin{aligned}\theta(g,h;\varphi)&=\sum_{x\in V(F)^2}\omega(g)\,\varphi(h^{-1}x)\\&=\sum_{w\in W(F)}\omega(h)\,\hat{\varphi}(w\cdot g).\end{aligned}$$

Fix a (real) archimedean place v of F and let $\mathfrak{g}_v=\mathrm{Lie}\,G_v$ be the Lie algebra of G_v. Let $U(\mathfrak{g}_v)$ be the universal enveloping algebra of $(\mathfrak{g}_v)_{\mathbb{C}}$ and let $\mathfrak{z}(\mathfrak{g}_v)$ be its center. Recall that, under the Harish-Chandra isomorphism [34],

$$\mathfrak{z}(\mathfrak{g}_v)\xrightarrow{\sim}\mathbb{C}[x_1,x_2]^W, \tag{3.5}$$

where W is the Weyl group of G_v. Thus $\mathfrak{z}(\mathfrak{g}_v)$ is isomorphic to the subalgebra of $\mathbb{C}[x_1,x_2]$ generated by $x_1^2+x_2^2$ and $x_1^2x_2^2$, the elementary symmetric polynomials in x_1^2 and x_2^2. Let $D\in\mathfrak{z}(\mathfrak{g}_v)$ be the element which corresponds to

$$x_1^2+x_2^2-5.$$

Now the action of D in $\mathbf{S}(W_v)$ is given by

Lemma 3.1 (i) *For* $\hat{\varphi}\in\mathbf{S}(\mathbb{R}^4)$

$$\omega_v(D)\,\hat{\varphi}=c\cdot E(E+4)\hat{\varphi}$$

where

$$E=\sum_{i=1}^{4} x_i \frac{\partial}{\partial x_i}$$

is the usual Euler operator on $\mathbb{R}^4$, *and* c *is a non-zero constant.*
(ii) $\omega_v(D)$ *commutes with the action of* H_v *and of* G_v *in* $\mathbb{S}(\mathbb{R}^4)$.
(iii) *For* $\phi \in \mathbb{S}(\mathbb{R}^4)$, *let* $\tilde{\phi}$ *denote its Fourier transform. Then* $\omega_v(D)$ *commutes with Fourier transform and*

$$\omega_v(D)\,\hat{\phi}(0)=\omega_v(D)\,\tilde{\phi}(0)=0.$$

Of course, $\omega(D)$ commutes with the action of $H(\mathbb{A})$ and of $G(\mathbb{A})$ in either model of the global oscillator representation.
For $h \in H(\mathbb{A})$ with $h \simeq (a, \xi) \in F_{\mathbb{A}}^{\times} \times \Xi$, define

$$|a(h)| = |a|_{\mathbb{A}}$$

so that the function $h \mapsto |a(h)|$ is right invariant under the maximal compact subgroup of $H(\mathbb{A})$. Also define a Haar measure on $H(\mathbb{A})$ by $d\mu(h) = d^{\times}a \cdot d\xi$ where $d^{\times}a$ is a Haar measure on $F_{\mathbb{A}}^{\times}$ and $d\xi$ is the Haar measure on Ξ with total measure 1.

We define an "Eisenstein series" $E(h, s)$ on $H(\mathbb{A})$ by

$$E(h,s)=\sum_{H^1(F)\backslash H(F)} |a(\gamma h)|^s = |a(h)|^s + |a(h)|^{-s}. \tag{3.6}$$

Note that

$$H^1(F)\backslash H(\mathbb{A}) \simeq F^{\times}\backslash F_{\mathbb{A}}^{\times} \times \Xi \simeq \mathbb{R}_+^{\times} \times F^{\times}\backslash F_{\mathbb{A}}^{1} \times \Xi,$$

where $F_{\mathbb{A}}^{1}$ is the subgroup of $a \in F_{\mathbb{A}}^{\times}$ with $|a|_{\mathbb{A}} = 1$.

Lemma 3.2 *As a function on* $H^1(F)\backslash H(\mathbb{A})$, $\theta(g, h; \omega(D)\varphi)$ *is rapidly decreasing at both* 0 *and* ∞.

Now consider the integral

$$\begin{aligned}\int_{H(F)/H(\mathbb{A})} \theta(g,h;\omega(D)\varphi)\,E(h,s)\,dh &= \int_{H^1(F)\backslash H(\mathbb{A})} \theta(g,h;\omega(D)\varphi)|a(h)|^s\,dh \\ &= \int_{F^{\times}\backslash F_{\mathbb{A}}^{\times}} \theta(g,h(a);\omega(D)\,\mathrm{pr}(\varphi))|a|^s\,d^{\times}a,\end{aligned} \tag{3.7}$$

where $\mathrm{pr}(\varphi)$ is defined by

$$\mathrm{pr}(\varphi)=\int_{\Xi} \omega(\xi)\,\varphi\,d\xi. \tag{3.8}$$

Continuing, we obtain for (3.7):

$$\begin{aligned}&\int_{F^{\times}\backslash F_{\mathbb{A}}^{\times}} \sum_{\substack{w\in F^4\\ w\neq 0}} \omega(D)\,\mathrm{pr}(\hat{\varphi})(awg)|a|^{s+2}\,d^{\times}a \\ &\quad=\sum_{\gamma\in P_1\backslash G} \int_{F_{\mathbb{A}}^{\times}} \omega(D)\,\mathrm{pr}(\hat{\varphi})(a(0,0,1,0)\gamma g)|a|^{s+2}\,d^{\times}a.\end{aligned} \tag{3.9}$$

Here P_1 is the stabilizer in G of the line $F\cdot(0,0,1,0)$. Now observe that if $X\in \mathrm{Lie}(H_v)$ denotes the Lie algebra vector associated with the 1-parameter subgroup $a(t)=e^t$ of $H_v\simeq \mathbb{R}^\times \rtimes\langle\sigma_v\rangle$, then

$$\omega(X)=E$$

and so

$$\omega(D)=c\,\omega(X(X+4)),$$

for some non-zero constant c. But then we have, for $\mathrm{Re}(s)>2$,

$$\int_{F_{\mathbb{A}}^\times} \omega(D)\,\mathrm{pr}(\hat{\varphi})(a(0,0,1,0)\gamma g)|a|^{s+2}\,d^\times a$$
$$=(s^2-4)\cdot \int_{F_{\mathbb{A}}^\times} \mathrm{pr}(\hat{\varphi})(a(0,0,1,0)\gamma g)|a|^{s+2}\,d^\times a.$$

If we finally set

$$F(g,s,\varphi)=\int_{F_{\mathbb{A}}^\times} \mathrm{pr}(\hat{\varphi})(a(0,0,1,0)\gamma g)|a|^{s+2}\,d^\times a \tag{3.10}$$

for $\mathrm{Re}(s)>2$, and extend $F(g,s,\varphi)$ to a meromorphic function of s, then, for $\mathrm{Re}(s)>2$, (3.7) becomes

$$(s^2-4)\sum_{\gamma\in P_1\backslash G} F(\gamma g,s,\varphi).$$

We now define

$$\mathscr{E}(g,s,\varphi)=\frac{1}{s^2-4}\int_{H(F)\backslash H(\mathbb{A})} \theta(g,h;\omega(D)\varphi)\,E(h,s)\,dh \tag{3.11}$$
$$=\sum_{\gamma\in P_1\backslash G} F(\gamma g,s,\varphi).$$

This is an Eisenstein series associated to the parabolic $P_1\subset G$. Since the integral above is entire (since $E(h,s)$ is), we conclude that $\mathscr{E}(g,s;\varphi)$ has simple poles at $s=\pm 2$. The value of $\mathscr{E}(g,s,\varphi)$ at $s=0$ will be of interest to us, and for this value we have the expression

$$\mathscr{E}(g,0,\varphi)=-\frac{1}{2}\int_{H(F)/H(\mathbb{A})} \theta(g,h;\omega(D)\varphi)\,dh. \tag{3.12}$$

In our present special case we may obtain another expression for $\mathscr{E}(g,s,\varphi)$ as follows. Let

$$\theta^+(g,h;\varphi)=\sum_{\substack{w\in F^4\\ w\neq 0}} \omega(h)\,\hat{\varphi}(w\cdot g) \tag{3.13}$$

and note that

$$\theta^+(g,h;\omega(D)\varphi)=\theta(g,h;\omega(D)\varphi)$$

by (iii) of Lemma 3.1. Then for $\mathrm{Re}(s)>2$, we have

$$\mathscr{E}(g,s,\varphi)=\int_{F^\times\backslash F_{\mathbb{A}}^\times}\theta^+(g,h(a);\mathrm{pr}(\varphi))|a|^s\,d^\times a.$$

Taking

$$F^\times\backslash F_{\mathbb{A}}^\times\xrightarrow{\sim}F^\times\backslash F_{\mathbb{A}}^1\times\mathbb{R}_+^\times$$

as in [38] and setting

$$C_1=F^\times\backslash F_{\mathbb{A}}^1\times(1,\infty)$$

and

$$C_0=F^\times\backslash F_{\mathbb{A}}^1\times(0,1),$$

we obtain, in the usual way,

$$\mathscr{E}(g,s,\varphi)=\int_{C_1}[\theta^+(g,h(a);\mathrm{pr}(\hat\varphi))|a|^s+\theta^+(g,h(a);\mathrm{pr}(\tilde{\hat\varphi}))|a|^{-s}]\,d^\times a+\left[\frac{\mathrm{pr}(\hat\varphi)(0)}{s-2}-\frac{\mathrm{pr}(\tilde{\hat\varphi})(0)}{s+2}\right]\mathrm{vol}(F^\times\backslash F_{\mathbb{A}}^1),\tag{3.14}$$

which holds for all s. At $s=0$ this gives

$$\mathscr{E}(g,0,\varphi)=\int_{C_1}\theta^+(g,h(a);\mathrm{pr}(\hat\varphi))+\theta^+(g,h(a);\mathrm{pr}(\tilde{\hat\varphi}))\,d^\times a-\tfrac{1}{2}[\mathrm{pr}(\hat\varphi)(0)+\mathrm{pr}(\hat\varphi)(0)].\tag{3.15}$$

On the other hand, by (3.12), we see that, for all $h\in H(\mathbb{A})$,

$$\mathscr{E}(g,0,\omega(h)\varphi)=\mathscr{E}(g,0,\varphi).\tag{3.16}$$

Define

$$I(g;\varphi)=\tfrac{1}{2}\mathscr{E}(g,0,\varphi).\tag{3.17}$$

The factor $\frac{1}{2}$ compensates for the fact that $E(h,0)=2$.

The map

$$\varphi\mapsto I(\varphi),$$

defined by (3.1) if V_0 is anistropic and by (3.17) if V_0 is split, defines an intertwining map from $S(V_0(\mathbb{A})^2)$ to $\mathscr{A}(G)$. Moreover, this map is H_v-invariant for all non-archimedean places and $(\mathfrak{h}_v,L_v)$-invariant for all archimedean places. Thus it factors through the quotient $\Pi(V_0)=\bigotimes_v R_v(V_0)$ of $S(V_0(\mathbb{A})^2)$. It is easy to check, by computing suitable Fourier coefficients for suitable φ, that the $I(g;\varphi)$'s are not all zero, and hence we have proved:

Proposition 3.3 *$\Pi(V_0)$ occurs in $\mathscr{A}(G)$ with multiplicity one.*

3.2 The case $m=4$

There are now two cases, depending on the Witt index r of V, ($r=1$ or 2). We refer to [21] for more details. The case $m=2$, $r=1$ discussed above should provide the necessary motivation. We will fix a Witt decomposition

$$V=V_r'+V_0+V_r'' \tag{3.18}$$

where $V_r'+V_r''$ is a hyperbolic plane of dimension $2r$ and V_0 is its orthogonal complement, which is anisotropic of dimension 2 if $r=1$ and is zero if $r=2$. Let Q_r be the maximal parabolic subgroup of $H=O(V)$ which stabilizes the isotropic r-plane V_r''. (When $r=2$ this is a slight abuse of terminology [16].) Let $L\subset H(\mathbb{A})$ be a maximal compact subgroup which is compatible with (3.18) in the sense that we have a global Iwasawa decomposition

$$H(\mathbb{A})=Q_r(\mathbb{A})\cdot L. \tag{3.19}$$

Write

$$Q_r=M_r U_r$$

where U_r is the unipotent radical of Q_r and M_r is the Levi component which preserves the decomposition (3.18). By restriction to V_0 and to $V/(V_0+V_r'')\simeq V_r'$, we have an isomorphism

$$M_r \xrightarrow{\sim} O(V_0)\times \mathrm{GL}(r), \tag{3.20}$$

and for $h_0\in O(V_0)$ and $a\in \mathrm{GL}(r)$ we denote the corresponding element of M_r by $m(h_0, a)$.

The oscillator representation for the dual pair (G, H) may be realized in the standard Schrödinger model (ω, S), $S=S(V(\mathbb{A})^2)$ or in a mixed model on the space

$$\hat{S}=S(V_0(\mathbb{A})^2)\otimes S(W(\mathbb{A})^r).$$

The two models are related by the partial Fourier transform

$$\hat{\varphi}(x_0, x, y)=\int_{M_{r,2}(\mathbb{A})} \psi(\mathrm{tr}({}^t u y))\varphi\begin{pmatrix} x \\ x_0 \\ u \end{pmatrix} du \tag{3.21}$$

where $x_0\in V_0(\mathbb{A})^2$ and $x, y, u\in M_{r,2}(\mathbb{A})$. Note that the action of $m(h_0, a)$ on $\hat{\varphi}\in\hat{S}$ is given by

$$\omega(m(h_0, a))\,\hat{\varphi}(x_0, x, y)=|a|^2\,\hat{\varphi}(h_0^{-1}x_0, {}^t a x, {}^t a y). \tag{3.22}$$

If $g\in G(\mathbb{A})$, then

$$\omega(g)\,\hat{\varphi}(x_0, x, y)=\omega_{V_0}(g)\,\hat{\varphi}(x_0,(x, y)\cdot g) \tag{3.23}$$

where ω_{V_0} denotes the oscillator representation of $G(\mathbb{A})$ associated to V_0 on $S(V_0(\mathbb{A})^2)$.

By Poisson summation we have

$$\theta(g,h;\varphi)=\sum_{x\in V(F)^2}\omega(g)\,\varphi(h^{-1}x)=\sum_{\substack{x_0\in V_0(F)^2\\ w\in W(F)^r}}\omega(g)\,\omega(h)\,\hat{\varphi}(x_0,w). \tag{3.24}$$

Again we want to apply a differential operator to improve the growth of $\theta(g,h;\varphi)$ on $H(F)\backslash H(\mathbb{A})$. Fix a real place v of F and let $D\in\mathfrak{z}(\mathfrak{g}_v)$ be the element which corresponds to

$$x_1^2x_2^2-x_1^2-x_2^2+1\in\mathbb{C}[x_1,x_2]^W \tag{3.25}$$

under the isomorphism (3.5).

Lemma 3.4 (i) *$\omega_v(D)$ commutes with G_v and H_v in S_v.*
(ii) *For $\hat{\varphi}\in S(V_{0,v}^2)\otimes S(M_{r,4}(\mathbb{R}))=\hat{S}_v$*

$$\omega_v(D)\,\hat{\varphi}(x_0,w)=0$$

for all $x_0\in V_{0,v}^2$ and $w\in M_{r,4}(\mathbb{R})$ such that rk $w<r$.

As a consequence, we have

Proposition 3.5 *For all $\varphi\in S(V(\mathbb{A})^2)$, $\theta(g,h;\omega(D)\varphi)$ is rapidly decreasing on $H(F)\backslash H(\mathbb{A})$.*

Next, for $h\in H(\mathbb{A})$, we write

$$h=u\cdot m(h_0,a)\cdot k$$

via the Iwasawa decomposition (3.19) and we let

$$|a(h)|=|\det a|_{\mathbb{A}}.$$

Then define the Eisenstein series

$$E(h,s)=\sum_{\gamma\in Q_r\backslash H}|a(\gamma h)|^{s+\rho'} \tag{3.26}$$

where $\rho'=2-\dfrac{r+1}{2}$. This series is absolutely convergent for $\mathrm{Re}(s)>\rho'$ and has a meromorphic analytic continuation. The first pole of $E(h,s)$ occurs at the point $s=\rho'$; it is simple with non-zero constant residue:

$$\kappa=\operatorname*{Res}_{s=\rho'}E(h,s).$$

It should be noted that the pullback of $E(h,s)$ to $\mathrm{GL}(2)\times\mathrm{GL}(2)$ (resp. $\mathrm{GL}(2)/F'$, F'/F quadratic) when $r=2$ (resp. $r=1$) is the usual Eisenstein series.

