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Non-minimal Yang-Mills fields and dynamics

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It is a long-standing question whether there exist Yang-Mills fields on Riemannian 4-manifolds that are neither self-dual nor anti-self-dual. From the variational viewpoint these fields correspond to non-minimal critical points of the Yang-Mills action function. This paper describes a dynamical systems approach to Yang-Mills fields, and uses it to construct non-minimal Yang-Mills fields. The first case considered is that of $S^3 \times S^1$:

Theorem 3.1 *Let $P \to S^1_L \times S^3$ be the trivial* SU(2) *bundle over the product of the circle of length L and the 3-sphere of unit radius. For each integer $N > 0$ there exists an L such that $S^1_L \times S^3$ admits at least N non-gauge-equivalent irreducible non-minimal Yang-Mills fields.*

The proof shows that one can take any $L > \pi N\sqrt{2}$, and that all but one of these Yang-Mills fields lies on a circle of non-gauge-equivalent irreducible non-minimal Yang-Mills fields with constant action. Note that, because the Yang-Mills equations are conformally invariant, these non-minimal fields are associated with the conformal (rather than metric) structure, and the conformal structure on $S^1_L \times S^3$ varies with L. These Yang-Mills fields are found by looking for solutions invariant under the action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $S^3 \times \mathbb{R}$. This reduces the Yang-Mills equations to a completely integrable Hamiltonian system (Eqs. (2.3), (2.4)), which can be explicitly solved in terms of Jacobi elliptic functions. Furthermore, these solutions can be neatly visualized as the motions of a point particle moving in one dimension in the potential $P(x) = -(x^2 - 1)^2$.

More generally, one can look for solutions that are invariant under only $\mathrm{SU}(2) \times S^1$, and one can also vary the metric on S^3 while preserving this symmetry. The Yang-Mills equations then reduce to a (non-autonomous) 4-dimensional dynamical system (Eqs. (4.12–4.14) and (6.3–6.4)). There is a 2-dimensional subsystem defined by the self-duality equations, and it is easy to identify the standard $k = \pm 1$ instantons as flow curves in this subsystem which start and end at the

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trivial connection. A very careful analysis of this system shows that one can choose a time-dependent metric such that there exists an additional flow line beginning and ending at the trivial connection and roughly resembling an instanton followed by an anti-instanton. This corresponds to an unstable Yang-Mills field on the trivial bundle over $S^3 \times \mathbb{R}$ with the product metric modified on a compact set. Finally, using a conformal transformation and Uhlenbeck's Removable Singularities Theorem one obtains a Yang-Mills field on the 4-sphere.

Theorem 8.3 *There exists a family of metrics on S^4 each of which admits an irreducible non-minimal Yang-Mills field on the trivial* SU(2) *bundle. There are metrics in this family arbitrarily close to the standard metric.*

This family of Yang-Mills fields on S^4 does not have a uniform limit as the metric approaches the standard metric. Instead, the action becomes increasingly concentrated near two antipodal points p and $\bar{p}$ and in the limit an instanton bubbles off at p and an anti-instanton bubbles off at $\bar{p}$.

L. Sibner, R. Sibner, and K. Uhlenbeck have found a different example of a non-minimal Yang-Mills field on the standard S^4 (announced in [SSU]). The most basic of the above examples on $S^3 \times S^1$ has been known to experts for some time; it is mentioned by Laquer ([L]) and Urakawa ([Ur]). Recently, G. Bor and R. Montgomery ([BM]), and L. Sadun and J. Segert ([SS1, SS2]) have proved the existence of non-minimal Yang-Mills fields on non-trivial bundles over S^4. Each of these arguments makes use of a group action. In fact, for certain types of group actions one can use general arguments to prove the existence of non-minimal Yang-Mills fields ([Pa]).

These other approaches are all based on variational methods or Morse theory. All begin by regarding the Yang-Mills action as a Lagrangian on the space $\mathscr{B}$ of connections modulo gauge transformations, or on a subspace $\mathscr{B}^H \subset \mathscr{B}$ invariant under an action of a group H. The symmetry reduces the problem to one on a lower dimensional manifold, where the Yang-Mills action gives a well-behaved variational problem. Finally, the fact that $\mathscr{B}^H$ has non-trivial homotopy type forces the existence of non-minimal critical points.

In contrast, the theorems stated above are *not* topological in nature. In the first theorem the number of (invariant) solutions grows with L, even though the space $\mathscr{B}$ is independent of L. Similarly, in the second theorem $\mathscr{B}$ is again independent of the metric, yet the solutions disappear at the standard metric. Thus the choice of conformal structure is crucial in both of these results.

The dynamical systems methods of this paper have several other advantages over variational approaches. First, the problem is presented in terms of the time evolution of a field on the 3-sphere and therefore reflects the physicists' view of Yang-Mills fields; this aids intuition and enables us to cast the problem in terms of Hamiltonian mechanics. Second, it is constructive — the Yang-Mills fields of the above theorems can be calculated explicitly on a computer, and many qualitative features of the solutions can be seen directly from the method of proof. Finally, this method provides interesting insight into the role of the metric in these non-minimal solutions. The metric appears as a "control parameter" in the ODE; one can allow the metric to vary with considerable latitude (maintaining the $\mathrm{SU}(2) \times S^1$ symmetry) and still get non-minimal Yang-Mills fields.

The dynamical systems considered here are finite-dimensional slices of the infinite-dimensional Yang-Mills flow on the space of all connections on S^3. The

qualitative features we study on the slices are valid for the infinite-dimensional system. For example, there are two "half-dimensional" invariant submanifolds whose flows are solutions of the self-duality equations, and non-minimal Yang-Mills fields of the above theorems are flow lines limiting to these submanifolds. It would be interesting to see other examples of this phenomenon.

Physicists have long viewed instantons as analogous to certain problems in classical mechanics (c.f. Coleman [C]). The key observation of this paper is that, when enough symmetry is imposed, Yang-Mills *becomes* a classical mechanics problem. This correspondence is described in the first two sections, and used in §3 to construct non-minimal Yang-Mills fields on $S^3 \times S^1$. We then reduce the symmetry group to $S^3 \times S^1$; this results in the more complicated dynamical system described in §4. The remaining sections are devoted to a detailed analysis of this system. The technical parts of this analysis appear in §5–§7, and the main argument for the existence of non-minimal Yang-Mills fields on S^4 is given in §8.