By Proposition 3.5 above, the integral

$$\int_{H(F)\backslash H(\mathbb{A})}\theta(g,h;\omega(D)\varphi)\,E(h,s)\,dh$$

is absolutely convergent for all s for which $E(h, s)$ is holomorphic. Moreover, by Howe's basic result [13], there exists a $D' \in \mathfrak{z}(\mathfrak{h}_v)$, where $\mathfrak{h}_v = \mathrm{Lie}(H_v)$ such that

$$\omega(D) = \omega(D'),$$

and hence we may write

$$\theta(g, h; \omega(D)\varphi) = \theta(g, h; \omega(D')\varphi) = \theta(g, h; \varphi) * D'.$$

Also, D' is self-adjoint for the Petersson inner product on $H(\mathbb{A})$, and we have

$$E(h, s) * D' = P(s) E(h, s) \tag{3.27}$$

with

$$P(s) = \begin{cases} -(s-1)(s+1) & r = 1 (\text{so } \rho' = 1) \\ (s-\frac{3}{2})(s-\frac{1}{2})(s+\frac{1}{2})(s+\frac{3}{2}) & r = 2 (\text{so } \rho' = \frac{1}{2}) \end{cases}. \tag{3.28}$$

We then define

$$\mathscr{E}(g, s, \varphi) = \frac{1}{P(s)} \int_{H(F)\backslash H(\mathbb{A})} \theta(g, h; \omega(D)\varphi) E(h, s)\, dh. \tag{3.29}$$

A more detailed discussion of a general construction of this type will be given in a future paper of the first two authors.

At the point $s = \rho'$, which is now of interest to us, we have the Laurent expansion

$$\mathscr{E}(g, s, \varphi) = \frac{B_{-2}(g, \varphi)}{(s-\rho')^2} + \frac{B_{-1}(g, \varphi)}{s-\rho'} + B_0(g, \varphi) + \cdots, \tag{3.38}$$

and the various terms $B_i(g, \varphi)$ provide us with a regularization of the theta integral. In fact, we will obtain an interpretation of B_{-2} and B_{-1}.

Note that if we write

$$\frac{1}{P(s)} = \frac{a}{s-\rho'} + b + O(s-\rho'), \tag{3.39}$$

then

$$B_{-2}(g, \varphi) = a\kappa \int_{H(F)\backslash H(\mathbb{A})} \theta(g, h; \omega(D)\varphi)\, dh \tag{3.40}$$

and

$$B_{-1}(g, \varphi) = \frac{b}{a} B_{-2}(g, \varphi) + a \int_{H(F)\backslash H(\mathbb{A})} \theta(g, h; \omega(D)\varphi)\, \kappa_0(h)\, dh \tag{3.41}$$

where

$$E(h, s) = \frac{\kappa}{s-\rho'} + \kappa_0(h) + O(s-\rho'). \tag{3.42}$$

We now restrict to the case $r=1$. Another expression for $\mathscr{E}(g, s, \varphi)$ will be useful later. Let $H_0=O(V_0)$. For $\phi\in S(V_0(\mathbb{A}))$ and $g\in \mathrm{Sp}(1, \mathbb{A})$, define a theta integral:

$$I_{V_0}(g, \phi):=I_{V_0}(\omega_{V_0}(g)\,\phi):=\int_{H_0(F)\backslash H_0(\mathbb{A})}\sum_{x\in V_0(F)}\omega_{V_0}(g)\phi(h^{-1}x)\,dh, \tag{3.30}$$

where ω_{V_0} denotes the oscillator representation of $\mathrm{Sp}(1, \mathbb{A})$ on $S(V_0(\mathbb{A}))$.

If, for large $\mathrm{Re}(s)$, we unfold (3.29), we obtain

$$\mathscr{E}(g, s, \varphi)=\sum_{\gamma\in P_1\backslash G}F(\gamma g, s, \mathrm{pr}_{K_H}(\varphi)) \tag{3.31}$$

where

$$\mathrm{pr}_{K_H}(\varphi)=\int_{K_H}\omega(k)\varphi\,dk, \tag{3.32}$$

is the projection of φ to the K_H invariants in $S(V(\mathbb{A})^2)$, and, for any $\varphi\in S(V(\mathbb{A})^2)$,

$$F(g, s, \varphi)=\int_{\mathbb{A}^\times} I_{V_0}(\omega(g)\,\hat{\varphi}(0, \cdot, a(0, 0, 1, 0)))|a|^{s+1}\,d^\times a, \tag{3.33}$$

(recall that $r=1$). Here, as above, P_1 is the standard maximal parabolic subgroup of G which stabilizes the line $F\cdot(0, 0, 1, 0)$. Details of this calculation will be found in [21]. Thus $\mathscr{E}(g, s, \varphi)$ is an Eisenstein series associated to P_1 and to a certain induced representation $I_{P_1}^G(s)$ of $G(\mathbb{A})$. More prescisely, note that $P_1=M_1N_1$ with $M_1\simeq \mathrm{GL}(1)\times \mathrm{Sp}(1)$. Since V_0 is anisotropic, the theta integrals (3.30) lie in $L^2(\mathrm{Sp}(1, F)\backslash \mathrm{Sp}(1, \mathbb{A}))$, and we let $\Pi^{(1)}(V_0)$ denote the irreducible automorphic representation of $\mathrm{Sp}(1, \mathbb{A})$ on the space which they span. The representation $I_{P_1}^G(s)$ is then realized on the space of functions

$$\Psi: N_1(\mathbb{A})\,M_1(F)\backslash G(\mathbb{A})\to\mathbb{C} \tag{3.34}$$

such that

(i) for all $g\in G(\mathbb{A})$, $g_1\in \mathrm{Sp}(1, \mathbb{A})$ and for $m_1(1, g_1)\in M_1(\mathbb{A})$ the corresponding element, the function $g_1\mapsto\Psi(m_1(1, g_1)\,g, s)$ lies in $\Pi^{(1)}(V_0)$.

(ii) For $a\in\mathbb{A}^\times$ and for $m_1(a, 1)\in M_1(\mathbb{A})$ the corresponding element,

$$\Psi(m_1(a, 1)\,g, s)=|a|^{s+2}\,\Psi(g, s).$$

We moreover require that Ψ be smooth and right K finite.

For any section $\Psi(s)\in I_{P_1}^G(s)$, we let

$$\mathscr{E}(g, s, \Psi(s))=\sum_{\gamma\in P_1(F)\backslash G(F)}\Psi(\gamma g, s) \tag{3.35}$$

be the associated Eisenstein series. Note that $F(s)$, defined by (3.33) for $\mathrm{Re}(s)>0$, defines an intertwining operator

$$F(s): S(V(\mathbb{A})^2)\to I_{P_1}^G(s), \tag{3.36}$$

and has a meromorphic analytic continuation to the whole s plane. Then, for any K-finite $\varphi \in S(V(\mathbb{A})^2)$, (3.31) may be restated as

$$\mathscr{E}(g, s, \varphi) = \mathscr{E}(g, s, F(s)(\mathrm{pr}_{K_H}(\varphi))). \tag{3.37}$$

We will return to this discussion in section 6, where a deeper analysis will be needed.

4 The residue of $E(g, s, \Phi)$ at its second pole

We will show that, for an arbitrary standard section Φ, the residue $A_{-1}(g, \Phi)$ of $E(g, s, \Phi)$ at $s = \frac{1}{2}$ is a finite linear combination of binary theta functions $I(g, \varphi)$. Actually our result is more precise. First we introduce some terminology concerning sections. From now on all our sections Φ are assumed to be K-finite. Let $\Phi = \bigotimes_v \Phi_v$ be a factorizable standard section. We call Φ homogeneous (relative to the point $s = \frac{1}{2}$) if for every place v there exists a quaternary quadratic space V_v such that $\Phi_v(\frac{1}{2}) \in R_v(V_v)$. The collection of V_v's may not be unique. We call Φ coherent (relative to the point $s = \frac{1}{2}$) if it is factorizable, homogeneous and if there exists a global quaternary space V such that $\Phi_v(\frac{1}{2}) \in R_v(V)$ for every place v. Note that then $\chi_V = \chi$. On the other hand, we will call Φ incoherent if it is factorizable and homogeneous, but *no such global space V exists.* If Φ is coherent, then there exists a function $\varphi = \bigotimes_v \varphi_v \in S(V(\mathbb{A})^2)$ such that

$$\Phi(g, \tfrac{1}{2}) = \omega(g)\,\varphi(0), \qquad \forall g \in G(\mathbb{A})$$

where V is a global space associated to Φ. Note that V and φ may not be unique.

Theorem 4.1 *Suppose that $\Phi = \bigotimes_v \Phi_v$ is a homogeneous standard section.*

(i) *If Φ is incoherent, then $A_{-1}(g, \Phi) = 0$, i.e., $E(g, s, \Phi)$ has no pole at $s = \frac{1}{2}$. Next, suppose that Φ is coherent and let V be an associated global quaternary space.*

(ii) *If V is anisotropic, then $A_{-1}(g, \Phi) = 0$.*

(iii) *If V is isotropic, write $V = V_0 + V_{1,1}$, where $V_{1,1}$ is a hyperbolic plane and V_0 is a binary space. For any $\varphi \in S(V(\mathbb{A})^2)$ and for $\Phi(s) \in I(s)$, the corresponding standard section, there exists a $\varphi' \in S(V_0(\mathbb{A})^2)$ such that*

$$A_{-1}(g, \Phi) = I(g, \varphi').$$

(iv) *Given any $V = V_0 + V_{1,1}$ as in* (iii), *there exists a standard section $\Phi = \bigotimes_v \Phi_v$ with $\Phi_v(\frac{1}{2}) \in R_v(V)$ for all v such that $A_{-1}(\Phi) \neq 0$.*

Remark. The dependence of φ' on φ may be described as follows: For each v, let $M_v^*(\frac{1}{2})$: $R_v(V) \to R_v(V_0)$ be the map given by the normalized intertwining operator as in Propositions 1.1 and 1.2. Let $\varphi_v' \in S(V_{0,v}^2)$ be any function whose image in $R_v(V_0)$ coincides with that of Φ_v under $M_v^*(\frac{1}{2})$. For almost all places

v we choose φ'_v to be the characteristic function of a fixed maximal lattice. Then there is a non-zero constant C_A, independent of φ, such that $\varphi' = C_A \cdot \bigotimes_v \varphi'_v$.

We will prove this theorem in several steps. First, consider the non-singular Fourier coefficients of $E(g, s, \Phi)$. For $\beta \in \mathrm{Sym}_2(F)$ with $\mathrm{rk}\ \beta = 2$, and for fixed $g \in G(\mathbb{A})$ and $\Phi = \bigotimes_v \Phi_v$, we have [30, p. 102]

$$E_\beta(g, s, \Phi) = \left(\prod_{v \in S} W^*_{\beta,v}(g_v, s, \Phi_v)\right) \frac{L(s+\frac{1}{2}, \chi\chi_\beta)}{L(s+\frac{3}{2}, \chi)\, L(2s+1)} \tag{4.1}$$

where

$$W^*_{\beta,v}(g_v, s, \Phi_v) = \frac{L_v(s+\frac{3}{2}, \chi)}{L_v(s+\frac{1}{2}, \chi\chi_\beta)} L_v(2s+1) \int_{\mathrm{Sym}_2(F_v)} \Phi_v(w n(b)\, g_v, s)\, \psi_v(-\mathrm{tr}(\beta b))\, db \tag{4.2}$$

is the normalized (generalized) Whittaker integral, and where the set S of exceptional places depends on g, Φ and β. The part $W_{\beta,v}(g, s, \Phi)$ of $W^*_{\beta,v}(g, s, \Phi)$ given by the integral is entire [15, 36]. Thus, at $s = \frac{1}{2}$ we see that

$$\operatorname*{ord}_{s=\frac{1}{2}} E_\beta(g, s, \Phi) \geqq \begin{cases} -1 & \text{if } \chi_\beta = \chi \\ 0 & \text{otherwise} \end{cases}. \tag{4.3}$$

Proposition 4.2 *For any standard section Φ, $E(g, s, \Phi)$ has at most a simple pole at $s = \frac{1}{2}$.*

Proof. Choose a place v and any homogeneous section $\Phi^0 = \bigotimes_w \Phi^0_w$, and consider the map

$$\Phi_v(s) \mapsto E(g, s, \Phi) = \sum_{j=-k}^{\infty} A_j(g, \Phi)(s-\tfrac{1}{2})^j,$$

where $\Phi = \Phi_v \otimes \Phi^{0,v}$ with $\Phi^{0,v} = \bigotimes_{w \neq v} \Phi^0_w$. The map

$$A_{-k} : I_v(\tfrac{1}{2}) \to \mathscr{A}(G)$$
$$\Phi_v \mapsto A_{-k}(\Phi)$$

is then intertwining, and, by (4.3), if $k > 1$ the composition $A_{-k,\beta} = W_\beta \circ A_{-k}$ is zero for all non-singular β. Since all constituents of the representation $I_v(\frac{1}{2})$ are non-singular, the map A_{-k} must be identically zero. Since the choice of Φ^0 was arbitrary, we conclude that $A_{-k}(g, \Phi) = 0$ for all $k > 1$, as claimed. □

Next, we apply the same considerations to A_{-1}. By the previous proposition, the map

$$I(\tfrac{1}{2}) \to \mathscr{A}(G)$$
$$\Phi \mapsto A_{-1}(\Phi)$$

is intertwining and, by (4.3), the composition $A_{-1,\beta}=W_\beta\circ A_{-1}$ vanishes for all β of rank 2 with $\chi_\beta\neq\chi$. Moreover, as remarked in [30, p. 98], Wallach [36] has shown that the functionals $A_{-1,\beta}$ extend to continuous functionals on $I^\infty(\frac{1}{2})$.

Lemma 4.3 *For any fixed place v, the restriction of A_{-1} to sections of the form $\Phi=\bigotimes_w \Phi_w$ with $\Phi_v(\frac{1}{2})\in R_v\cap R_v'$ is zero. Here $R_v\cap R_v'$ is the unique maximal submodule of $I_v(\frac{1}{2})$ as in Propositions* 1.1 *and* 1.2.

Proof. Fix $\Phi^0=\bigotimes_w \Phi_w^0$ and consider the map

$$I_v(\tfrac{1}{2})\to\mathscr{A}(G)$$
$$\Phi_v\mapsto A_{-1}(\Phi)$$

with $\Phi=\Phi_v\otimes\Phi^{0,v}$. Suppose that β has rank 2. For any $g\in G(\mathbb{A})$ with $g_v=1$, the map

$$\Phi_v\mapsto A_{-1,\beta}(\Phi)(g)$$

defines a linear functional which factors through $(R_v\cap R_v')_{N,\beta}$ if v is non-archimedean and lies in $(R_v^\infty\cap(R_v')^\infty)'_\beta$ if v is archimedean. If $\chi_{\beta,v}\neq\chi_v$, this functional is zero by the remarks above. If $\chi_{\beta,v}=\chi_v$, then this functional is zero by Lemma 1.4 and Proposition 1.9. As before, by the density of $G(F)\cdot\prod_{w\neq v} G_v$ and smoothness of $A_{-1,\beta}(\Phi)$, we conclude that $A_{-1,\beta}(\Phi)=0$ for all β of rank 2. But the representation $R_v\cap R_v'$ is non-singular. For example, if v is non-archimedean, there exists a function $f\in S(N_v)$ with supp $\hat{f}\subset\{n(b)|\text{rk } b=2\}$ which does not act by zero in $R_v\cap R_v'$. By the same argument as in the proof of Lemma 2.5, we conclude that $A_{-1}(\Phi)$ is zero, as claimed. □

We may now give the proof of Theorem 4.1.

Proof. By Lemma 4.3, we conclude that A_{-1} factors through the quotient of $I(\frac{1}{2})$ which is the sum of all the spaces of the form $\bigotimes_v R_v(V_{0,v})$ for collections of binary spaces $\{V_{0,v}\}$ with $\chi_{V_{0,v}}=\chi_v$ for all v. More precisely, if $\Phi=\bigotimes_v \Phi_v$ is a homogeneous section in $I(s)$, with $\Phi_v(\frac{1}{2})\in R_v(V_v)$ for all v, then $A_{-1}(\Phi)$ depends only on the image of $\Phi_v(\frac{1}{2})$ in $R_v(V_v)/(R_v\cap R_v')$. If V_v is anisotropic for some v, then $A_{-1}(\Phi)$ is zero. This proves (ii).

We may then assume that V_v is isotropic for all v, and we fix the isomorphism

$$i_v=M_v^*(\tfrac{1}{2})\colon R_v(V_v)/(R_v(V_v)\cap R_v(V_v'))\xrightarrow{\sim} R_v(V_{0,v}).$$

Then we obtain an intertwining map

$$A_{-1}\circ(\bigotimes_v i_v^{-1})\colon \bigotimes_v R_v(V_{0,v})\to\mathscr{A}(G).$$

This map is zero if $\prod_v \varepsilon_{V_{0,v}}=-1$ (Proposition 2.6), i.e., if Φ is incoherent. This proves (i).

Now by the uniqueness of Theorem 2.2, there exists a constant c such that, for all Φ of the type under discussion,

$$A_{-1}(\Phi)=c\cdot I(\varphi^0)$$

where $\varphi^0=\bigotimes_v \varphi_v^0 \in S(V_0(\mathbb{A})^2)$ is any Schwartz function such that the image of φ_v^0 in $R_v(V_{0,v})$ is $i_v(\Phi_v)$ for all v.

Finally we must prove that A_{-1} is not identically zero. For this we consider the β-th Fourier coefficent of $E(g, s, \Phi)$, which is given by (4.1) for non-singular β. It suffices to show that for each exceptional place v, the integral in (4.2) is non-zero for some choice of data g_v, Φ_v. Note that we may as well assume that $\beta\in \mathrm{Sym}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of F. First assume that v is a finite place. For any positive integer n, let $K_v(n)$ be the subgroup of $K_v=\mathrm{Sp}(2, \mathcal{O}_v)$ consisting of elements which are congruent to the identity modulo $\mathcal{P}_v^n$. If we take n sufficiently large, e.g., larger than the conductor of χ_v, then we may take Φ_v to be supported in the set $P_v K_v(n)$, to be right $K_v(n)$ invariant, and to have value 1 at $g=e$. We may also take $g_v=w^{-1}$ so that the argument of $\Phi_v(s)$ in the integral will be

$$w n(b) w^{-1}=\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}.$$

Then

$$\Phi(w n(b) w^{-1}, s)=\begin{cases} 1 & \text{if } b\in \mathcal{P}_v^n\cdot \mathrm{Sym}_2(\mathcal{O}_v) \\ 0 & \text{otherwise} \end{cases}.$$

Since ψ_v is trivial on $\mathcal{O}_v$ the factor $\psi_v(-\mathrm{tr}(\beta b))=1$, for $b\in \mathcal{P}_v^n \mathrm{Sym}_2(\mathcal{O}_v)$, and the integral is then a non-zero constant, independent of s.

Next suppose that v is a real place, and for convenience omit if from the notation. Let $X=V^2$ and let

$$X_{\mathrm{reg}}=\{x\in X \mid \mathrm{rank}\, x=2, \quad \text{and} \quad \mathrm{rank}(x, x)=2\}.$$

Here the rank of x is the dimension of the subspace of V spanned by the two components of x. Also let $Y=\mathrm{Sym}_2(\mathbb{R})$, let

$$\mu: X\to Y \quad x\mapsto \mu(x)=\tfrac{1}{2}(x, x)$$

be the moment map, and let

$$Y_{\mathrm{reg}}=\mu(X_{\mathrm{reg}}).$$

Note that the restriction of μ to X_{reg} is everywhere submersive, and that both X_{reg} and Y_{reg} are orientable manifolds. A basic result of Harish-Chandra [8, §§2, 3, 5; 10, p. 49, Theorem 11] then asserts that there is a surjective map

$$C_c^\infty(X_{\mathrm{reg}})\to C_c^\infty(Y_{\mathrm{reg}})$$
$$\varphi\mapsto M_\varphi$$

with the property that for any locally L^1 function α on Y_{reg}

$$\int_{X_{\mathrm{reg}}} \varphi(x)\,\alpha(\mu(x))\,dx = \int_{Y_{\mathrm{reg}}} M_\varphi(y)\,\alpha(y)\,dy.$$

Here, since X_{reg} and Y_{reg} are actually open submanifolds of the Euclidean spaces X and Y respectively, we may take dx and dy to be the restrictions of Lebesque measure.

Now for large $\mathrm{Re}(s)$,

$$\begin{aligned} W_\beta(e, s, \Phi) &= \int_Y \Phi(wn(b), s)\,\psi(-\mathrm{tr}(\beta b))\,db \\ &= \int_Y \int_X \psi(\mathrm{tr}(b\mu(x))\,\varphi(x)\,dx\,|a(wn(b))|^{s-\frac{1}{2}}\,\psi(-\mathrm{tr}(\beta b))\,db. \end{aligned}$$

Taking $\varphi \in C_c^\infty(X_{\mathrm{reg}}) \subset S(X)$ the inner integral becomes

$$\int_{Y_{\mathrm{reg}}} \psi(\mathrm{tr}(by))\,M_\varphi(y)\,dy = \hat{M}_\varphi(b),$$

and, since $M_\varphi \in C_c^\infty(Y_{\mathrm{reg}})$, this function lies in $S(Y)$. Thus our original integral is absolutely convergent for all s, and, setting $s = \frac{1}{2}$, we obtain

$$W_\beta(e, s, \Phi) = \int_Y \hat{M}_\varphi(b)\,\psi(-\mathrm{tr}(\beta b))\,db = M_\varphi(\beta).$$

Thus, for $\beta \in Y_{\mathrm{reg}}$, the surjectivity of the map $\varphi \mapsto M_\varphi$ yields the desired non-vanishing.

This completes the proof of the theorem. □

5 The leading term of the regularized theta integral

In this section we begin our interpretation of the terms $B_{-2}(g, \varphi)$ and $B_{-1}(g, \varphi)$. Again, the key is the behavior of the non-singular Fourier coefficients.

Fix an isotropic quaternary space V, $(\,,\,)$ and let $r = 1$ or 2 be the Witt index of V, as in §3. Write

$$V = V_0 + V_{1,1},$$

where $V_{1,1} = V' + V''$ is a hyperbolic plane and $\dim V_0 = 2$. Let $\chi = \chi_V = \chi_{V_0}$ and note that V and V_0 are split if and only if $\chi = 1$.

As in section 2, let

$$\mathcal{O}_{V_0} = \{\beta \in \mathrm{Sym}_2(F) \mid \exists\, x \in V_0(F)^2 \quad \text{with} \quad (x, x) = 2\beta \quad \text{and} \quad \det \beta \neq 0\},$$

and observe that $\mathcal{O}_{V_0}$ is a $\mathrm{GL}(2, F)$-orbit in $\mathrm{Sym}_2(F)$.

For any $\beta \in \mathrm{Sym}_2(F)$, recall that

$$\theta_\beta(g, h; \varphi) = \int_{N(F)\backslash N(\mathbb{A})} \theta(ng, h, \varphi)\,\overline{\psi_\beta(n)}\,dn = \sum_{\substack{x \in V(F)^2 \\ (x,x) = 2\beta}} \omega(g)\,\varphi(h^{-1}x). \tag{5.1}$$

Proposition 5.1 *If* rk $\beta=2$ *and* $\beta\notin\mathcal{O}_{V_0}$, *then*

$$\int_{H(F)\backslash H(\mathbb{A})} \theta_\beta(g,h;\varphi)\,dh$$

is absolutely convergent for all $\varphi\in S(V(\mathbb{A})^2)$. *In fact,* $\theta_\beta(g,h;\varphi)$ *is rapidly decreasing on* $H(F)\backslash H(\mathbb{A})$.

Proof. We let $T\subset H$ be the maximal F-split torus which is adapted to the decomposition $V=V_0+V_{1,1}$ where $V_{1,1}=V'+V''$ for isotropic lines V' and V''. More precisely, if V_0 is anisotropic, $T\simeq F^\times$ and $F^\times$ acts on V' (resp. V'') by the character $t\mapsto t$ (resp. $t\mapsto t^{-1}$). If V_0 is split, then $T\simeq(F^\times)^2$ and $t=(t_1,t_2)\in(F^\times)^2$ acts on V' (resp. V'') by the character $t\mapsto t_1$ (resp. $t\mapsto t_1^{-1}$) and in a similar way on $V_0=V_0'+V_0''$ relative to the t_2 variable.

If V_0 is anisotropic, the weights of T acting on $V(F)^2$ are X_1,X_2,X_3 with $X_1(t)=t$, $X_2(t)=t^{-1}$, and $X_3(t)=1$, while if V_0 is isotropic, then the weights of T on $V(F)^2$ are X_1,X_2,X_3,X_4 with $X_1(t)=t_1$, $X_2(t)=t_1^{-1}$, $X_3(t)=t_2$ and $X_4(t)=t_2^{-1}$. Now if $x\in V(F)^2$ with $(x,x)=2\beta$ for $\beta\notin\mathcal{O}_{V_0}$, it follows that x must have a nonzero component $x(X)$ in $V(F)_X^2$ for $X=X_1$ and X_2 if V_0 is anisotropic, or for $X=X_1,X_2,X_3$ and X_4 if V_0 is split. Indeed, if V_0 is anisotropic, then $x(X_1)=0$ or $x(X_2)=0$ would imply that $\frac{1}{2}(x,x)\in\mathcal{O}_{V_0}$, and a similar argument works if V_0 is split.

Now let $\mathfrak{S}_\varepsilon$ be a Siegel set for the group $H(\mathbb{A})$ relative to a Borel subgroup B with T in a Levi factor. We may write

$$\mathfrak{S}_\varepsilon=\omega A_\varepsilon^+ L$$

where L is our fixed maximal compact subgroup, ω is a compact subset of $B(\mathbb{A})$, and

$$A_\varepsilon^+=\begin{cases}\{t\in T(F_v)\mid t_\infty^+\geqq\varepsilon\} & \text{if } V_0 \text{ is anisotropic}\\ \{t\in T(F_v)\mid t_1t_2^{-1}\geqq\varepsilon,\ t_1t_2\geqq\varepsilon\} & \text{if } V_0 \text{ is split}\end{cases}.$$

Here we have fixed a real archimedean place v of F. For a sufficiently small ε,

$$H(\mathbb{A})=H(F)\cdot\mathfrak{S}_\varepsilon.$$

Next we apply the remarks and the standard trick given in [37], that for each $\varphi\in S(V(\mathbb{A})^2)$ and each $m>0$ there is a $\varphi_0\in S(V(\mathbb{A})^2)$ with $\varphi_0(x)\geqq 0$ for all x and such that

$$|\theta_\beta(uak,\varphi)|\leqq\sum_{\substack{x\in V(F)^2\\(x,x)=2\beta}}\omega(a)\,\varphi_0(x)\leqq C_\alpha\cdot\alpha(a)^{-m}.$$

Here $u\in\omega$, $a\in A_\varepsilon^+$ and

$$\alpha(a)=\begin{cases}t & \text{if } V_0 \text{ is anisotropic}\\ t_1t_2 \quad\text{or}\quad t_1t_2^{-1} & \text{if } V_0 \text{ is split}\end{cases}.$$

Here, $k\in L$, and the constant $C>0$ is independent of u, k and a. We note that $\{a^{-1}ua\mid a\in A_\varepsilon^+ \text{ and } u\in\omega\}$ is a relatively compact set in $B(\mathbb{A})$.

Thus, in any case, we deduce immediately that $\theta_\beta(L,h;\varphi)$ is rapidly decreasing on $\mathfrak{S}_\varepsilon$ and hence on $H(F)\backslash H(\mathbb{A})$ as claimed. □

Corollary 5.2 *If* rk $\beta=2$ *and* $\beta\notin\mathcal{O}_{V_0}$, *then* $B_{-2,\beta}(g,\varphi)=0$.

Proof. By Proposition 5.1 and (3.40) we have

$$B_{-2,\beta}(g,\varphi)=a\kappa\int_{H(F)\backslash H(\mathbb{A})}\theta_\beta(g,h;\omega(D)\varphi)\,dh$$
$$=a\kappa\int_{H(F)\backslash H(\mathbb{A})}\theta_\beta(g,h;\varphi)(D')^*\{1\}\,dh=0 \tag{5.2}$$

since $(D')^*=D'$ and $D'\{1\}=0$. Here note that the constant function 1 is the residue of $\kappa^{-1}E(h,s)$ at the point $s=\rho'$. Thus, by (3.27) and (3.28), $E(h,s)*D'$ is holomorphic at $s=\rho'$. This shows that $D'\{1\}=0$. □

Next, consider $\beta\in\mathcal{O}_{V_0}$. Note that for $a\in\mathrm{GL}(2,F)$ and for $m(a)\in G(F)$,

$$B_{-2,\beta}(m(a)\,g,\varphi)=B_{-2,\beta[a]}(g,\varphi),$$

where $\beta[a]={}^t a\beta a$. Also, for $g\in G(\mathbb{A})$,

$$B_{-2}(g,\varphi)=B_{-2}(e,\omega(g)\varphi) \tag{5.3}$$

and hence

$$B_{-2,\beta}(g,\varphi)=B_{-2,\beta}(e,\omega(g)\varphi). \tag{5.4}$$

First fix a place v of F and a factorizable vector $\varphi^0=\bigotimes_w\varphi_w^0\in S(V(\mathbb{A})^2)$ and an element $g\in G(\mathbb{A})$ with $g_v=1$. Define a linear functional on $S_v=S(V_v^2)$ by

$$\mu_v:\varphi_v\mapsto B_{-2,\beta}(g,\varphi) \tag{5.5}$$

with $\varphi=\varphi_v\otimes\varphi^{0,v}$ and $\varphi^{0,v}=\bigotimes_{w\neq v}\varphi_w^0$. Then it is easily checked that

$$\mu_v(\omega(n)\,\varphi_v)=\psi_\beta(n)\,\mu_v(\varphi_v)\quad\forall n\in N_v, \tag{5.6}$$

and

$$\mu_v(\omega(h)\,\varphi_v)=\mu_v(\varphi_v)\quad\forall h\in H_v=O(V_v). \tag{5.7}$$

Note that if v is archimedean, we may extend μ_v to a continuous linear functional on $S_v=S(V_v^2)$ satisfying (5.6) and (5.7). Thus μ_v factors through the quotient $R_v(V)$ of S_v and, by the first of these identities, defines an element of $R_v^\infty(V)'_\beta$. In particular, $R_v\cap R'_v$ lies in the kernel of μ_v, via Lemma 1.4 or Proposition 1.9.

We conclude that $B_{-2,\beta}$ is zero for any $\varphi=\bigotimes_w\varphi_w$ for which, for some place v, the image of φ_v in $R_v(V)$ lies in $R_v\cap R'_v$. The proof of Lemma 2.5 then implies that B_{-2} itself vanishes for such φ's.

Now fix isomorphisms

$$i_v=M_v^*(\tfrac{1}{2}):R_v(V)/(R_v\cap R'_v)\xrightarrow{\sim}R_v(V_0)$$

as in the proof of Theorem 4.1, and define an embedding

$$\Pi(V_0) \to \mathscr{A}(G)$$
$$\Phi_0 \mapsto B_{-2}(\varphi),$$

where, for given $\Phi_0 = \bigotimes_v \Phi^0_{0,v}$ we choose any $\varphi = \bigotimes_v \varphi_v \in S(V(\mathbb{A})^2)$ such that for all v, the image of φ_v in $R_v(V)/(R_v \cap R'_v)$ is equal to $i_v^{-1}(\Phi^0_v)$. By the previous discussion and, by the uniqueness theorem of Sect. 2, we obtain

Theorem 5.3 *There exists a constant $c_B \neq 0$ such that, for any $\varphi = \bigotimes_v \varphi_v \in S(V(\mathbb{A})^2)$ and for any $g \in G(\mathbb{A})$*

$$B_{-2}(g, \varphi) = c_B I(g, \varphi_0),$$

where $\varphi_0 = \bigotimes_v \varphi_{0,v} \in S(V_0(\mathbb{A})^2)$ is any Schwartz function whose local components $\varphi_{0,v}$ are such that their image in $R_v(V_0)$ are equal to i_v of the image of φ_v in $R_v(V)/(R_v \cap R'_v)$.