1 Bi-invariant connections on the 3-sphere

The 3-sphere $S^3 = \mathrm{SU}(2)$ is a Lie group, and hence SU(2) acts on it on the left and the right. These actions lift to the bundle of frames $P \to S^3$, which is a principal SU(2) bundle. This section gives an elementary description of the Yang-Mills function on the space of bi-invariant connections on P.

To set up the calculations we choose a basis $\{e_i\}$ of the Lie algebra $\mathfrak{su}(2)$ — thought of as the left-invariant forms on SU(2) — satisfying

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1 \,. \tag{1.1}$$

Minus one eighth of the Killing form then defines a metric $\frac{1}{8}\kappa$ on SU(2) with respect to which the $\{e_i\}$ are orthonormal; it is easy to verify that this makes SU(2) isometric to the unit sphere in $\mathbb{R}^4$.

In terms of the trivialization provided by the $\{e_i\}$ every left-invariant connection A on P is given by a constant matrix $A : \mathfrak{su}(2) \to \mathfrak{su}(2)$. Such a connection is bi-invariant if A is Ad-invariant. Because SU(2) is simple any Ad-invariant A is a multiple of the identity. Thus the space X of bi-invariant connections on S^3 is one-dimensional, and any $A \in X$ can be written explicitly as

$$A = \left(\frac{x+1}{2}\right) \sum e^i \otimes e_i \tag{1.2}$$

where $\{e^i\}$ are the left-invariant 1-forms dual to the $\{e_i\}$, and where $x \in \mathbb{R} = X$ is a constant. The corresponding covariant derivative satisfies

$$\nabla_{e_i} e_j = e_i \cdot e_j + \frac{x+1}{2}[e_i, e_j] = \frac{x+1}{2}[e_i, e_j] \,. \tag{1.3}$$

Calculating $[\nabla_{e_i}, \nabla_{e_j}] - \nabla_{[e_i, e_j]}$ one finds that the curvature tensor is

$$F_{ij} = \left(\frac{x^2-1}{4}\right)[e_i, e_j] \in \mathfrak{su}(2) \tag{1.4}$$

and hence, using the metric $\frac{1}{8}\kappa$,

$$|F|^2 = |F_{12}|^2 + |F_{13}|^2 + |F_{23}|^2 = \tfrac{3}{4}(x^2-1)^2 \,. \tag{1.5}$$

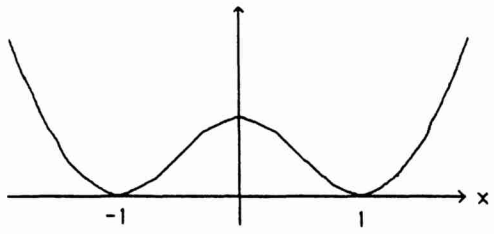

Fig. 1

We can now regard the Yang-Mills action as a function on the space X of bi-invariant connections. Since the density $|F|^2\,dv$ is left-invariant the integration over S^3 is simply multiplication by the volume, which is $2\pi^2$. Thus the Yang-Mills action is

$$YM(A) = \frac{1}{4\pi^2}\int_{S^3} |F|^2\,dv = \frac{3}{8}(x^2-1)^2\,. \tag{1.6}$$

This shows that the Yang-Mills function restricts to the fourth-degree polynomial on X depicted in Fig. 1. The critical points on X occur a $x = 0, \pm 1$.

The connections at $x = \pm 1$ are flat, and hence are Yang-Mills. In fact, since S^3 is simply connected the only flat connection in the trivial one, so the ± 1 connections are reducible and gauge equivalent. The connection $x = 0$ is the *canonical connection* on the group $\mathrm{SU}(2) = S^3$. To identify it, note that the Levi-Civita connection of the standard metric on S^3 is a torsion-free bi-invariant connection on P. From (1.3) the torsion $\nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j] = x[e_i, e_j]$ vanishes only for $x = 0$. Thus the canonical connection is the Levi-Civita connection. It therefore has constant curvature, so $\nabla^0 F_{A_0} \equiv 0$ and hence $D^*_{A_0} F_{A_0} = 0$, i.e. the canonical connection is Yang-Mills. Furthermore, formula (1.4) also shows that the vector space span of the curvature for $x = 0$ is all of $\mathfrak{su}(2)$, so by the holonomy theorem ([KN]) the canonical connection is (even locally) irreducible. Thus there are two gauge equivalence classes of Yang-Mills connections on P: the trivial connection (corresponding to $x = \pm 1$) and the canonical connection, which is not flat or reducible. In particular, *the canonical connection is an irreducible non-minimal Yang-Mills field on* S^3.

Remark 1.1 (1) This is the simplest case of a general phenomenon: the canonical connection on any homogeneous space is Yang-Mills [L].

(2) While the connections $x = \pm 1$ are gauge equivalent, the connections $\pm x$ are not gauge equivalent for any other value of x [L].

In the next two sections we will use these connections on S^3 to construct Yang-Mills fields on four-dimensional manifolds. The following proposition shows that the simplest such construction already yields an interesting result.

Proposition 1.2 *There exists an irreducible Yang-Mills field on the trivial bundle over* $S^3 \times S^1$ *that is neither self-dual nor anti-self-dual.*

Proof. The sum $\omega = A_0 + \theta$ of the canonical connection A_0 on S^3 and the trivial connection θ on S^1 is a connection on the trivial bundle over $S^3 \times S^1$ with

curvature (1.3) (the remaining curvatures vanish), and with $\nabla^\omega F_\omega = 0$ as above. This connection ω is therefore Yang-Mills, irreducible, and not flat (since these were true locally for $A_0 + \theta$), and hence is non-minimal. □

To obtain more interesting examples we can consider *time-dependent* bi-invariant connections on $\mathbb{R} \times S^3$. We will do this next by viewing the Yang-Mills equations as a classical mechanical system on the space of connections on S^3.

2 The equations of motion

Let P be the trivial SU(2) bundle over $S^3 \times \mathbb{R}$ with its usual metric and with orientation form $e^1 \wedge e^2 \wedge e^3 \wedge dt$. Each connection on P has the form $\nabla = \partial + \omega$ where ω is an $\mathfrak{su}(2)$-valued 1-form on $S^3 \times \mathbb{R}$. The group SU(2) acts isometrically on $S^3 \times \mathbb{R}$ on the right and the left, and this action lifts (trivially) to P. Each bi-invariant connection restricts to an invariant connection on each slice $S^3_t = S^3 \times \{t\}$, so has the form

$$\omega = A(t) + \phi(t)\,dt \tag{2.1}$$

where, for each t, $A(t)$ is an element of the space X of §1, and $\phi(t) \in \mathfrak{su}(2)$. The next lemma shows that this gauge freedom can be used to transform ω into a connection form with $\phi \equiv 0$ (this is the "temporal gauge"; it does not require that ω be invariant).

Lemma 2.1 *For each connection ω on P there exists a path $g = g(t) \in \mathrm{SU}(2)$, unique up to a constant $g \in \mathrm{SU}(2)$, that defines a gauge transformation taking ω to $\omega' = A'(t)$.*

Proof. Write ω as in (2.1). Then ϕ defines a time-dependent left-invariant vector field on SU(2) by $\phi_t(g) = g\phi(t)$. Let $g(t)$, $-\infty \leqq t \leqq \infty$, be the integral curve with initial condition $g(0) = \mathrm{Id}$. Then $g(t) \in \mathrm{SU}(2)$ satisfies $\dot{g}(t) = g(t)\phi(t)$, so gives a gauge transformation that is constant on each sphere S^3_t and satisfies $(dg)g^{-1} = g\phi g^{-1}\,dt$. Applying this gauge transformation to ω gives

$$g \cdot \omega = (\mathrm{Ad}(g))(A + \phi dt) + g dg^{-1} = (\mathrm{Ad}(g))A\,.$$

Conversely, if the dt component of $g \cdot \omega$ vanishes for some $g = g(t)$ then we must have $\dot{g}(t) = g(t)\phi(t)$, so g is unique up to the choice of $g(0) \in \mathrm{SU}(2)$. □

Using the gauge of Lemma 2.1, we can regard the connection $\omega = A(t)$ as a path in the space X of invariant connections on S^3. Its curvature is $F^\omega = dt \wedge \dot{A} + F^A$ where F^A is the curvature of $A \in X$ on S^3. Hence $|F^\omega|^2 = |\dot{A}|^2 + |F^A|^2$ where both $|\dot{A}|^2$ and $|F^A|^2$ are constant functions on each S^3_t. Thus

$$YM(A(t)) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{S^3} |\dot{A}|^2 + |F^A|^2\,dv\,dt = \frac{1}{2} \int_{-\infty}^{\infty} |\dot{A}|^2 + P(A(t))\,dt \tag{2.2}$$

where $P(A)$ is the fourth degree polynomial in A given by (1.5). This integral has a familiar interpretation in classical mechanics: it is the Lagrangian of a point particle moving in X in the potential *minus* P.

To emphasize this viewpoint we adopt the notation of (1.2) and (1.5). Then $|\dot{A}(t)|^2 = \frac{3}{4}\dot{x}^2$ and the Yang-Mills Lagrangian is simply

$$YM(x(t)) = \frac{3}{8}\int_{-\infty}^{\infty} \dot{x}^2 + P(x)\,dt \tag{2.3}$$

where $P(x) = (x^2 - 1)^2$. The corresponding equations of motion are

$$\ddot{x} = 2x(x^2 - 1)\,, \tag{2.4}$$

and for any solution the energy $E = \dot{x}^2 - P(x)$ is constant.

The following theorem shows the correspondence between solutions of (2.4) and bi-invariant Yang-Mills fields on $S^3 \times S^1$. It is an immediate consequence of the symmetric criticality principle ([P]), but we will give a direct proof.

Theorem 2.2 *For any solution $x(t)$ of* (2.4) *the connection given by* (1.2) *in the temporal gauge is a Yang-Mills field on $S^3 \times \mathbb{R}$. If the action* (2.3) *is finite then the connection extends via stereographic projection to a connection on S^4.*

Proof. In the temporal gauge the Yang-Mills equation $D^*F^\omega = 0$ becomes $\ddot{A} = -\sum i_k \nabla_k F$, where i_k denotes contraction by e_k. Writing $F = \frac{1}{2}\sum_{i,j} e^i \wedge e^j \otimes F_{ij}$, it becomes

$$\ddot{A} = \sum_{i,j} \frac{1}{2}\,\nabla^0_k(e^i \wedge e^j) \otimes F_{ij} + e^j \otimes \nabla_k F_{kj}$$

where ∇^0 is the Riemannian connection. Substituting in (1.4) and differentiating using (1.3) reduces this to

$$\ddot{A} = \left(\frac{x^2 - 1}{8}\right)\sum_{i,j} [e^i, e^j] \otimes [e^i, e^j] - 8(x + 1)e^j \otimes e_j = x(1 - x^2)\sum e^i \otimes e_i\,.$$

Comparing this with (1.2), we see that the Yang-Mills equation is exactly the equation of motion (2.4). Now consider the conformal map (stereographic projection)

$$\Psi : S^3 \times \mathbb{R} \to S^4 \quad \text{by} \quad \Psi(x, t) = (x\,\mathrm{sech}(t), \tanh(t)) \in \mathbb{R}^5\,. \tag{2.5}$$

The Yang-Mills equation is conformally invariant, so the pullback by Ψ^{-1} is a Yang-Mills field on S^4 with two points removed; if the action is finite this extends to a smooth field on S^4 by Uhlenbeck's Removable Singularities Theorem ([U]). □

We can also express the instanton number and the self-dual equations in terms of the mechanical system (2.3)–(2.4). First, since $F^\omega = dt \wedge \dot{A} + F^A$ the Pontryagin form is the real-valued 4-form $\frac{1}{4\pi^2}\langle F^\omega \wedge F^\omega\rangle\,dt = \frac{1}{2\pi^2}\langle \dot{A} \wedge F^\omega\rangle\,dt$, where $\langle\,,\rangle$ denotes our inner product on $\mathfrak{su}(2)$. From (1.1)–(1.4) one finds that the instanton number is

$$k = \frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{S^3} \langle \dot{A} \wedge F^\omega\rangle\,dt = \frac{3}{4}\int_{-\infty}^{\infty}(1 - x^2)\frac{dx}{dt}\,dt = \frac{1}{4}(3x - x^3)\Big|_{x_0}^{x_1} \tag{2.6}$$

where x_0 and x_1 are the initial and final points of the trajectory. This function $\mathrm{CS}(x) = \frac{1}{4}(3x - x^3)$ is the Chern-Simons function on X; it is well-defined up to an additive constant.