Corollary 5.4 *If $\varphi = \bigotimes_v \varphi_v \in S(V(\mathbb{A})^2)$ and if $\Phi = \bigotimes_v \Phi_v$ is the corresponding standard section of $I(s)$, then there exists a constant $C \neq 0$, independent of φ, such that*

$$B_{-2}(g, \varphi) = C \cdot A_{-1}(g, \Phi)$$

for all $g \in G(\mathbb{A})$.

Proof. In view of Theorems 4.1 and 5.3 the only point to check is the non-vanishing of A_{-1}, but this is given by (iv) of Theorem 4.1. □

6 The relation between the next terms of the Laurent expansion

We now come to the heart of the paper. Fix V as in §5 and retain the notation of that section.

Our main result is

Theorem 6.1 *There exists a constant c' such that for all $\varphi \in S(V(\mathbb{A})^2)$*

$$A_0(g, \Phi) = c' B_{-1}(g, \varphi) + B_{-2}(g, \varphi'),$$

for some function $\varphi' \in S(V(\mathbb{A})^2)$. Here $\Phi(s) \in I(s, \chi)$ is the standard section determined by φ.

Remark. This result may be thought of as a kind of limit formula which expresses the *second term* in the Laurent expansion of $E(g, s, \Phi)$ at $s = \frac{1}{2}$ in terms of theta series.

The proof of Theorem 6.1 will occupy the rest of this section, and we separate the cases in which the Witt index r of V is 1 or 2.

6.1

We begin with the case in which $r=1$ so that V_0 is anisotropic and $\chi \neq 1$. First we compare non-singular Fourier coefficients.

Proposition 6.2 *If $\beta \in \mathrm{Sym}_2(F)$ with* rk $\beta=2$, *and $\beta \notin \mathcal{O}_{V_0}$, then there exists a constant c', independent of β, such that, for all $\varphi \in S(V(\mathbb{A}^2))$,*

$$A_{0,\beta}(g, \Phi)=c' \cdot B_{-1,\beta}(g, \varphi).$$

Here Φ is the standard section in $I(s)$ associated to φ.

Proof. This is a consequence of Propositions 4.1 and 4.2 of [30] as follows: Since $\beta \notin \mathcal{O}_{V_0}$, we have

$$A_{0,\beta}(g, \Phi)=E_\beta(g, \tfrac{1}{2}, \Phi). \tag{6.1}$$

On the other hand, by Proposition 5.1, we have

$$\begin{aligned} \mathscr{E}_\beta(g, s, \varphi) &= \frac{1}{P(s)} \int_{H(F)\backslash H(\mathbb{A})} \theta_\beta(g, h; \omega(D)\varphi)\, E(h, s)\, dh \\ &= \frac{1}{P(s)} \int_{H(F)\backslash H(\mathbb{A})} \theta_\beta(g, h; \varphi)\, E(h, s) * D'\, dh \\ &= \int_{H(F)\backslash H(\mathbb{A})} \theta_\beta(g, h; \varphi)\, E(h, s)\, dh. \end{aligned} \tag{6.2}$$

Hence, if β is represented by V,

$$\begin{aligned} B_{-1,\beta}(g, \varphi) &= \kappa \cdot \int_{H(F)\backslash H(\mathbb{A})} \theta_\beta(g, h; \varphi)\, dh \\ &= \kappa \cdot \mathrm{vol}(H_x(F)\backslash H_x(\mathbb{A})) \int_{H_x(\mathbb{A})\backslash H(\mathbb{A})} \omega(g)\, \varphi(h^{-1} x)\, dh, \end{aligned} \tag{6.3}$$

where $x \in V(F)^2$ with $(x, x)=2\beta$ and $H_x \subset H$ is the stabilizer of x. Thus $B_{-1,\beta}(g, \varphi)$ is κ times the expression $(*)$ at the top of p. 113 of [30]. Proposition 4.2, p. 113 of [30], gives the required proportionality when β is represented by V. If β is not represented by V, we obtain $B_{-1,\beta}(g, \varphi)=0$. In this case, Proposition 4.1 on p. 103 of [30] yields the required vanishing of $A_{0,\beta}(g, \Phi)$. □

For c' as in Proposition 6.2, we define

$$D: S(V(\mathbb{A})^2) \to \mathscr{A}(G) \tag{6.4}$$

by

$$D(g, \varphi)=A_0(g, \Phi)-c' \cdot B_{-1}(g, \varphi). \tag{6.5}$$

Note that

$$D_\beta(g, \varphi)=0 \tag{6.6}$$

for all β with rk $\beta=2$ and $\beta \notin \mathcal{O}_{V_0}$. It is important to note that the map D is neither G_v (resp. $(\mathfrak{g}_v, K_v)$) intertwining – because A_0 is not – nor H_v (resp.

$(\mathfrak{h}_v, L_v)$) invariant – since B_{-1} is not. This makes the remaining part of the argument rather delicate.

At this point it might be helpful to explain the reason for the somewhat intricate argument which follows. Suppose for a moment that the map D just defined were $H(\mathbb{A})$ invariant and $G(\mathbb{A})$ intertwining (*which it is not*!). Then D would factor through the quotient $\bigotimes_v R_v(V)$ of $S(V(\mathbb{A})^2)$, and, by the results of Sect. 1, the fact that $D_\beta = 0$ for non-singular $\beta \notin \mathcal{O}_{V_0}$ would then force D to further factor through the quotient $\bigotimes_v R_v(V_0) = \Pi(V_0)$. The uniqueness Theorem 2.2 would then yield what we want. Unfortunately, this suggestive argument cannot be made and we must proceed in a more roundabout fashion.

As explained in [17, §2], we do have

Proposition 6.3 *Let* $\mathrm{Im}(A_{-1}) \subset \mathscr{A}(G)$ *be the image of the map*

$$A_{-1} : I(\tfrac{1}{2}) \to \mathscr{A}(G),$$

described in §4. *Then the map*

$$\bar{D} : S(V(\mathbb{A})^2) \to \mathscr{A}(G)/\mathrm{Im}(A_{-1})$$

is G_v (*resp.* $(\mathfrak{g}_v, K_v)$) *intertwining.*

Thus we will consider the extension

$$0 \to \mathrm{Im}(A_{-1}) \to Y \to \mathrm{Im}(\bar{D}) \to 0,$$

so that Y is the inverse image in $\mathscr{A}(G)$ of $\mathrm{Im}(\bar{D}) \subset \mathscr{A}(G)/\mathrm{Im}(A_{-1})$, and will show, ultimately, that Y coincides with $\mathrm{Im}(A_{-1})$. The key technique – which is in the spirit of [35] – will be a comparison of 'automorphic exponents' with local v-exponents provided by Proposition 6.9.

First we show that, at each place v, the obstruction to H_v-invariance of the map B_{-1} and hence of D is carried by a certain submodule X_v of $I^{G_v}_{P_{1,v}}(1)$ (Proposition 6.5). It will turn out that, for a place v for which $\chi_v \neq 1$, the 'local v-exponents' of X_v are incompatible with the automorphic exponents of B_{-1} (proof of Proposition 6.14), and this forces such an X_v to lie in the kernel of $\bar{B}_{-1}$. Hence we recover H_v-invariance at such a place (Corollary 6.15). But then the restriction of $\bar{D}$ to $S(V_v^2)$ must factor through $R_v(V)$ and, by singularity, through $R_v(V_0)$ – cf. (6.45). Thus, as a representation of G_v, the space Y is an extension of multiples of $R_v(V_0)$ and of $R_v(V_0')$ where V_0' is the other binary quadratic space with character χ_v.

Next, comparing automorphic exponents of Y with its local v-exponents and applying the standard criterion for square integrability, we conclude that Y lies in L^2, and hence is completely reducible. In particular, each irreducible constituent π of Y must have local component $\pi_v \simeq R_v(V_0)$ or $R_v(V_0')$.

Finally we show, again by relating automorphic exponents of π to the local exponents at an arbitrary place, that $\pi \simeq \Pi(V_{00})$ for some binary space V_{00} with character χ_V. The uniqueness Theorem 2.2 and a consideration of Fourier coefficients then completes the proof.

In fact, we must overcome the additional difficulty that it is necessary to consider the whole family of Eisenstein series associated to the global induced representation $I^G_{P_1}(s)$. To do this, we must reconsider the intertwining operator

$F(s)$ defined in Sect. 3. Recall that the Levi factor M_1 of P_1 is isomorphic to $\mathrm{GL}(1)\times \mathrm{Sp}(1)$. We will denote the second factor by g_1, and we let B_1 denote its Borel subgroup obtained by intersecting with our fixed Borel of G.

First, for any place v, let $R_v^{(1)}(V_0)$ denote the image of $S(V_{0,v})$ in the induced representation $I_{B_{1,v}}^{G_{1,v}}(\chi_v)$. If $\chi_v=1$, $R_v^{(1)}(V_0)$ is the full (irreducible) induced representation, while, if $\chi_v\neq 1$, $R_v^{(1)}(V_0)$ is one of the two irreducible summands of this induced representation. Note that $I_{B_{1,v}}^{G_{1,v}}(\chi_v)$ is unitarizable.

Now

$$\Pi^{(1)}(V_0)\simeq \bigotimes_v R_v^{(1)}(V_0), \tag{6.7}$$

and hence,

$$I_{P_1}^{G}(s)\simeq \bigotimes_v I_{P_{1,v}}^{G_v}(s), \tag{6.8}$$

where

$$I_{P_{1,v}}^{G_v}(s)=\mathrm{Ind}_{P_{1,v}}^{G_v}(\chi|\ |^s\otimes R_v^{(1)}(V_0)). \tag{6.9}$$

Under the isomorphism (6.8), we have

$$F(s)=\bigotimes_v F_v(s) \tag{6.10}$$

with $F_v(s)\colon S(V_v^2)\to I_{P_{1,v}}^{G_v}(s)$ given by

$$F_v(g,s,\varphi)=\int_{F_v^\times} \omega(g)\,\hat{\varphi}(0,0;(0,0,a,0))|a|^{s+1}\,d^\times a. \tag{6.11}$$

Recall that we are interested in the value $s=1$. We define an intertwining operator

$$T_v: I_{P_{1,v}}^{G_v}(1)\to I_v(1/2) \tag{6.12}$$

by

$$T_v(\Psi)(g)=\int_{F_v} \Psi\left(m\left(\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\right)g,1\right)du. \tag{6.13}$$

Note that this integral converges absolutely for all Ψ. It is then easy to verify that

Lemma 6.4 *For all v, the composition $T_v\circ F_v(1)\colon S(V_v^2)\to I_v(1/2)$ is given by*

$$(T_v\circ F_v(1))(\varphi)(g)=\omega(g)\,\varphi(0).$$

Proof. We calculate the composition, setting $g=e$ since both maps are intertwining:

$$(6.14)\quad \begin{aligned} T_v(F_v(e)(\varphi)) &= \int_{F_v}\int_{F_v^\times} \omega\left(m\left(\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\right)\right)\hat{\varphi}(0,0;(0,0,a,0))|a|^2\, d^\times a\, du \\ &= \int_{F_v}\int_{F_v^\times} \hat{\varphi}(0,0;(0,0,a,-au))|a|^2\, d^\times a\, du \\ &= \int_{F_v}\int_{F_v^\times} \hat{\varphi}(0,0;(0,0,a,-u))|a|\, d^\times a\, du \\ &= \varphi(0). \end{aligned}$$

Here we use Fourier inversion, recalling that by (3.21)

$$(6.15)\quad \hat{\varphi}(0,0;(0,0,x_1,x_2)) = \int_{F_v^2} \varphi\left(\begin{pmatrix}0 & 0\\ 0 & 0\\ z_1 & z_2\end{pmatrix}\right)\psi(z_1x_1+z_2x_2)\,dz_1\,dz_2.$$

Using this result we obtain

Proposition 6.5 *Let $X_v=\ker(T_v)$. Then*
(i) *The sequence*

$$0\to X_v\to I_{P_{1,v}}^{G_v}(1)\xrightarrow{T_v} R_v(V)\to 0$$

is exact. In particular, the image of T_v is $R_v(V)$.
(ii) *The map*

$$S(V_v^2)\to I_{P_{1,v}}^{G_v}(1)/X_v\simeq R_v(V)$$

induced by $F_v(1)$ is H_v invariant.
(iii) *$F_v(1)$ is surjective.*

Proof. Lemma 6.4 implies that $\mathrm{Im}(T_v)\supset R_v(V)$. If $\chi_v=1$ then T_v is surjective by Propositions 1.1 and 1.2. If $\chi_v\neq 1$, then $\mathrm{Im}(T_v)$ must be either $R_v(V)$ or $I_v(\frac{1}{2})$, by the maximality of $R_v(V)$. Since the induction data $R_v^{(1)}(V_0)$ is tempered, $I_{P_{1,v}}^{G_v}(1)$ has a unique Langlands quotient, and thus $\mathrm{Im}(T_v)$ must also have a unique quotient. But $I_v(\frac{1}{2})$ has two irreducible quotients by Propositions 1.1 and 1.2, and thus $\mathrm{Im}(T_v)=R_v(V)$. This proves (i). Part (ii) follows immediately. Finally, $I_{P_{1,v}}^{G_v}(1)$ is generated by any vector whose image in the Langlands quotient is non-zero. By Lemma 6.4, the Langlands quotient must be isomorphic to $R_v(V_0)$, onto which the image of $F_v(1)$ surjects. This proves (iii). □

Corollary 6.6 (i) $F(1):S(V(\mathbb{A})^2)\to I_{P_1}^G(1)$ *is surjective.*
(ii) *In a neighborhood of $s=1$ the analytic behavior of the family of Eisenstein series formed from arbitrary holomorphic sections of $I_{P_1}^G(s)$ is the same as that of the family of Eisenstein series formed from sections of the form $F(g,s,\varphi)$ for $\varphi\in S(V(\mathbb{A})^2)$.*

We will need some facts about exponents and square integrability.

For a moment, let G be an arbitrary connected reductive group over F, let $K=K_\infty\cdot K_f$ be a maximal compact subgroup of $G(\mathbb{A})$, and, following the conventions and notation of [4], let $\mathscr{H}=\mathscr{H}_\infty\otimes\mathscr{H}_f$ be the global Hecke algebra

of $G(\mathbb{A})$. Let $X \subset \mathscr{A}(G)$ be a $\mathscr{H}$ invariant subspace of the space of automorphic forms on $G(\mathbb{A})$. Let $P \subset G$ be a parabolic subgroup with a Levi decomposition $P = MN$ and let A be the split component of M. Let $\mathfrak{a}_P = X(A) \bigotimes_{\mathbb{Z}} \mathbb{C}$ be its complexified Lie algebra, as in [14, p. 183]. Also as in [14] we let A_∞^+ denote the subgroup of A_∞ consisting of elements such that for every $\alpha \in X(A)$ and for every archimedean place v the value $\alpha(a_v) \in \mathbb{R}_+^\times$ and is independent of v. Then $M(\mathbb{A}) = A_\infty^+ \cdot M(\mathbb{A})^0$ where $M(\mathbb{A})^0$ is the set of all $m \in M(\mathbb{A})$ such that $|\alpha(m)|_{\mathbb{A}} = 1$ for all $\alpha \in X(M)$. Then, as in [14, (9), p. 186], if $f \in X$, we write the constant term of f along N as

$$f_P(g) = \sum_i P_i(H_P(a)) \cdot a^{\mu_i + \rho_P} [\sum_j \varphi_{i,j}(m) f_{i,j}(k)], \tag{6.16}$$

where $g = namk$ with $n \in N(\mathbb{A})$, $a \in A_\infty^+$, $m \in M^0(\mathbb{A})$ and $k \in K$. Here the $\varphi_{i,j}$ are automorphic forms on $M^0(\mathbb{A})$ and the f_{ij} are K-finite functions. In particular, there exists a compact open subgroup $K_f' \subset K_f$ such that the functions $f_{i,j}$ are all bi-K_f'-invariant. We call the μ_i's the automorphic exponents of f along P and we call the set of all μ_i's which occur as we vary $f \in X$ the *automorphic exponents* of X along P.