Next note that the solutions with energy $E = 0$ satisfy $\dot{x}^2 = P(x) = (1 - x^2)^2$, so $\dot{x} = \pm(1 - x^2)$. From (1.2) and (1.4) we have

$$\dot{A} = \frac{\dot{x}}{2}\sum e^i \otimes e_i \qquad *F = \frac{(x^2 - 1)}{2}\sum e^i \otimes e_i \tag{2.7}$$

where $*$ is the star operator on S^3, so the self-duality equation $\dot{A} + *F = 0$ becomes $\dot{x} = (1 - x^2)$. Thus the zero-energy trajectories correspond to self-dual or anti-self-dual fields on $S^3 \times \mathbb{R}$.

The critical paths of the Lagrangian (2.3) are easily visualized by picturing a ball rolling in the potential-P (Fig. 2). Consider the solutions for which the Lagrangian is finite. For any such solution, both $\dot{x}^2$ and $P(A)$ vanish as $t \to \pm\infty$, so we must have $E = 0$ and $x(\pm\infty) = \pm 1$. There are three possible trajectories:

(1) The ball remains stationary, balanced on top of the hill at $x = \pm 1$.
(2) The ball begins on the top of the hill at $x = -1$, rolls down the hill, up the other side, and comes to rest at $x = 1$; this is the solution $x(t) = \tanh(t)$ of (2.4).
(3) The ball begins at $x = 1$, rolls down, and comes to rest at $x = -1$.

Of course, (2) and (3) are actually 1-parameter families of solutions which differ by time translation.

Each of these trajectories corresponds to a Yang-Mills field on $S^3 \times \mathbb{R}$; these can be interpreted as follows:

(1) These two Yang-Mills fields are gauge equivalent and represent the trivial connection on $S^3 \times \mathbb{R}$.
(2) This Yang-Mills field is the standard $k = 1$ t'Hooft instanton on $S^3 \times \mathbb{R}$. Its energy density $|F^{\omega}(t)|^2 = 2\,\mathrm{sech}^4(t)$ is a bumplike function concentrated around $t = 0$ (or the translation of this function). This property of being concentrated in time is what originally led to the term "instanton".
(3) This is the same as case (2) with the orientation reversed, and corresponds to the standard $k = -1$ anti-instanton.

Remarks. (1) The t'Hooft instantons are usually considered as $k = 1$ self-dual connections on S^4. There is a unique "standard" one with constant energy density on the unit S^4 in $\mathbb{R}^5$. This pulls back to the solution (2) under the stereographic projection (2.5).

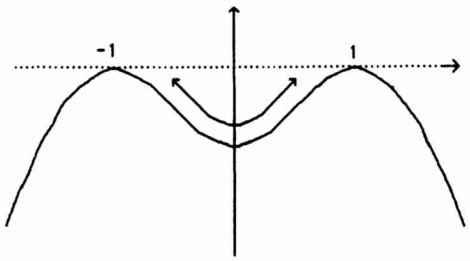

Fig. 2

(2) One can vary the metric on $S^3 \times S^1$ and maintain bi-invariance by taking the metric to be of the form $\bar{g} = R(t)^2 g_0 + dt^2$ where g_0 is the metric on the unit 3-sphere. This changes the equations of motion (2.4). However, this metric is conformal to $g_0 \oplus R(t)^{-2} dt^2 = g_0 \oplus d\tau^2$ where $\tau = \int R(t)^{-1} dt$ is a new time coordinate. Thus $\bar{g}$ is equivalent to the original metric by a conformal change and a diffeomorphism. Since the Yang-Mills equations are invariant under conformal diffeomorphisms, nothing new is gained.

The above solutions are well-known, and there are no additional zero-energy trajectories with finite action. There are, however, many periodic solutions. We will next examine these in detail.

3 Non-minimal solutions on $S^3 \times S^1$

The solutions of the classical mechanics problem (2.3)–(2.4) can be expressed in terms of Jacobi elliptic functions ([D], Chaps. 6, 7). Recall that the elliptic function $\mathrm{sn}(u, k)$ gives the inverse of the elliptic integral of the first kind: if

$$u = F(\phi, x) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \qquad k^2 < 1 \tag{3.1}$$

then $\mathrm{sn}(u, k) = \sin \phi$. For each k, it is an odd doubly periodic function with periods $4F\left(\frac{\pi}{2}, k\right)$ and $4iF\left(\frac{\pi}{2}, \sqrt{1 - k^2}\right)$. As k varies the family $\mathrm{sn}(u, k)$ interpolates between $\mathrm{sn}(u, 0) = \sin(u)$ and $\mathrm{sn}(u, 1) = \tanh(u)$. The general solution of (2.4) is then

$$x(t) = C\,\mathrm{sn}(\lambda(t - t_0), k) \tag{3.2}$$

where λ and t_0 are constants and $C^2 = 2 - \lambda^2$ and $k^2 = (2\lambda^{-2} - 1)$. We can translate the solution to insure that $x(0) = 0$; this is done by setting $t_0 = 0$. Evaluating the derivatives at $t = 0$ gives $E = (\dot{x})^2 - 1 = C^2\lambda^2 - 1$, so $\lambda^2 = 1 \pm \sqrt{-E}$. Now if the particle starts at rest at position $-x_0$, then its energy is $E = -P(x_0) = -(1 - x_0^2)^2$. There are four points x_0 with this potential; we are interested in the one with $0 \leqq x_0 \leqq 1$. Solving, we find that the solution with turning point x_0 is

$$x(t) = x_0\,\mathrm{sn}(\lambda t, x_0\lambda^{-1}) \quad \text{with} \quad \lambda = \sqrt{2 - x_0^2} \tag{3.3}$$

and that its period is $L = 4\lambda^{-1} F\left(1, \frac{x_0}{\lambda}\right) = 4\int_0^{\pi/2} (2 - x_0^2 + x_0^2 \sin^2\theta)^{-1/2}\, d\theta$. The period increases monotonically from $\sqrt{2}\pi$ when $x_0 = 0$ to infinity when $x_0 = 1$.

We can now enumerate the periodic solutions according to their period L. There are always the stationary solutions $x = \pm 1$ (corresponding to the trivial connection). Additional solutions describe the (periodic) motions of a particle with turning point x_0, $0 < x_0 < 1$; for each x_0 these form a family parameterized by S^1 (acting by time translation).

(1) For $0 < L \leqq \pi\sqrt{2}$ there are only the stationary solutions $x \equiv 0$ and $x \equiv \pm 1$.
(2) For $\pi\sqrt{2} < L \leqq 2\pi\sqrt{2}$ there are the stationary solutions: $x \equiv 0, \pm 1$ and a circle of solutions with turning point x_1.
(3) For $2\pi\sqrt{2} < L \leqq 3\pi\sqrt{2}$ there are the stationary solutions, a circle of solutions with turning point at some x_1, and a second circle of solutions of period $L/2$ and turning point at some $x_2, 0 < x_2 < x_1$.
(4) Similarly, For $k\pi\sqrt{2} < L \leqq (k+1)\pi\sqrt{2}$ there are exactly $k+1$ distinct circles of solutions.

Thus as L increases we see more and more "Fourier modes".

By Theorem 2.2 each of the above solutions describes a Yang-Mills field on $S^3 \times \mathbb{R}$ that is periodic of period L. Restricting to one period and identifying the bundles and connections over the ends gives a Yang-Mills field on the trivial bundle over the product $S^1_L \times S^3$ of the circle of length L and the unit 3-sphere. Of course, any self-dual field on the trivial bundle is flat, so these are non-minimal Yang-Mills fields. Each restricts to the canonical connection on $S^3 \times \{t\}$ for at least one t, and hence cannot be reducible. Finally, these Yang-Mills fields are not gauge equivalent because their energy densities $|F^\omega|(t)$ are distinct functions on $S^3 \times \mathbb{R}$. Thus we have:

Theorem 3.1 *Let P be the trivial principal* SU(2) *bundle over $S^1_L \times S^3$. For each integer $N > 0$ there exists L such that P admits at least N non-gauge-equivalent irreducible non-minimal Yang-Mills fields.*

This analysis gives a nice picture of the moduli space $\mathcal{M}^{\text{bi}}$ of bi-invariant Yang-Mills fields modulo gauge transformations. Since S^1 acts isometrically on $S^3 \times S^1$ each of these Yang-Mills fields (except those corresponding to the stationary solutions $x \equiv 0, 1$ above) lies on a circle of non-gauge-equivalent irreducible non-minimal Yang-Mills fields (the orbit of S^1 modulo a finite isotropy subgroup). Thus $\mathcal{M}^{\text{bi}}$ is a disjoint union of two points and N circles. This moduli space depends on the conformal structure of $S^3 \times S^1$, which is specified by L. The above description clearly shows this dependence on L: the number of circles in $\mathcal{M}^{\text{bi}}$ grows as L gets larger, and the moduli space bifurcates when L is a multiple of $\pi\sqrt{2}$.

4 $SU(2) \times S^1$ invariance

In this section we revise the arguments of the first two sections, now allowing any connection on P and any metric on S^3 that is invariant under the subgroup $H = SU(2) \times S^1$ of $SU(2) \times SU(2)$.

Every left-invariant connection A on P is given by a constant matrix $A: \mathfrak{su}(2) \to \mathfrak{su}(2)$ in the trivialization $\{e_i\}$ of §1. Right invariance under S^1 is then equivalent to $\mathrm{Ad}(S^1)$ invariance. We can choose the $\{e_i\}$ so that this Adjoint action fixes e_1 and rotates e_2 and e_3 into one another. A straightforward matrix calculation then shows that the space W of H-invariant connections on S^3 is three-dimensional, and any $A \in W$ can be written explicitly as

$$A = \left(\frac{x+1}{2}\right)\mathrm{Id.} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & z \\ 0 & -z & y \end{pmatrix} : \mathfrak{su}(2) \to \mathfrak{su}(2) \tag{4.1}$$

where $(x, y, z) \in \mathbb{R}^3 = W$. The corresponding covariant derivatives are

$$\nabla_{e_i} = \nabla^x_{e_i} + \Phi_i$$

where ∇^x is the covariant derivative of the connection (1.2)–(1.4), and Φ is the "Higgs field" $\Phi_1 = 0$, $\Phi_2 = ye_2 - ze_3$ and $\Phi_3 = ze_2 + ye_3$. The curvature formula $[\nabla_{e_i}, \nabla_{e_j}] - \nabla_{[e_i, e_j]}$ then becomes

$$F_{ij} = F^x_{ij} + (\nabla^x_i \Phi_j) - (\nabla^x_j \Phi_i) + [\Phi_i, \Phi_j] - \Phi_{[e_i, e_j]} .$$

This is easily computed using (1.3) and (1.4), yielding

$$\text{(4.2)} \qquad \begin{cases} F_{12} = (x-1)\left[ze_2 + \left(\dfrac{x+1}{2} + y\right) e_3 \right] \\ F_{13} = (1-x)\left[\left(\dfrac{x+1}{2} + y\right) e_2 - ze_3 \right] \\ F_{23} = \left[2\left(\dfrac{x+1}{2} + y\right)^2 - (x+1) + 2z^2 \right] e_1 . \end{cases}$$

We can also replace the Riemannian metric $\frac{\kappa}{8}$ with a general H-invariant metric. Such a metric is also determined by a symmetric 3×3 matrix invariant under conjugation by S^1; the matrix calculation that gave (4.1) now implies that the metric has the form $\operatorname{diag}(\lambda^2, \mu^2, \mu^2)$. But the Yang-Mills equations are conformally invariant, so we are free to normalize μ to 1. Thus the most general H-invariant Riemannian metric is given by

$$\text{(4.3)} \qquad |e_1|^2 = \lambda^2, \qquad |e_2|^2 = |e_3|^2 = 1 .$$

Geometrically, this amounts to considering S^3 as foliated by the circles of the Hopf fibration and changing the length of the circles. The volume form of this metric is $\lambda e^1 \wedge e^2 \wedge e^3$, and we have $|e^1 \wedge e^2|^2 = |e^1 \wedge e^3|^2 = \lambda^{-2}$, $|e^2 \wedge e^3|^2 = 1$. Hence

$$\text{(4.4)} \qquad |F|^2 = \lambda^{-2}|F_{12}|^2 + \lambda^{-2}|F_{13}|^2 + |F_{23}|^2 .$$

Integrating over S^3 then gives the Yang-Mills Lagrangian as a fourth degree polynomial on $W = \mathbb{R}^3$.