On the other hand, for any non-archimedean place v, we may consider the Jacquet functor associated to N_v applied to X. Note that by Proposition 4.5 of [4], for any function $f \in X$ the $\mathscr{H}_v$-submodule $f * \mathscr{H}_v$ generated by f is admissible and hence the Jacquet module $(f * \mathscr{H}_v)_{N_v}$ is a finitely generated admissible M_v-module, hence has finite length. In particular, there is a finite set of characters ξ of A_v which occur as restrictions to A_v of the central characters of the irreducible components of the semi-simplification of $(f * \mathscr{H}_v)_{N_v}$ as an M_v-module. Note that we are taking the normalized Jacquet functor here so that the characters which occur in the natural action of A_v in the quotient $(f * \mathscr{H}_v)/N_v(f * \mathscr{H}_v)$ have the form $\xi \cdot |\ |^{\rho_P}$. For any character ξ of A_v as above, we write ξ in the form

$$\xi(a) = |\xi(a)| \cdot \xi'(a) = a^{v + \rho_P} \cdot \xi'(a). \tag{6.17}$$

We call the resulting $\{v\}$ as we vary ξ the set of local v-exponents of f along P, and we call the union of these sets as f varies in X the *local v-exponents* of X along P. Note that by definition these v's are real.

Suppose that $f \in X$ is concentrated along P as in [14]. The square integrability criterion [14] asserts that f is square integrable provided all of its automorphic exponents μ_i along P satisfy $\mathrm{Re}(\mu_i) \in -\mathscr{C}(\mathfrak{a}_P)$. Here $\mathscr{C}(\mathfrak{a}_P)$ is defined as in [14, p. 187].

Note that in our present case, taking $P = B$, our fixed Borel subgroup, we have $M = A$ and

$$M(\mathbb{A})^0 = A(\mathbb{A})^0 \simeq F_{\mathbb{A}}^1 \times F_{\mathbb{A}}^1 \tag{6.18}$$

where $F_{\mathbb{A}}^1$ is the kernel of the norm map $|\ |_{\mathbb{A}} : F_{\mathbb{A}}^\times \to \mathbb{R}_+^\times$. Thus there are isomorphisms

$$F_{\mathbb{A}}^\times \simeq \mathbb{R}_+^\times \times F_{\mathbb{A}}^1 \tag{6.19}$$

(here we follow the convention of [14] and embed $\mathbb{R}_+^\times$ in $F_{\mathbb{A}}^\times$ as the set of ideles whose archimedean components are all equal to the positive d-th root of the given $t \in \mathbb{R}_+^\times$) and

$$A(\mathbb{A}) \simeq A_\infty^+ \times A(\mathbb{A})^0 \simeq (\mathbb{R}_+^\times)^2 \times (F_{\mathbb{A}}^1)^2. \tag{6.20}$$

In this case,

$$\mathcal{C}(\mathfrak{a}_B) \simeq \{r_1(1,-1)+r_2(0,2) \mid r_1, r_2 > 0\}. \tag{6.21}$$

Also observe that the $\varphi_{i,j}$'s in (6.16) are products of characters of $F^\times \backslash F_{\mathbb{A}}^1$.

Applying the square integrability criterion we have:

Lemma 6.7 *Assume that $r=1$. Then*

$$\mathrm{Im}(A_{-1}) \subset L^2(G(F)\backslash G(\mathbb{A})).$$

Proof. As remarked in [18], $E(g, s, \Phi)$ and, hence, all of its residues are concentrated on the Borel B of G, in the terminology of, say [14, 22]. We compute the constant term of $E(g, s, \Phi)$ along U, the unipotent radical of B in two steps. First, for N the unipotent radical of P, as usual, we have

$$E_P(g, s, \Phi) = \Phi(g, s) + E_P^1(g, s, \Phi) + M(s)\Phi, \tag{6.22}$$

as in [18, p. 14].

If we further compute the constant term along U, we find that there are four exponents:

$$(s-\tfrac{1}{2}, s+\tfrac{1}{2}), \quad (s-\tfrac{1}{2}, -s-\tfrac{1}{2}), \quad (-s-\tfrac{1}{2}, s-\tfrac{1}{2}), \text{ and } (-s-\tfrac{1}{2}, -s+\tfrac{1}{2}). \tag{6.23}$$

Since $\chi \neq 1$ in the case we are considering, we find that the terms corresponding to the first three of these exponents are holomorphic at $s=\frac{1}{2}$, while the fourth has a simple pole. Thus $\mathrm{Im}(A_{-1})$ has $(-1, 0)$ as its only exponent. Since $(-1, 0) \in -\mathcal{C}(\mathfrak{a}_B)$, it follows from the square integrability criterion that $\mathrm{Im}(A_{-1})$ lies in L^2 as claimed. □

Note that, by the same calculation, the possible exponents of A_0 are

$$(0, 1), \quad (0, -1), \quad (-1, 0), \quad (-1, 0), \tag{6.24}$$

and that, among these, only $(-1, 0)$ lies in $-\mathcal{C}(\mathfrak{a}_B)$.

Lemma 6.8 *If v is a non-archimedean place at which $V_{0,v}$ is anisotropic, then the set of local v-exponents of $\Pi(V_0)$ is $\{(-1, 0)\}$. In fact the only character ξ which occurs is*

$$\xi(a_1, a_2) = \chi_v(a_1 a_2)|a_1|_v^{-1}.$$

Proof. Recall that $\Pi_v(V_0) = R_v(V_0)$ is a quotient of $S(V_{0,v}^2)$. Also, since $V_{0,v}$ is anisotropic, the map

$$S(V_{0,v}) \to S(V_{0,v})_{U_v} \simeq \mathbb{C}$$

may be realized as evaluation at 0, i.e., as

$$\varphi_v \mapsto \varphi_v(0),$$

cf. Lemma 1.3. In particular, if $a=\mathrm{diag}(a_1, a_2)$ for $a_1, a_2 \in F_v^\times$, so that $m(a) \in A_v$, then

$$\omega(m(a))\,\varphi(0) = \chi(a_1 a_2)|a_1 a_2|_v\,\varphi(0),$$

and hence A_v acts in $\Pi_v(V_0)_{U_v}$ by the character $\chi_v \cdot |\ |_v$. Thus $\nu + \rho_B = (1, 1)$ and $\nu = (-1, 0)$, as claimed. □

The relation between automorphic and local v-exponents evident here is actually quite general. Returning to the general connected reductive group G:

Proposition 6.9 (i) *Suppose that X is a space of automorphic forms concentrated on P, as above. For every automorphic exponent μ of X along P, and for every non-archimedean place v, there exists a local v-exponent ν of X along P such that*

$$\mathrm{Re}(\mu) = \nu.$$

(ii) *For every automorphic exponent μ of X along P and for every archimedean place v, the generalized eigenspace $H_0(\mathfrak{n}_v, X)_\mu$ is non-zero. Here $\mathfrak{n}_v$ is the Lie algebra of N_v, the unipotent radical of P_v, and*

$$H_0(\mathfrak{n}_v, X) = X/\mathfrak{n}_v X$$

is the 0-th $\mathfrak{n}_v$ homology of X.

Proof. First fix a non-archimedean place v and an automorphic exponent μ of X. Choose $f \in X$, such that the exponent $\mu = \mu_i$ occurs in the expression (6.16) for f_P. Note that, by replacing f by some right translate if necessary, we may assume that the coefficient $\sum_j \varphi_{i,j} f_{i,j}$ of $a^{\mu+\rho_P}$ is non-zero at some point $m \cdot k$ where $m_v = 1$ and $k_v = 1$.

Fix an isomorphism $A \simeq \mathbb{G}_m^r$ where r is the split rank of P. Let h be the narrow class number of F and fix $\alpha \in F^\times$ such that α is totally positive and $(\alpha) = \mathfrak{P}_v^h$ where $\mathfrak{P}_v$ is the prime ideal associated to v. Write $\alpha = \alpha_\infty \cdot \alpha_f \in \mathbb{A}^\times = \mathbb{A}_\infty^\times \times \mathbb{A}_f^\times$. Let $\varpi_v \in F_v^\times$ be an element with $\mathrm{ord}_v\, \varpi_v = 1$, and let

$$\alpha_0 = \alpha_f \cdot \varpi_v^{-h} \tag{6.25}$$

where we view ϖ_v as an idèle with components equal to 1 away from v. Thus we have

$$\alpha^{-1} \cdot \alpha_0 \cdot \alpha_\infty = \varpi_v^{-h}. \tag{6.26}$$

Now for $m_1, \ldots, m_r \in \mathbb{Z}$, let

$$\begin{aligned} a_v &= (\varpi_v^{-hm_1}, \ldots, \omega_v^{-hm_r}) \\ \gamma &= (\alpha^{-m_1}, \ldots, \alpha^{-m_r}) \in A(F) \\ a_0 &= (\alpha_0^{m_1}, \ldots, \alpha_0^{m_1}) \in A_f \cap K_f \end{aligned}$$

and

$$a_\infty = (\alpha_\infty^{m_1}, \ldots, \alpha_\infty^{m_r}). \tag{6.27}$$

Finally, let $q_v = N(\mathscr{P}_v)$ and let α_∞^+ be the idele with component $q_v^{h/d}$, $d = |F:\mathbb{Q}|$, at each archimedean place and 1 elsewhere, and write

$$\alpha_\infty = \alpha_\infty^+ \cdot \alpha_\infty^1. \tag{6.28}$$

Then we can write

$$a_\infty = a_\infty^+ \cdot a_\infty^1, \tag{6.29}$$

with

$$a_\infty^+ = ((\alpha_\infty^+)^{m_1}, \ldots, (\alpha_\infty^+)^{m_r}) \in A_\infty^+. \tag{6.30}$$

Then

$$a_v = \gamma \cdot a_\infty^+ \cdot a_\infty^1 \cdot a_0. \tag{6.31}$$

As noted above the functions $f_{i,j}$ which occur in the expression for f_P above are bi-K_f'-invariant for a certain compact open subgroup K_f'. Thus there exists an integer $m_0 \geqq 0$ such that for $m_1, \ldots, m_r \equiv (m_0)$, we have $a_0 \in A_f \cap K_f'$. We will assume, for the moment, that this condition holds.

Let

$$G(\mathbb{A})^v = P_v \cdot \prod_{w \neq v}{}' G_w, \tag{6.32}$$

and let f_P^v denote the restriction of f_P to $G(\mathbb{A})^v$. Note that, by our choice of f, the coefficient of $a^{\mu_i + \rho_P}$ in the expression (6.16) for f_P^v does not vanish identically.

Let m and k be as above, i.e., such that the coefficient of $a^{\mu+\rho_P}$ in f_P^v is non-zero at the point $m \cdot k$ and for any $a' \in A_\infty^+$, consider the map which sends $\phi \in f * \mathscr{H}_v$ to $\phi_P^v(a' m \cdot k)$. This map actually factors through the projection $f * \mathscr{H}_v \to (f * \mathscr{H}_v)_{N_v}$, and thus defines a linear functional $\xi_{a'}$ on this Jacquet module. Let $\phi \to \bar{\phi}$ denote the projection $f * \mathscr{H}_v \to (f * \mathscr{H}_v)_{N_v}$. Then, for a_v as above, and for $m \in M(\mathbb{A})^0$ and $k \in K$ with $m_v = 1$ and $k_v = 1$,

$$\begin{aligned}
\xi_{a'}(r(a_v)\bar{f}) &= (r(a_v)f)_P^v(a' m k) \\
&= f_P(a' a_v m k) \\
&= f_P(a' \gamma a_\infty^+ a_\infty^1 m a_0 k) \\
&= f_P((a' a_\infty^+)(a_\infty^1 m) k), \\
&= \sum_i P_i(H_P(a' a_\infty^+))(a' a_\infty^+)^{\mu_i + \rho_P} \sum_j \varphi_{ij}(a_\infty^1 m) f_{ij}(k)
\end{aligned}$$

by the various conditions which we have imposed. Note that the quantity $\xi_{a'}(r(a_v)\bar{f}) = \langle r(a_v)\bar{f}, \xi_{a'} \rangle$ may be viewed as the value at a_v of a matrix coefficient of $(f * \mathscr{H}_v)_{N_v}$. By varying the choice of a' and via independence of characters of A_∞^+, we may separate terms in $\xi(r(a_v)\bar{f})$ with different automorphic exponents. On the other hand, by considering the dependence on a_v and hence on a_∞^+, and comparing possible exponents in $(f * \mathscr{H}_v)_{N_v}$ with those of f_P^v, we find that, if $\mathrm{Re}(\mu_i) = \mathrm{Re}(\mu) = (x_1, \ldots, x_r)$,

$$(a_\infty^+)^{\mu + \rho_P} = q_v^{h(m_1 x_1 + \ldots + m_r x_r + \rho_P \cdot m)} \tag{6.33}$$

must be equal to

$$|\varpi|_v^{-h(m_1y_1+\ldots+m_ry_r+\rho_P\cdot m)}=q_v^{h(m_1y_1+\ldots+m_ry_r+\rho_P\cdot m)} \tag{6.34}$$

where $v=(y_1, \ldots, y_r)$ is a local v-exponent of $(f*\mathscr{H}_v)_{N_v}$. Since this equality must hold for all sufficiently large $m_1, \ldots, m_r$, we conclude that $\mathrm{Re}(\mu)=v$, as claimed.

Next to prove (ii) we suppose that v is an archimedean place and again observe that the map $f\mapsto f_P^v(m\cdot k)$, for $m\cdot k\in G(\mathbb{A})^v$ as above, factors through the quotient $X/\mathfrak{n}_v X=H_0(\mathfrak{n}_v, X)$. Again taking a right translate of f if necessary, we may assume that the term with exponent μ in f_P^v is non-zero for some $f\in X$. Since

$$H_0(\mathfrak{n}_v, X)=\bigotimes_v H_0(\mathfrak{n}_v, X)_v \tag{6.35}$$

as v runs over $\mathfrak{a}_v^*$ [12, (2.29), p. 60], we conclude that for every automorphic exponent μ, $H_0(\mathfrak{n}_v, X)_\mu\neq 0$, as required. Note that we are again using the fact that for any $f\in X$ the Harish-Chandra module $f*U(\mathfrak{g})$ is admissible and finitely generated, so that [12, (2.29)] applies and the action of $\mathfrak{a}$ on the $\mathfrak{n}$-homology any such submodule and hence on that of all of X can be put in Jordan canonical form. □

We now return to $G=\mathrm{Sp}(2)$ etc., and consider the behavior of the family series $\mathscr{E}(g, s, \Psi)$ at $s=1$. Note that, because of the presence of the projection operator pr_{K_H} in (3.31), (3.37) does not give sufficient control. But, in fact, we have

Proposition 6.10 *For an arbitrary section $\Psi(s)\in I_{P_1}^G(s)$ which is holomorphic in a neighhorbood of $s=1$,*

$$\operatorname*{ord}_{s=1}\mathscr{E}(g, s, \Psi(s))\geqq -2.$$

Moreover, the possible exponents of $\mathscr{E}(g, s, \Psi)$ with respect to B are $(s, 0)$, $(-s, 0)$, $(0, s)$, *and* $(0, -s)$.

Proof. First note that $\mathscr{E}$ is concentrated on the Borel. We consider the P_1 constant term $\mathscr{E}_{P_1}(g, s, \Psi(s))$ and find, by the usual coset analysis, that it is the sum of three terms. The first of these is holomorphic at $s=1$ while the other two have at most a double and a simple pole there respectively. Moreover, if we further compute the constant terms with respect to the Borel, we obtain the exponents

$$(s, 0), \quad (-s, 0), \quad (0, s), \quad \text{and} \quad (0, -s), \tag{6.36}$$

and the terms involving these have order at least 0, -2, -1 and -1 respectively at $s=1$. We omit the details, since a more general calculation of this type will be contained in a forthcoming paper of the first two authors. □

Now for an arbitrary holomorphic section $\Psi(s)\in I_{P_1}^G(s)$ we write

$$\mathscr{E}(g, s, \Psi)=\frac{C_{-2}(g, \Psi)}{(s-1)^2}+\frac{C_{-1}(g, \Psi)}{(s-1)}+O(1), \tag{6.37}$$

and we note that by (3.37),

$$B_{-2}(g,\varphi)=C_{-2}(g,\Psi), \tag{6.38}$$

when $\Psi(g,s)=F(g,s,\mathrm{pr}_{K_H}(\varphi))$.

By the remarks about the exponents of $\mathscr{E}(g,s,\Psi)$ contained in the proof of Proposition 6.10, we have

Corollary 6.11 *The exponent of $C_{-2}(g,\Psi)$ is $(-1,0)$, and thus*

$$\mathrm{Im}(C_{-2})\subset\mathscr{A}_2(G),$$

the space of square integrable automorphic forms.

Remark. The possible exponents of $C_{-1}(g,\Psi)$ are $(0,1)$, $(0,-1)$, and $(-1,0)$. Thus this function need not be in L^2.

Corollary 6.12 (i) *The map*

$$I_{P_1}^G(1)\xrightarrow{C_{-2}}\mathscr{A}(G)$$

is G_v (resp. $(\mathfrak{g}_v,K_v)$) intertwining at a non-archimedean (resp. an archimedean) place v.

(ii)

$$\mathrm{Im}(C_{-2})=\mathrm{Im}(B_{-2})\subset\mathrm{Im}(A_{-1}).$$

(iii) *The map*

$$\bar{C}_{-1}: I_{P_1}^G(1)\to\mathscr{A}(G)/\mathrm{Im}(C_{-2})$$

induced by C_{-1} is G_v (resp. $(\mathfrak{g}_v,K_v)$) intertwining at a non-archimedean (resp. an archimedean) place v.

Proof. Part (i) is clear. For part (ii) we observe that since $\mathrm{Im}(C_{-2})\subset\mathscr{A}_2(G)$, this image is completely reducible. Since, for each place v, $I_{P_{1,v}}^{G_v}(1)$ has a unique irreducible quotient $R_v(V_0)$, we must have

$$\mathrm{Im}(C_{-2})\simeq\bigotimes_v R_v(V_0)=\Pi(V_0). \tag{6.39}$$

Theorem 2.2 then yields (ii). Finally, (iii) is just the analogue of Proposition 6.3 in this situation. □

By Proposition 6.5, the exponents of $I_{P_{1,v}}^{G_v}(1)$ are given as follows:

Proposition 6.13 *If v is non-archimedean,*

(i) *Suppose that $\chi_v\neq 1$. Then the exponents of $R_v(V)$ are $(0,-1)$, $(-1,0)$ and $(0,1)$. Also the exponent of X_v is $(1,0)$. In particular, X_v is irreducible in this case.*

(ii) *Suppose that $\chi_v=1$. Then the exponents of $R_v(V)$ are $(0,-1)$, $(-1,0)$ (with multiplicity 2), and $(0,1)$. Also the exponents of X_v are $(1,0)$ (with multiplicity 2), $(0,1)$ and $(0,-1)$.*

Proof. In both cases (i) and (ii), the exponents of $R_v(V)$ are computed in [20]. On the other hand, in case (i) the inducing data on $M_{1,v}$ is one component of an induced representation from the Borel subgroup of this Levi factor, and

the two components are switched by the outer automorphism of Sp(1) given by conjugation by a matrix $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ where $\alpha \in F_v^\times$ is an element with $\chi_v(\alpha) = -1$. In turn, the induced representation $I_{P_{1,v}}^{G_v}(1)$ is 'half' of a full induced representation $I_{B_v}^{G_v}(\lambda, \chi_v)$ where $\lambda = (1,0)$, and the other 'half' is obtained by applying the outer automorphism given by conjugation by $\begin{pmatrix} 1_2 & 0 \\ 0 & \alpha \cdot 1_2 \end{pmatrix}$ on G_v. The full induced representation $I_{B_v}^{G_v}(\lambda, \chi_v)$ has 4 exponents $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, each with multiplicity 2 (this set is just the Weyl orbit of $(1,0)$, counted with multiplicity). Since the outer automorphism just described preserves exponents, the exponents of the two halves of the full induced must be the same and must occur with multiplicity one. Thus X_v must have the exponent $(1,0)$ not accounted for by $R_v(V)$, and hence must be irreducible. This finishes the proof in case (i). In case (ii), $R_v^{(1)}(V_0)$ is actually a full irreducible unitarizable induced representation of $M_{1,v}$ and so $I_{P_{1,v}}^{G_v}(1) = I_{B_v}^{G_v}(\lambda)$ has the full set of 4 exponents, each with multiplicity 2. Again, X_v must have the exponents not already accounted for by $R_v(V)$. □

For any fixed place v, let

$$I_{P_1}^G(1)[v] = X_v \otimes \Big(\bigotimes_{w \neq v} I_{P_{1,w}}^{G_w}(1)\Big), \tag{6.40}$$

be the subrepresentation of $I_{P_1}^G(1)$ whose v component is required to lie in X_v.

Proposition 6.14 *Suppose that v is non-archimedean place at which $\chi_v \neq 1$. Then $I_{P_1}^G(1)[v]$ lies in the kernel of the map $\bar{C}_{-1}$ defined in Corollary* 6.12.

Proof. Suppose that $\bar{C}_{-1}$ does not vanish identically on $I_{P_1}^G(1)[v]$, and let $Y[v] \subset \mathscr{A}(G)$ be the inverse image in $\mathscr{A}(G)$ of the image $\bar{C}_{-1}(I_{P_1}^G(1)[v]) \subset \mathscr{A}(G)/\mathrm{Im}(C_{-2})$. Note that the space $Y[v]$ is concentrated on the Borel. Since $Y[v]$ is thus an extension

$$0 \to \Pi(V_0) \to Y[v] \to \bar{C}_{-1}(I_{P_1}^G(1)[v]) \to 0, \tag{6.41}$$

the local v-exponents of $Y[v]$ lie in the set $\{(-1,0), (1,0)\}$, where $(-1,0)$ is the local v-exponent of $R_v(V_0)$ and $(1,0)$ is that of X_v. On the other hand, since $Y[v]$ is generated by functions in $\Pi(V_0)$ and functions in the image of C_{-1}, the automorphic exponents of $Y[v]$ must have real parts lying in the set $\{(-1,0), (0,1), (0,-1)\}$, by Proposition 6.10 and the proof of Lemma 6.7. Comparing these and applying Proposition 6.9, we conclude that all automorphic exponents of $Y[v]$ must have real part $(-1,0)$. Thus, by the square integrability criterion, $Y[v] \subset \mathscr{A}_{(2)}(G)$, and hence is completely reducible as a representation of $G(\mathbb{A})$. In particular, the extension (6.41) must split, and we must have a non-zero intertwining map from $I_{P_1}^G(1)[v]$ to $\mathscr{A}_{(2)}(G)$. Any irreducible constituent of the image would then have to have X_v as its local component at v. But, since the local v-exponent of this constituent would then have to be $(1,0)$, square integrability, together with Proposition 6.9 leads to a contradiction. Thus C_{-1} must vanish identically on $I_{P_1}^G(1)[v]$. □

Corollary 6.15 (i) *For every non-archimedean place v at which $\chi_v \neq 1$, $\bar{C}_{-1}$ is H_v invariant.*

(ii) *For every such* v,

$$\bar{B}_{-1}\colon\ S(V(\mathbb{A})^2)\to\mathscr{A}(G)/\mathrm{Im}(A_{-1})$$

is H_v *invariant.*

Proof. Part (i) follows from Proposition 6.14 together with (ii) of Proposition 6.5. Then (ii) results from the relation

$$B_{-1}(g,\varphi)=C_{-1}(g,\Psi) \tag{6.42}$$

with $\Psi(g,s)=F(g,s,\mathrm{pr}_{K_H}(\varphi))$, and the fact that $\mathrm{Im}(C_{-2})\subset\mathrm{Im}(A_{-1})$. □

At this point we return to the map $\bar{D}$ defined in Proposition 6.3 above, and note that it too is H_v-invariant for each non-archimedean place v at which $\chi_v\neq 1$. Fix one such place v chosen so that χ_v is unramified. Then, setting

$$S^v=\bigotimes_{w\neq v}{}' S(V_w^2), \tag{6.43}$$

the map

$$\bar{D}: S(V_v^2)\otimes S^v\to\mathscr{A}(G)/\mathrm{Im}(A_{-1}) \tag{6.44}$$

factors through $R_v(V)\otimes S^v$. Moreover, since $\bar{D}(\varphi)_\beta=0$ for all $\beta\notin\mathcal{O}_{V_0}$ with $\det\beta\neq 0$, $\bar{D}$ must further factor through the quotient $R_v(V_0)\otimes S^v$, i.e., we may write $\bar{D}$ as the composition of the projection $S(V_v^2)\otimes S^v\to R_v(V_0)\otimes S^v$ with a map

$$\bar{\bar{D}}: R_v(V_0)\otimes S^v\to\mathscr{A}(G)/\mathrm{Im}(A_{-1}). \tag{6.45}$$

Let Y denote the inverse image in $\mathscr{A}(G)$ of the image of $\bar{D}$ in $\mathscr{A}(G)/\mathrm{Im}(A_{-1})$, and note that, as a representation of G_v, Y is an extension of multiples of $R_v(V_0)$ and $R_v(V_0')$. Thus Y has $(-1,0)$ as its only local v-exponent along B. By Proposition 6.9, we must have $Y\subset\mathscr{A}_{(2)}(G)$ so that Y is completely reducible as a representation of $G(\mathbb{A})$. Moreover, every irreducible constituent π of Y must have local component $R_v(V_0)$ or $R_v(V_0')$ at v. In particular, every automorphic exponent μ of π along B must have $\mathrm{Re}(\mu)=(-1,0)$, again by Proposition 6.9.

Now let π be any irreducible constituent of Y and (changing notation from above) fix a place v. Write $\pi=\pi_v\otimes\pi^v$ where $\pi^v=\bigotimes_{w\neq v}\pi_w$.

First assume that v is non-archimedean. Then, since π is concentrated on B, there is some $f\in\pi$ such that $f_B\neq 0$, and if μ is any automorphic exponent of π along B, Proposition 6.9 implies that $(-1,0)=\mathrm{Re}(\mu)=\nu$ for some local v-exponent of π_v along B. But, by the original construction of Y, π_v is either a constituent of $I_{P_{1,v}}^{G_v}(1)$ or is isomorphic to $R_v(V_0')$. The later case can only occur if $\chi_v\neq 0$. By Proposition 6.13 the only such constituent which has an exponent $(-1,0)$ is $R_v(V_0)$. Thus $\pi_v\simeq R_v(V_0)$ or $R_v(V_0')$.

Next suppose that v is an archimedean place. By (ii) of Proposition 6.9 we conclude that for every automorphic exponent μ of π along B

$$H_0(\mathfrak{u}_v,\pi)_\mu\neq 0.$$

But, since π_v is a constituent of $I_{P_{1,v}}^{G_v}(1)$, it has infinitesimal character $\chi_{(1,0)}$ in the usual Harish-Chandra notation. By [12, Corollary 3.32, p. 68] the only possible v's for which $H_0(\mathfrak{u}_v, \pi)_v \neq 0$ must lie in the set

$$\{(-1,0),\quad (0,-1),\quad (0,1),\quad (1,0)\}, \tag{6.46}$$

i.e., in the Weyl orbit of the infinitesimal character. Thus, recalling the fact noted above that every automorphic exponent of π must have $\mathrm{Re}(\mu)=(-1,0)$, we conclude that $\mu=(-1,0)$. Finally, the representation $R_v(V_0)$ is the Langlands quotient of $I_{P_{1,v}}^{G_v}(1)$, and hence, by [12, (8.48), p. 140] it is only the constituent with $(-1,0)$ as a *leading* exponent. Since $(-1,0)$ is the minimal element in the set (6.46) with respect to the ordering in question [12, (2.31), p. 60], and since all of the leading exponents of π_v must occur in $H_0(\mathfrak{u}_v, \pi_v)$, i.e., must be v's with $H_0(\mathfrak{u}_v, \pi_v)_v \neq 0$ [12, Theorem 4.25, p. 84], we conclude that $(-1,0)$ is the unique leading exponent of π_v and hence that $\pi_v \simeq R_v(V_0)$ or $R_v(V_0')$ when $\chi_v \neq 1$. Thus we have proved that every irreducible constituent of Y is isomorphic to $\Pi(V_{00})$ for some binary quadratic space V_{00} with $\chi_{V_{00}}=\chi_V$. Applying the multiplicity one result of Theorem 2.2, we obtain

$$Y=\mathrm{Im}(A_{-1})=\oplus\, \Pi(V_{00}). \tag{6.47}$$

But this implies that

$$\bar{D}=0.$$

Thus, for every $\varphi \in S(V(\mathbb{A})^2)$ the function

$$D(g,\varphi)=A_0(g,\Phi)-c'\cdot B_{-1}(g,\varphi)$$

lies in the space $\oplus\, \Pi(V_{00})$. On the other hand, since $D_\beta(g,\varphi)=0$ for $\beta \notin \mathcal{O}_{V_0}$ with $\det \beta \neq 0$, we conclude that this function must actually lie in $\Pi(V_0)$. Hence there exists a $\varphi' \in S(V(\mathbb{A})^2)$ such that

$$D(g,\varphi)=A_0(g,\Phi)-c'\cdot B_{-1}(g,\varphi)=B_{-2}(g,\varphi'). \tag{6.48}$$

This completes the proof of Theorem 6.1 in the case $r=1$ and $\chi \neq 1$.

6.2

We now turn to the case in which $r=2$ so that V and V_0 are split and $\chi_V=1$. The series $\mathscr{E}(g,s,\varphi)$ is associated to the Siegel parabolic in this case and is, in fact, closely related to $E(g,s,\Phi)$. Thus a more direct proof of Theorem 6.1 will be possible in this case.

For $\varphi \in S(V(\mathbb{A})^2)$ and for $\hat{\varphi}$ as in §3.2, let

$$\begin{aligned} F(g,s,\varphi) &= \int_{\mathrm{GL}(2,\mathbb{A})} \omega(g)\,\mathrm{pr}(\hat{\varphi})((0,{}^t a))|a|^{s+(3/2)}\,da \\ &= \int_{\mathrm{GL}(2,\mathbb{A})} \mathrm{pr}(\hat{\varphi})((0,{}^t a)\,g)|a|^{s+(3/2)}\,da, \end{aligned} \tag{6.49}$$

where

$$\mathrm{pr}(\hat{\varphi}) = \int_L \omega(k)\,\hat{\varphi}\,dk \tag{6.50}$$

is the projection of $\hat{\varphi}$ to the space of L-invariants. It is then not difficult to check that $F(s,\varphi)\in I(s)$, and that for $\mathrm{Re}(s)>\frac{3}{2}$

$$\mathscr{E}(g,s,\varphi) = \sum_{\gamma\in P(F)\backslash G(F)} F(\gamma g,s,\varphi). \tag{6.51}$$

If $\varphi = \bigotimes_v \varphi_v$, then

$$F(s,\varphi) = \bigotimes_v F_v(s,\varphi_v) \tag{6.52}$$

where

$$F_v(g,s,\varphi_v) = \int_{\mathrm{GL}(2,F_v)} \mathrm{pr}_v(\hat{\varphi}_v)((0,{}^t a)\,g)|a|^{s+(3/2)}\,da. \tag{6.53}$$

Moreover, if $\hat{\varphi}_v = \hat{\varphi}_v^0$ is the characteristic function of $M_{2,4}(\mathcal{O}_v)$ and if $\varphi_v^0\in S(V_v^2)$ corresponds to $\hat{\varphi}_v^0$, then

$$F_v(g,s,\varphi_v^0) = L_v(s+\tfrac{3}{2})\,L_v(s+\tfrac{1}{2})\cdot\Phi_v^0(g,s) \tag{6.54}$$

where $\Phi_v^0(s)\in I_v(s)$ is the K_v-invariant vector with $\Phi_v^0(e)=1$. Thus we may write

$$F(s,\varphi) = L(s+\tfrac{3}{2})\,L(s+\tfrac{1}{2})\Big[\bigotimes_{v\in S} F_v^*(s,\varphi_v)\Big]\otimes\Phi_S^0(s) \tag{6.55}$$

where

$$F_v^*(s,\varphi_v) = \frac{1}{L_v(s+\frac{3}{2})\,L_v(s+\frac{1}{2})}\,F_v(s,\varphi_v). \tag{6.56}$$

Note that the integral defining $F_v(s,\varphi_v)$ is absolutely convergent for $\mathrm{Re}(s)>-\frac{1}{2}$. Moreover, it is not difficult to check that the map

$$\varphi_v\mapsto F_v(\tfrac{1}{2},\varphi_v) \tag{6.57}$$

is H_v-invariant, and hence factors through $R_v(V)$. Since V is split, the module $R_v(V)$ is cyclic, generated by the image of φ_v^0 where φ_v^0 is defined as above for v non-archimedean and is the Gaussian defined in §1 if v is archimedean. Since the map (6.53) is G_v (resp. $(\mathfrak{g}_v, K_v)$) intertwining if v is non-archimedean (resp. archimedean), we conclude that, if

$$\Phi_v(g) = \omega(g)\,\varphi_v(0), \tag{6.58}$$

as usual, then

$$F_v^*(g,\tfrac{1}{2},\varphi_v) = c_v\,\Phi_v(g), \tag{6.59}$$

for all $\varphi_v\in S(V_v^2)$. Here $c_v=1$ for almost all v, and $c_v\neq 0$ in general.