To make this a time-dependent problem we again consider the trivial SU(2) bundle P over $S^3 \times \mathbb{R}$, now with the metric

$$\text{(4.5)} \qquad g = \lambda^2(t) e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + dt \otimes dt$$

which, as above, represents the most general H-invariant conformal class on $S^3 \times \mathbb{R}$. We will assume that $\lambda > 0$ is a bounded function. Lemma 2.1 again shows that after a gauge transformation the connection form becomes $\omega = \sum e^i \otimes A_i(t)$ where

$$\text{(4.6)} \qquad A_i(t) = \left[\left(\frac{x(t)+1}{2}\right) e_i + \Phi_i(t) \right] .$$

Then

$$(4.7)\quad |\dot{A}|^2 = \lambda^{-2}|\dot{A}_1|^2 + |\dot{A}_2|^2 + |\dot{A}_3|^2 = \lambda^{-2}\left(\frac{\dot{x}}{2}\right)^2 + 2\left(\frac{\dot{x}}{2} - \dot{y}\right)^2 + 2(\dot{z})^2 .$$

Substituting (4.2), (4.4), (4.6) and the volume form into (2.2) converts the Yang-Mills action into the Lagrangian of a 3-dimensional mechanical system — albeit a complicated one.

We can simplify the Lagrangian using the methods of classical mechanics. For this we introduce new coordinates (w, r, θ) defined by

$$(4.8)\quad \begin{cases} x = 2w - 1 , \\ w + y = r\cos\theta , \\ z = r\sin\theta . \end{cases}$$

Then $(\dot{w} + \dot{y})^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2$ and the Yang-Mills action (2.2) becomes

(4.9)

$$YM(w, r, \theta) = \int_{-\infty}^{\infty} \frac{1}{2}\lambda^{-1}\dot{w}^2 + \lambda\dot{r}^2 + \lambda r^2\dot{\theta}^2 + 4\lambda^{-1}r^2(w-1)^2 + 2\lambda(r^2 - w)^2\, dt .$$

We now follow the usual procedure of Hamiltonian mechanics. From the Lagrangian density L (the integrand in the above expression) we obtain conjugate momenta

$$(4.10)\qquad p_w = \frac{\partial L}{\partial \dot{w}} = \lambda^{-1}\dot{w}, \quad p_r = \frac{\partial L}{\partial \dot{r}} = 2\lambda\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2\lambda r^2\dot{\theta} .$$

The Hamiltonian $H = \sum p_i\dot{q}_i - L$ is then

$$(4.11)\qquad H = \tfrac{1}{2}\lambda^{-1}\dot{w}^2 + \lambda\dot{r}^2 + \lambda r^2\dot{\theta}^2 - 4\lambda^{-1}r^2(w-1)^2 - 2\lambda(r^2 - w)^2 .$$

The equations of motion are then obtained by Hamilton's equations. In particular, $\dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0$, so $p_\theta = 2\lambda r^2\dot{\theta}$ is constant — conservation of "angular momentum" for this system. But if the action is finite then

$$4\int_{-\infty}^{\infty} \lambda r^2\dot{\theta}^2\, dt = \int_{-\infty}^{\infty} \frac{p_\theta^2}{\lambda r^2}\, dt = p_\theta^2 \int_{-\infty}^{\infty} \frac{dt}{\lambda r^2}$$

is finite. Because λ is bounded this means that either $p_\theta = 0$ or $r(t) \to \infty$ as $t \to \pm\infty$. Thus any finite action solution that is bounded (as a path in W) has $p_\theta \equiv 0$. But then $\theta = $ const. and L and H are independent of θ. Thus we can restrict attention to one value of θ, say $\theta = 0$. This is equivalent to taking $z = 0$.

Remark 4.1 This conserved angular momentum arises because there is a subgroup of the gauge group that preserves the space of $S^3 \times S^1$-invariant connections. This subgroup is isomorphic to S^1 and acts by the angle θ in (4.8).

This reduction leaves us with a mechanical system in the variables are w and $r = w + y$; we will henceforth denote this reduced configuration space by $W = \mathbb{R}^2$.

To simplify notation we will write the corresponding momenta as $p = p_w$ and $q = p_r$. The Lagrangian is the Yang-Mills action

$$YM(w, r) = \int_{-\infty}^{\infty} \frac{1}{2}\lambda^{-1}\dot{w}^2 + \lambda\dot{r}^2 + 4\lambda^{-1}r^2(w-1)^2 + 2\lambda(r^2 - w)^2\, dt\,, \tag{4.12}$$

the Hamiltonian is

$$H = \frac{\lambda}{2}p^2 + \frac{q^2}{4\lambda} - 4\lambda^{-1}r^2(w-1)^2 - 2\lambda(r^2 - w)^2\,, \tag{4.13}$$

and the equations of motion are

$$\begin{cases} \dot{w} = \lambda p \\ \dot{r} = \dfrac{q}{2\lambda} \end{cases} \quad \begin{cases} \dot{p} = 8\lambda^{-1}r^2(w-1) - 4\lambda(r^2 - w) \\ \dot{q} = 8r[\lambda^{-1}(w-1)^2 + \lambda(r^2 - w)]\,. \end{cases} \tag{4.14}$$

Note that the Hamiltonian is a function of time, so the energy $E = H$ will not be conserved unless λ is constant.