We now write

$$F_v^*(s,\varphi_v)=\sum_i \alpha_{i,v}(s)\,\Psi_v^i(s) \tag{6.60}$$

where the $\Psi_v^i(s)\in I_v(s)$ are standard sections and the $\alpha_{i,v}(s)$ are meromorphic functions with no poles in the half plane $\mathrm{Re}(s)>-\frac{1}{2}$. Note that since $\chi_v=1$ and V_v is split we have $I_v(\frac{1}{2})=R_v(V)$ for all places v. Thus the $\Psi_v^i(\frac{1}{2})$'s all lie in $R_v(V)$.

In particular, we have

$$F_v^*(s,\varphi_v)=c_v\,\Phi_v(s)+\sum_i(\alpha_{i,v}(s)-\alpha_{i,v}(\tfrac{1}{2}))\,\Psi_v^i(s). \tag{6.61}$$

Substituting these expressions into (6.55) and expanding, we have

$$F(s,\varphi)=L(s+\tfrac{3}{2})\,L(s+\tfrac{1}{2})[c_S\,\Phi(s)+\sum_i \beta_i(s)\,\Psi^i(s)] \tag{6.62}$$

where $c_S=\prod_{v\in S} c_v$ and

$$\operatorname*{ord}_{1/2}\beta_i(s)\geqq|T_i|\geqq 1 \tag{6.63}$$

with $T_i=\{v\,|\,\Psi_v^i(s)\neq\Phi_v(s)\}$. Thus we obtain

$$\mathscr{E}(g,s,\varphi)=L(s+\tfrac{3}{2})\,L(s+\tfrac{1}{2})[c_S\,E(g,s,\Phi)+\sum_i \beta_i(s)\,E(g,s,\Psi^i)]. \tag{6.64}$$

Writing

$$L(s)=\frac{\eta}{s-1}+\eta'+O(s-1) \tag{6.65}$$

we conclude that

$$B_{-2}(g,\varphi)=c_S\,L(2)\,A_{-1}(g,\Phi) \tag{6.66}$$

and

$$\begin{aligned}B_{-1}(g,\varphi)=&L(2)\,\eta\,c_S\,A_0(g,\Phi)+(L'(2)\,\eta+L(2)\,\eta')\,c_S\,A_{-1}(g,\Phi)\\&+L(2)\eta\sum_i \beta_i'(\tfrac{1}{2})\,A_{-1}(g,\Psi_i).\end{aligned} \tag{6.67}$$

But now we may find $\varphi'\in S(V(\mathbb{A})^2)$ whose image in $I(s)$ is C times the standard section

$$c_S^{-1}\sum_i \beta_i'(\tfrac{1}{2})\,\Psi_i(s)+(L(2)\,\eta\,c_S)^{-1}(L'(2)\eta+L(2)\eta')\,\Phi(s), \tag{6.68}$$

where C is as in Corollary 5.4, and thus

$$A_0(g,\Phi)=(L(2)\,\eta\,c_S)^{-1}\cdot B_{-1}(g,\varphi)+B_{-2}(g,\varphi') \tag{6.69}$$

as claimed. In particular, $C'=L(2)\,\eta\,c_S$ in the split case. □

Note that the function φ' depends on φ in a rather complicated way via (6.68) and that the constants of proportionality involve the first two terms in the expansion of $L(s)$ at the points $s=1$ and $s=2$.

7 Applications to poles and values of L-functions

In this section we apply the results of the first six sections to give an interpretation of the pole or value at the point $s=1$ of the degree 5 Langlands L-function associated to a cusp form on G. This interpretation is based on the Andrianov integral representation of this L-function as discussed in [26].

First note that for $G=\mathrm{Sp}(2)$ over F, ${}^LG^0=\mathrm{SO}(5,\mathbb{C})$ and ${}^LG={}^LG^0\times\Gamma$, where $\Gamma=\mathrm{Gal}(\bar{F}/F)$. Let $r:{}^LG\to\mathrm{GL}(5,\mathbb{C})$ be the standard representation, trivial on Γ.

Let π be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$, and let $S(\pi)$ be the set of all archimedean places together with the set of all non-archimedean places at which π_v does not have a K_v fixed vector. Choose a cusp form f in the space of $\pi=\bigotimes_v \pi_v$ and assume that f corresponds to a factorizable vector $\bigotimes_v f_v$ and that f_v is K_v-invariant for all $v\notin S(\pi)$. Let $T\in\mathrm{Sym}_2(F)$ be non-singular and assume that the T-th Fourier coefficient f_T of f is non-zero, as it must be for at least one T [24]. Here we are using the convention of [26] so that

$$f_T(g)=\int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\,\psi(\mathrm{tr}(T\cdot b))\,db.$$

Let V_T be the binary quadratic space defined by T. For a Schwartz function $\phi\in S(V_T(\mathbb{A})^2)$ let

$$\theta^T(g,\phi)=\sum_{x\in V_T(F)^2}\omega(g)\phi(x)$$

be the corresponding theta function. For a quadratic character χ let $\Psi(s)\in I(s,\chi)$ be a standard factorizable section and let $E(g,s,\Psi)$ be the Eisenstein series, as in Sect. 1. Let $S=S(\pi,T,\chi)$ be the finite set of places consisting of

(i) all archimedean places and all places v such that $v|2$,

(ii) all finite places at which π_v, χ_v or $\chi_{T,v}$ are ramified or at which $\chi_v\neq 1$ and $V_{T,v}$ is not isomorphic to the norm form of the quadratic extension of F_v determined by χ_v,

(iii) all finite places at which ψ_v is not of order 1.

Note that the set S depends only on π, T, χ and our fixed ψ.

Now let

$$L_S(s,\chi,\pi)=\prod_{v\notin S}L_v(s,\chi_v,\pi_v),$$

be the twist by χ of the Langlands L-function associated to π and the standard representation r of ${}^LG$. Assume that for $v\notin S$, ϕ_v is the characteristic function of a standard maximal lattice in $V_{T,v}$ and Ψ_v in the normalized K_v fixed vector

in $I_v(s, \chi_v)$. Then the main result of [26] asserts that there is a meromorphic function $\Lambda_S(s)$ depending on the choices of f, Ψ, T, ϕ, etc., such that

$$(7.1) \qquad b_S(s) \int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)E(g,s,\Psi)\,dg = \Lambda_S(s)\,L_S(s+\tfrac{1}{2}, \chi_T\,\chi, \pi),$$

where

$$b_S(s) = \prod_{v \notin S} b_v(s).$$

with

$$b_v(s) = L_v(s+\tfrac{3}{2}, \chi_v)\,L_v(2s+1).$$

Moreover, given any s_0, there exists a suitable choice of data ϕ_v and Ψ_v for $v \in S$ such that $\Lambda_S(s_0) \neq 0$. Note that in [26] χ is trivial, but their argument immediately allows a twist by an arbitrary χ.

It is further proved in [26] that the first possible pole of $L_S(s, \chi_T, \pi)$ occurs at the point $s=2$ and that, if the residue at this point is non-zero, then π has a non-trivial lift to $O(V_T)$.

Our first main result is an analogue for the point $s=1$.

Theorem 7.1 *Let π be an irreducible automorphic cuspidal representation of* $\mathrm{Sp}(2, \mathbb{A})$ *and let $f \in \pi$ be an automorphic form as above which has a non-vanishing T-th Fourier coefficient for some non-singular T. Let S be as above.*

(i) *Then $L_S(s, \chi_T, \pi)$ has at most a simple pole at $s=1$, and if*

$$\operatorname*{Res}_{s=1} L_S(s, \chi_T, \pi) \neq 0,$$

then π has a non-trivial theta lift to $O(V)$ where

$$V = V_T + V_{1,1},$$

that is, the four dimensional quadratic space obtained by adding a hyperbolic plane to V_T.

(ii) *Moreover, if χ is a non-trivial quadratic character of $F^\times \backslash F_{\mathbb{A}}^\times$, and if*

$$\operatorname*{Res}_{s=1} L_S(s, \chi_T\,\chi, \pi) \neq 0,$$

then π has a non-trivial theta lift to $O(V')$ where

$$V' = V_T + V_{T'}$$

for some binary quadratic space $V_{T'}$ over F with character χ.

Proof. Suppose that $L_S(s, \chi_T\,\chi, \pi)$ has a non-zero residue at $s=1$. We choose ϕ_v and Ψ_v for $v \in S$ so that $\Lambda_S(\frac{1}{2}) \neq 0$, and we conclude from (7.1) that

$$(7.2) \qquad \operatorname*{Res}_{s=1} L_S(s, \chi\chi_T, \pi) = \frac{L(2,\chi)\,L(2)}{\Lambda_S(1/2)} \int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)\,A_{-1}(g,\Psi)\,dg.$$

In particular, the integral on the right side here does not vanish.

First suppose that $\chi=1$ and let V be the split space of dimension four. Then for all places v, $I_v(\frac{1}{2})=R_v(V)$ and so $\Psi=\bigotimes_v \Psi_v$ is a homogeneous standard section associated to V, in the terminology of §4. By Theorem 4.1, we have

$$A_{-1}(g,\Psi)=I(g,\varphi)$$

for some $\varphi\in S(V_0(\mathbb{A})^2)$, where V_0 is the split binary space. Using this identity, the integral in (7.2) can be written as

$$\begin{aligned}&\int_{G(F)\backslash G(A)} f(g)\,\theta^T(g,\phi)I(g,\varphi)\,dg\\ &\quad=-\tfrac{1}{4}\int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)\Big[\int_{H(F)\backslash H(\mathbb{A})}\theta(g,h;\omega(D)\varphi)\,dh\Big]\,dg\\ &\quad=-\tfrac{1}{4}\int_{H(F)\backslash H(\mathbb{A})}\Big[\int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)\theta(g,h;\omega(D)\varphi)\,dg\Big]\,dh,\end{aligned}$$

where $H=O(V_0)$. The inner intergral in this last expression is the integral of f against a theta kernel attached to the function

$$\phi\otimes\omega(h)\omega(D)\varphi\in S(V_T(\mathbb{A})^2)\otimes S(V_0(\mathbb{A})^2).$$

Thus we conclude that f has a non-zero θ-lift to the orthogonal group of the quadratic space $V_T+V_0=V_T+V_{1,1}$ as claimed in (i).

Remark. The theta lift of f actually has a non-zero integral over $H(F)\backslash H(\mathbb{A})$, where we view H as a subgroup of $O(V_T+V_{1,1})$.

Next suppose that $\chi\neq 1$, and let

$$\Psi=\sum_i \Psi^i, \tag{7.3}$$

where $\Psi^i=\bigotimes_v \Psi_v^i$ is a homogeneous standard section. Note that by (i) of Theorem 4.1 the incoherent Ψ^i's make no contribution to $A_{-1}(g,\Psi)$. For each i such that Ψ^i is coherent, let V^i be the corresponding global four-dimensional quadratic space. Note that by (ii) of Theorem 4.1 only the V^i's which are isotropic can contribute to $A_{-1}(g,\Psi)$. Let V_0^i be the associated anisotropic binary spaces. Then by (iii) of Theorem 4.1 we have

$$A_{-1}(g,\Psi)=\sum_i I(g,\varphi^i)$$

where i runs over the set of indices in (7.3) such that Ψ^i is coherent associated to an isotropic global space V^i and $\varphi^i\in S(V_0^i(\mathbb{A})^2)$. By the same argument as before, we conclude that at least one of the integrals

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)I(g,\varphi^i)\,dg$$

must be non-zero, and thus f has a non-zero theta lift to one of the spaces $V_T+V_0^i$. This proves (ii). □

Remark. It remains an open problem to determine which of the spaces $V_T + V_0^i$ actually allow a non-zero theta lift of f. Presumably an answer can be given in terms of the local components of π.

We now assume that for π, f, T, and χ as above, the L-function $L_S(s, \chi_T \chi, \pi)$ has no pole at $s=1$, and we would like to interpret its value. If $\chi \neq 1$ the method we are using would require sufficient information (i.e., an analogue of Theorem 6.1) about $A_0(g, \Psi)$ for an arbitrary section Ψ, including a possibly *incoherent* one. Thus, Theorem 6.1 is not sufficient.

We do have, however, the following somewhat surprising result:

Theorem 7.2 *Let π, f, and T, be as in Theorem* 7.1. *Then π has a non-zero theta lift to $O(U)$ where U is the six dimensional quadratic space*

$$U = V_T + V_{2,2}$$

with $V_{2,2}$ the split four dimensional quadratic space.

Proof. Take $\chi = 1$ and choose data Φ, ϕ, etc, as above so that the integral of (7.1) does not vanish identically, i.e.,

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)\, \theta^T(g, \phi) E(g, s, \Phi)\, dg \neq 0 \tag{7.4}$$

as a meromorphic function of s. Note that we may take $\Phi(s)$ to be a standard section associated to a function $\varphi \in S(V_{2,2}(\mathbb{A})^2)$. But now we may solve (6.64) for $E(g, s, \Phi)$:

$$E(g, s, \Phi) = c_S^{-1} L(s+\tfrac{3}{2})^{-1} L(s+\tfrac{1}{2})^{-1} \mathscr{E}(g, s, \varphi) - \sum_i \beta_i(s) E(g, s, \Psi^i). \tag{7.5}$$

Note that the meromorphic functions $\beta_i(s)$ all have at least a simple zero at $1/2$ and that the sections $\Psi^i(s)$ are standard sections. We may apply the same construction for each of the sections $\Psi^i(s)$ and obtain an expansion

$$\begin{aligned} E(g, s, \Phi) = {} & c_S^{-1} L(s+\tfrac{3}{2})^{-1} L(s+\tfrac{1}{2})^{-1} \mathscr{E}(g, s, \varphi) \\ & - \sum_i \beta_i(s)\, c_{S_i}^{-1} L(s+\tfrac{3}{2})^{-1} L(s+\tfrac{1}{2})^{-1} \mathscr{E}(g, s, \varphi^i) + O((s-\tfrac{1}{2})^2). \end{aligned} \tag{7.6}$$

This process can be continued indefinitely. But the meromorphic function defined by (7.4) has some non-zero term in its Laurent expansion, we conclude that there exists a function $\varphi' \in S(V_{2,2}(\mathbb{A})^2)$ such that

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)\, \theta^T(g, \phi) \mathscr{E}(g, s, \varphi')\, dg \neq 0. \tag{7.7}$$

This is turn implies that f has a non-zero integral against the function

$$\theta^T(g, \phi)\theta(g, h; \varphi'),$$

and hence has a non-zero theta lift to $O(V_T + V_{2,2})$, as claimed. □

Remark. (7.1) and the identities derived from it, e.g., (7.2) actually yield formulas for the residue or special value of the L-function in question. These formulas obviously involve the integral of a theta lift of f to some $O(4)$ (resp. some

$O(6)$) over a subgroup $O(2)$ (resp. $O(4)$). If f or π is "motivic", then these formulas should have some number theoretic consequences, cf. [11] where an analogous case is discussed.

8 Locally and globally generic forms

In this section we will show that if π is an automorphic cuspidal representation of G such that $\pi = \bigotimes_v \pi_v$ with π_v generic for all v, and if the L-function $L_S(s, 1, \pi)$ is non-zero at $s=1$, then π is globally generic. More precisely,

Theorem 8.1 *Let $\pi = \bigotimes \pi_v$ be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$ and let $S(\pi)$ be the ramified set of π as in Sect. 7. Suppose that for all v, π_v has a standard Whittaker model, i.e., that π_v is generic. Assume that $L_{S(\pi)}(s, 1, \pi)$ has either a pole or a non-zero value at $s=1$. Then π has a global Whittaker model, i.e., is globally generic, and π has a non-zero theta lift to the split $O(6)$.*

Remark. In fact, a yet to be published result of Jacquet and Shalika asserts that if π is an automorphic cuspidal representation of GSp(2), then π is (globally) generic if and only if it corresponds, via Langlands functoriality, to an automorphic representation of GL(4). Recall that $SL(4) \simeq Spin(3, 3)$, the spin group of the split form of dimension six. Their proof involves the theta correspondence.

Proof of Theorem 8.1 Let $f \in \pi$, T, $\chi = \chi_T$, and $S = S(\pi, \chi, T)$ be as in Sect. 7. In order to use the integral representation discussed in that section, we must replace the L-function $L_{S(\pi)}(s, \pi, 1)$ with the more restricted L-function $L_S(s, \pi, 1)$. Since

$$L_{S(\pi)}(s, 1, \pi) = \Big(\prod_{v \in S - S(\pi)} L_v(s, 1, \pi_v) \Big) \cdot L_S(s, 1, \pi), \tag{8.1}$$

the assumption that $L_S(s, 1, \pi)$ has either a pole or non-zero value at $s=1$ would be equivalent to the analogous assumption on $L_{S(\pi)}(s, 1, \pi)$ provided the local factors $L_v(s, 1, \pi_v)$ were known to be holomorphic at $s=1$. If we already knew π to be globally generic, then Shahidi's estimate [31] would provide precisely the holomorphy required. But in fact, Li (unpublished) has recently classified all irreducible admissible representations π_v of $Sp(2, F_v)$ such that π_v is unramified, unitary and generic. It follows from his description that $L_v(s, 1, \pi_v)$ is indeed holomorphic for $\mathrm{Re}(s) \geqq 1$. Thus we may assume that $L_S(s, 1, \pi)$ has either a pole or non-zero value at $s=1$.

First we suppose that T is split.

Lemma 8.2 *If $f \in \pi$ is such that $f_T \neq 0$ for a split binary form T with* det $T \neq 0$, *then f is generic.*

Proof. Recall that

$$f_T(g) = \int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\psi(\mathrm{tr}(Tb))\, db. \tag{8.2}$$

Moreover, since for $\alpha \in \mathrm{GL}_2(F)$

$$f_T(m(\alpha)g) = f_{T[\alpha]}(g) \tag{8.3}$$

where $T[\alpha] = {}^t\alpha T \alpha$, we may assume that $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The non-vanishing of f_T then implies that the integral

$$f_Z(g) = \int_{F\backslash \mathbb{A}} f\left(n\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} g\right) dx. \tag{8.4}$$

is non-zero. Note that the subgroup

$$Z = \left\{ n\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{8.5}$$

is the center of the unipotent radical N_1 of the maximal parabolic subgroup P_1 with Levi subgroup isomorphic to $\mathrm{GL}(1) \times \mathrm{Sp}(1)$. Moreover, we may further expand f along N_1 to obtain

$$f_Z(g) = \sum_{\substack{\xi \in F^2 \\ \xi \neq 0}} f_{Z,\xi}(g) \tag{8.6}$$

where, for $\xi = {}^t(\alpha, \beta) \in F^2$,

$$f_{Z,\xi}(g) = \int_{N_1(F)Z(\mathbb{A})\backslash N_1(\mathbb{A})} f_Z(n_1(x, y)g)\, \psi(\alpha x + \beta y)\, dx\, dy. \tag{8.7}$$

Here, for $x, y \in \mathbb{A}$,

$$n_1(x, y) = \begin{pmatrix} 1 & x & 1 & y \\ & 1 & y & \\ & & 1 & \\ & & -x & 1 \end{pmatrix}. \tag{8.8}$$

Note that the term with $\alpha = \beta = 0$ vanishes since f is cuspidal. For $\gamma \in \mathrm{SL}_2(F)$ and for $m_1(\gamma) \in M_1$, the corresponding element of the Levi factor, we have

$$m_1(\gamma)\, n_1(x, y)\, m_1(\gamma)^{-1} = n_1((x, y)\gamma^{-1}) \tag{8.8}$$

and thus

$$f_{Z,\xi}(m_1(\gamma)g) = f_{Z,\gamma\cdot\xi}(g). \tag{8.10}$$

Since $f_{Z,\xi}$ is non-zero for some ξ, we conclude that $f_{Z,\xi_0} \neq 0$ for $\xi_0 = (\alpha, 0)$ for some $\alpha \in F^\times$. Finally we expand f_{Z,ξ_0} along the subgroup $\left\{ n\begin{pmatrix} 0 & \\ & x \end{pmatrix} \right\}$, i.e., write

$$f_{Z,\xi_0}(g) = \sum_{\eta \in F^\times} f_{Z,\xi_0,\eta}(g) \tag{8.11}$$

with

$$f_{Z,\xi_0,\eta}(g)=\int_{F\backslash\mathbb{A}} f_{Z,\xi_0}\left(n\begin{pmatrix}0 & \\ & x\end{pmatrix}g\right)\psi(\eta x)\,dx. \tag{8.12}$$

Again the term with $\eta=0$ vanishes since f is cuspidal. But then the Whittaker function associated to the generic character $\alpha\cdot(\varepsilon_1-\varepsilon_2)+\eta(2\varepsilon_2)$ must be non-zero, as claimed. □

Note that as a corollary of the proof we have shown that the existence of a global Whittaker model is equivalent to the non-vanishing of the integral f_Z of (8.4).

Before continuing, we recall some results of [29]. For any positive integer k let $I_k\subset L^2_{\text{cusp}}(G(\mathbb{A}))$ denote the kernel of the theta lifting to $O(V_{k,k})$, where, as before, $V_{k,k}$ denotes the split quadratic space with dimension $2k$. Then define subspaces $R_k\subset L^2_{\text{cusp}}(G(\mathbb{A}))$ inductively as follows: $R_1=(I_1)^\perp$ and

$$R_k=(R_1\oplus R_2\oplus\cdots\oplus R_{k-1}\oplus I_k)^\perp. \tag{8.13}$$

Note that this makes sense since it is shown in [29] that $I_{k+1}\subset I_k$. Also note that

$$I_k=\bigoplus_{\ell>k} R_\ell. \tag{8.14}$$

Then by Theorem I.2.1 and Sect. I.3 of [29], we have $R_k=0$ for $k=1$ and for $k>4$, and

$$L^2_{\text{cusp}}(G(\mathbb{A}))=R_2\oplus R_3\oplus R_4. \tag{8.15}$$

Finally, it is proved in Sect. I.3 of [29] that the space R_4 is characterized by the vanishing of the integral f_Z, and hence, by the remark above about Whittaker models, R_4 is precisely the subspace of $L^2_{\text{cusp}}(G(\mathbb{A}))$ consisting of those cuspidal automorphic representations which do not have (global and automorphic) Whittaker models. Note that the criterion for the existence of a Whittaker model given here depends *a priori* on the embedding of π into $L^2_{\text{cusp}}(G(\mathbb{A}))$ since it depends on the integral f_Z for various functions in π.

As another preliminary, we note the following restriction which is imposed on a local component π_v by the locally generic hypothesis.

Lemma 8.3 *Let v be a non-archimedean place of F. Suppose that σ is an irreducible admissible representation of G_v and that V is a 4 dimensional anisotropic inner product space over F_v. Also suppose that there exists a non-zero G_v-invariant bilinear form $\lambda:\sigma\otimes S(V^2)\to\mathbb{C}$. Then σ is not generic, i.e., has no standard Whittaker model.*

Proof. The map $\varphi\mapsto\lambda_\varphi$ where for $f\in\sigma$, $\lambda_\varphi(f)=\lambda(f\otimes\varphi)$, defines a non-zero G_v intertwining map from $S=S(V^2)$ to the contragredient $\tilde\sigma$ of σ. This map is surjective since σ is irreducible. Also observe that σ is generic if and only if its contragredient $\tilde\sigma$ is. Now let U be the unipotent radical of B_v, the fixed Borel subgroup of G_v, and consider a non-degenerate character

$$\psi_{a,b}: U\to\mathbb{C}$$

where

$$\psi_{a,b}\left(m\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)n\left(\begin{pmatrix}0 & 0\\ 0 & v\end{pmatrix}\right)\right)=\psi(au+bv),$$

with $a, b\in F_v^\times$. Here we write ψ for ψ_v. We consider the twisted Jacquet spaces

$$S_{U,\psi_{a,b}}\to\tilde{\sigma}_{U,\psi_{a,b}}\to 0,$$

and we compute $S_{U,\psi_{a,b}}$ in two steps. First let

$$U^0=\left\{n\left(\begin{pmatrix}z_1 & z_2\\ z_2 & 0\end{pmatrix}\right)\right\},$$

and note that U^0 lies in the kernel of $\psi_{a,b}$. Then the quotient $S\to S_{U^0}$ is realized concretely via restriction of $\varphi\in S$ to the set

$$\left\{x\in V^2\,|\,(x,x)=\begin{pmatrix}0 & 0\\ 0 & *\end{pmatrix}\right\}.$$

Since V is anisotropic, the first component of any x in this set is forced to be zero. Once we have passed to S_{U^0} we must further take the maximal quotient of this space on which the abelian group U/U^0 acts by the character $\psi_{a,b}$. But

$$\omega\left(m\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\right)\varphi(0,x_2)=\varphi\left((0,x_2)\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)=\varphi(0,x_2),$$

so that the image of the subgroup of such $m\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)$'s in U/U^0 acts trivially on S_{U^0}. Since a is non-zero, the quotient $S_{U,\psi_{a,b}}$ is zero, and hence so is $\tilde{\sigma}_{U,\psi_{a,b}}$. Thus σ cannot have a non-trivial Whittaker functional for any non-degenerate character $\psi_{a,b}$, as claimed. □

Remark. In fact the same result holds if v is an archimedean place, since, at such a place, the representation of G_v on $S(V^2)$ for an anisotropic inner product space V is a direct sum of either holomorphic or antiholomorphic representations of G_v, and such representations can never be generic.

Returning to the proof we next consider the case in which $f_T\neq 0$ for some T which is anisotropic, so that $\chi_T\neq 1$. We take ϕ_v and Ψ_v so that the factor $\Lambda_S(s)$ is holomorphic and non-zero at $s=\frac{1}{2}$. Then, via (7.1), our hypothesis that $L_S(s,1,\pi)$ is not zero at $s=1$ implies that either

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)\,A_{-1}(g,\Psi)\,dg\neq 0 \tag{8.16}$$

in the case of a pole, or

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)\,A_0(g,\Psi)\,dg\neq 0, \tag{8.17}$$

in the case of a non-zero value.

Assume first, then, that for some choice of f_v, ϕ_v and Ψ_v for $v\in S$, the integral (8.16) is non-zero. We may assume that Ψ_v is homogeneous, and hence coherent,

associated to an isotropic four-dimensional $V=V_0+V_{1,1}$ via (i) and (ii) of Theorem 4.1. By (iii) of Theorem 4.1 there exists a $\varphi\in S(V_0(\mathbb{A})^2)$ such that $A_{-1}(g,\Phi)=I(g,\varphi)$, and thus, substituting this into (8.16), we conclude that f has a non-zero theta lift to the four-dimensional space $U=V_T+V_0$. Note that $\chi_{V_0}=\chi=\chi_T$. Then we may assume that there are factorizable functions $f=\bigotimes_v f_v$ and $\varphi=\bigotimes_v \varphi_v$ for which the theta lift $\theta_\varphi(f)$ is non-zero when evaluated at the identity element of $O(U)(\mathbb{A})$. If we vary only the v components of f and φ leaving all other components fixed, we obtain a non-zero G_v-invariant bilinear form on $\pi_v\otimes S(U_v^2)$. Now if there exists a place v such that $V_{0,v}\neq V_{-T,v}$, then U_v is anisotropic and by Lemma 8.3 and the remark following it, π could not be locally generic at v. Therefore, we must have $V_{0,v}=V_{-T,v}$ for all v and so $V_0=V_{-T}$ and U is split. Thus our function f has a non-zero theta lift to $O(V_{2,2})$ and so does not lie in $I_2=R_3\oplus R_4$ the kernel of this lifting. In particular, $f\notin R_4$ and hence admits a non-vanishing global Whittaker function by our criterion above. Thus π is globally generic, as claimed.

Next suppose that the integral in (8.16) vanishes for all choices of f_v, ϕ_v and Ψ_v for $v\in S$, but that there exists a choice for which (8.17) holds.

Let $V=V_{-T}+V_{1,1}$ be the four-dimensional space obtained by adding a hyperbolic plane to V_{-T}. Note that for $v\notin S$ we have $\Psi_v=\Psi_v^0$ and that $\Psi_v^0(\frac{1}{2})\in R_v(V)$. In fact, it is at this point in the argument that we must have the condition that V_{T_v} be the norm form for $v\notin S$, [29].

For any choice of f_v, ϕ_v and Ψ_v for $v\in S$, we call a place $v\in S$ exceptional if $\Psi_v(\frac{1}{2})\notin R_v(V)$. Among all choices of $\{f_v,\phi_v,\Psi_v|v\in S\}$ for which (8.17) holds we fix a choice for which the number of exceptional places is a minimum, and call the corresponding functions ${}^0f$, ${}^0\phi$ and ${}^0\Psi$. If the number of exceptional places is not zero, fix one such place v, let V_v' be defined as in section 1 after Lemma 1.3, and define a trilinear form

$$\lambda_v:\ \pi_v\otimes\omega_{T_v}\otimes R_v(V_v')\to\mathbb{C} \tag{8.18}$$
$$f_v,\phi_v,\Psi_v\mapsto \int_{G(F)\backslash G(\mathbb{A})} f(g)\,\theta^T(g,\phi)\,A_0(g,\Psi)\,dg,$$

where $f=f_v\otimes {}^0f^v$, $\phi=\phi_v\otimes {}^0\phi^v$ and $\Psi=\Psi_v\otimes {}^0\Psi^v$. Here ${}^0f^v=\bigotimes_{w\neq v} {}^0f_w$, etc., and ω_{T_v} denotes the local oscillator representation of G_v (resp. $(\mathfrak{g}_v,K_v)$) on $S(V_{T,v}^2)$. By the choice of ${}^0f$, ${}^0\phi$, and ${}^0\Psi$, λ_v is non-zero. Moreover, since the map

$$\Psi\mapsto A_0(g,\Psi) \tag{8.19}$$

is intertwining *modulo* the image of A_{-1}, and since, by our assumption that the integral in (8.16) vanishes for all choices, $f\cdot\theta^T(\phi)$ is orthogonal to $\mathrm{Im}(A_{-1})$ for all f and ϕ, λ_v is actually G_v-invariant (resp. $(\mathfrak{g}_v,K_v)$-invariant) if v is non-archimedean (resp. archimedean). Finally,

Lemma 8.4 *λ_v is zero on the space $\pi_v\otimes\omega_{T_v}\otimes(R_v(V_v')\cap R_v(V))$.*

Proof. If λ_v were non-zero for some f_v, ϕ_v and Ψ_v with $\Psi_v\in R_v(V')\cap R_v(V)$, then we could use that choice in place of ${}^0f_v$, ${}^0\phi_v$, ${}^0\Psi_v$ while retaining other choices, and thus obtain a smaller exceptional set, i.e., remove v from the exceptional set. This would contradict minimality. □

By Lemma 8.4, λ_v actually defines a non-zero invariant trilinear form

$$\lambda_v : \pi_v \otimes \omega_{T_v} \otimes R_v(V'_{-T,v}) \to \mathbb{C} \tag{8.20}$$

and hence, also

$$\lambda_v : \pi_v \otimes \omega_{T_v} \otimes \omega_{(-T_v)'} \to \mathbb{C}. \tag{8.21}$$

Since $\omega_{T_v} \otimes \omega_{(-T_v)'} = \omega_{T_v+(-T_v)'}$ and $T_v+(-T_v)'$ is an anisotropic four-dimensional space, by Lemma 8.3, the representation π_v would not then have a local Whittaker model, contrary to our assumption. Thus we conclude that the minimal exceptional set must actually be empty, i.e., that we may choose $\{f_v, \phi_v, \Psi_v | v \in S\}$ for which (8.17) holds and such that Ψ is coherent, associated to $V = V_{-T} + V_{1,1}$.

Now we apply Theorem 6.1, Corollary 5.4 and the assumption that the integrals as in (8.16) all vanish to conclude that f has a non-zero theta lift to $O(U)$ where

$$U = V_T + V$$

is the split six-dimensional space $V_{3,3}$. Again by the results of [29] reviewed above, this implies that f has a non-vanishing Whittaker function and hence π is globally generic. This completes the proof of Theorem 8.1 □

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