Theorem 4.2 *For any solution* (w, r) *of* (4.14) *the connection given by* $A(t) = w(t)e_1 + r(t)(e_2 + e_3)$ *in the temporal gauge has curvature*

$$F = 2r(w-1)(e^1 \wedge e^2 \otimes e_3 - e^1 \wedge e^3 \otimes e_2) + 2(r^2 - w)e^2 \wedge e^3 \otimes e_1 \tag{4.15}$$

and is a Yang-Mills field on $S^3 \times \mathbb{R}$ *with respect to the metric* (4.5). *If the action* (4.12) *is finite and* $\lambda(t) \equiv 1$ *off a compact set, then the energy density* $|F^\omega|^2 = |\dot{A}|^2 + |F^A|^2$ *decays exponentially fast as* $t \to \pm\infty$ *and the connection extends via stereographic projection to a connection on* S^4.

Proof. The first statement again follows from the symmetric criticality principle, or directly from a calculation similar to the proof of Theorem 2.2. We omit the details. If the action is finite and $\lambda(t) \equiv 1$ off a compact set then (2.5) gives a conformal transformation between $(S^3 \times \mathbb{R}, g)$ and S^4 with two points removed and with the metric $(\Psi^{-1})^* g$. The metric extends over the two points and, by Uhlenbeck's theorem, so does the connection, giving a Yang-Mills field on a bundle over S^4 with this metric. Since $|F^\omega|^2$ is bounded on S^4 its pullback $|F^\omega|^2 = |\Psi^* F^\omega|^2 = (\det \Psi_*)^{-4}|F^\omega|^2$ is bounded on $S^3 \times \mathbb{R}$, so $|F^\omega|^2$ decays exponentially. □

Since the solution extends over S^4 its instanton number k is an integer. As in (2.6) k is found by evaluating the Chern-Simons function between the initial and final points of the trajectory. Specifically, from (4.1), (4.2) and (4.8) we obtain

$$k = \frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{S^3} \langle F^A \wedge \dot{A}\rangle\, dt = \mathrm{CS}(r, w)\Big|_{(w_0, r_0)}^{(w_1, r_1)} \tag{4.16}$$

where the Chern-Simons function is $\mathrm{CS}(r, w) = w^2 - 2r^2(w-1)$.

Again the solutions can be visualized as the motions of a ball rolling in the potential $-P = 4\lambda^{-1}r^3(w-1)^2 + 2\lambda(r^2 - w)^2$ (Fig. 3). Note that the potential is invariant under the reflection $r \mapsto -r$; this is the gauge symmetry discussed in the remark above after restricting to $z = 0$. The points $(w, r) = (0, 0)$, $(1, 1)$ and $(1, -1)$ are gauge equivalent and correspond to the trivial connection. Any finite action

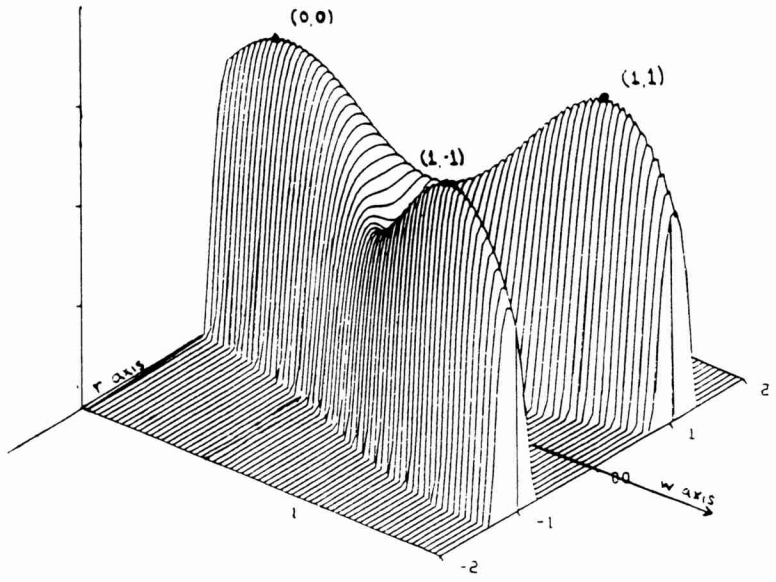

Fig. 3

Yang-Mills field on $S^3 \times \mathbb{R}$ must be flat at infinity, so corresponds to a solution curve beginning at one of these three points and ending at another. The aim of the next several sections is to find such solutions.

5 Anti-self-dual solutions

The Eqs. (4.12)–(4.14) are a 4-dimensional Hamiltonian system on the phase space $T^* W = T^*\mathbb{R}^2$. There are two 2-dimensional invariant subsystems corresponding to the solutions of the self-dual (SD) and anti-self-dual (ASD) equations. We examine these next, focusing on the ASD system (the SD system is entirely similar).

For the metric (4.5), the star operator on two-forms is given by $*(e^1 \wedge e^2) = \lambda^{-1} e^3$, $*(e^1 \wedge e^3) = -\lambda^{-1} e^2$, and $*(e^2 \wedge e^3) = \lambda e^1$. The connection of Theorem 4.2 therefore has

$$\dot{A}(t) = \dot{w} e^1 \otimes e_1 + \dot{r}(e^2 \otimes e_2 + e^3 \otimes e_3) \tag{5.1}$$

and

$$*F = 2r(w-1)\lambda^{-1}(e^2 \otimes e_2 + e^3 \otimes e_3) + 2\lambda(r^2 - w)e^1 \otimes e_1 \,, \tag{5.2}$$

so the duality equations $\dot{A} \pm *F = 0$ (the top sign for self-dual, the bottom sign for anti-self-dual) are

$$\begin{cases} \dot{w} = \mp 2\lambda(r^2 - w) \\ \dot{r} = \mp 2\lambda^{-1} r(w-1) \end{cases} \qquad \begin{cases} p = \mp 2(r^2 - w) \\ q = \mp 4r(w-1) \,. \end{cases} \tag{5.3}$$

Substituting into the Hamiltonian (4.13) shows that $H \equiv 0$ for self-dual solutions.

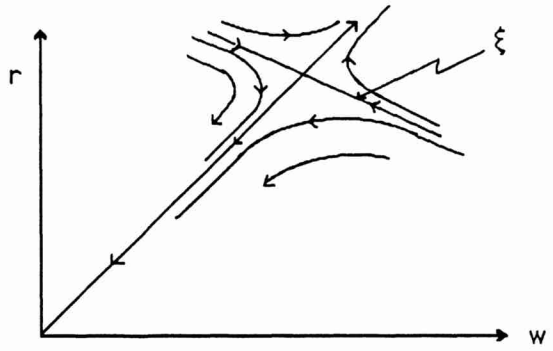

Fig. 4

When $\lambda \equiv 1$ we already know several solutions form §2: $(w, r) = (0, 0)$ or (1 ± 1) (all corresponding to the trivial connection), the instanton solution $w = r = \frac{1}{2}(1 + \tanh(t))$ from $(0, 0)$ to $(1, 1)$, and the anti-instanton solution $w = r = \frac{1}{2}(1 + \tanh(-t))$ from $(1, 1)$ to $(0, 0)$. The flow portrait the ASD system near the point $(1, 1)$ — which will be of interest later — is found by linearizing (5.3); the result is shown in Fig. 4. There are two ASD flow lines coming out of $(1, 1)$: the instanton solution above, and the unbounded solution $w = r = \frac{1}{2}(1 + \coth(t))$. There are also two flow curves coming into $(1, 1)$ with tangent directions $\pm(2, -1)$; for later use we denote the one coming in from the lower right by ξ.

The following lemma proves the existence of H-invariant ASD solutions for quite general H-invariant metrics.

Lemma 5.1 *If $0 < \lambda(t) \leqq 1$ and $\lambda \equiv 1$ off a compact set, then there is a solution curve of the system (5.3) from $A = (1, 1)$ to $B = (0, 0)$ with $0 \leqq r(t) \leqq w(t)$ and $\dot{w}(t) \leqq 0$ for all t.*

Proof. Let T be the largest time such that $\lambda \equiv 1$ for all $t \leqq T$. Then for $t \leqq T$, $w(t) = r(t) = \frac{1}{2}(1 + \tanh(t))$ is a solution curve of (5.3) starting at A and flowing to some point $a = (w(T), w(T))$.

Now set $C = (0, 1)$, $\varepsilon = \operatorname{dist}(A, a)$, and choose $0 < \delta \leqq [\sup(\lambda + \lambda^{-1})]^{-1}$. Let V be the closed triangle ABC with the balls $B(A, \varepsilon)$ and $B(B, \delta)$ removed. The boundary of V is the union of the closed arcs $\alpha = \partial B(A, \varepsilon) \cap V$ and $\beta = \partial B(B, \delta) \cap V$ and the segments I_1, I_2, and I_3 (see Fig. 5). The (time-dependent) vector field (5.3) vanishes nowhere on V, points outward along α, and points inward (or is tangent) along the rest of the boundary.

The flow from a stays in V and eventually intersects β at some point q (otherwise the backward flow gives a continuous map from the connected set β to the disconnected set $(I_1 \cup \alpha \cup I_2 \cup I_3) - \{a\}$). Since δ is arbitrarily small, this same argument shows that the forward flow from q stays in the sector $B(B, \delta) \cap$ ABC. To prove convergence, consider the "Liapunov function" $f(t) = w^2 + r^2$ which satisfies $\dot{f} = 4wr^2(\lambda + \lambda^{-1}) - \lambda(w^2 + r^2)$. Then for $w \geqq 0$ we have $-\lambda f \leqq \dot{f} \leqq -\frac{\lambda}{2} f$ and, after integrating, $Ce^{-\lambda t} \leqq f(t) \leqq Ce^{-\lambda t/2}$. Hence the flow from q converges to the origin as $t \to \infty$. This gives the desired solution, and (5.3) implies that $\dot{w} \leqq 0$ along this solution curve. □

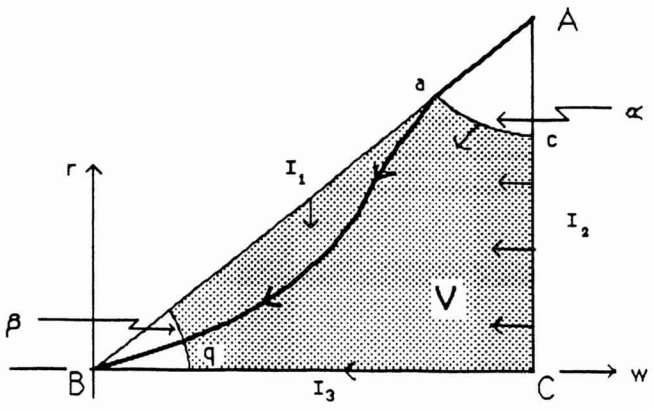

Fig. 5

6 The full Yang-Mills system with $\lambda = 1$

Introduce new "momentum" variables in phase space T^*W, replacing p and q by

$$\begin{cases} \phi = p - 2(r^2 - w) \\ \psi = q - 4r(w - 1) \ . \end{cases} \tag{6.1}$$

Then $dw \wedge d\phi + dr \wedge d\psi = dw \wedge dp + dr \wedge dq$, so $(w, r, p, q) \mapsto (w, r, \phi, \psi)$ is a canonical transformation. Of course, we could equally well work with the variables

$$\begin{cases} \phi^+ = p + 2(r^2 - w) \\ \psi^+ = q + 4r(w - 1) \ . \end{cases} \tag{6.2}$$

and make the corresponding canonical transformation. We will make the first choice, henceforth using the coordinates (w, r, ϕ, ψ) on T^*W and regarding ϕ^+ and ψ^+ as auxiliary functions that help simplify formulas. Let π denote the projection $\pi : (w, r, \phi, \psi) \to (w, r)$.

In these new coordinates, the Hamiltonian (4.13) becomes

$$\begin{aligned} H &= \frac{\lambda}{2}\phi[\phi + 4(r^2 - w)] + \frac{1}{4\lambda}\psi[\psi + 8r(w - 1)] \\ &= \frac{\lambda}{2}\phi\phi^+ + \frac{1}{4\lambda}\psi\psi^+ \end{aligned} \tag{6.3}$$

and the equations of motion (calculated from (4.14) or directly from the above H) are

$$\begin{aligned} \dot{w} &= \frac{\partial H}{\partial \phi} = \lambda(\phi + 2(r^2 - w)) \qquad \dot{\phi} = -\frac{\partial H}{\partial \omega} = 2\lambda\phi - \frac{2}{\lambda} r\psi \\ \dot{r} &= \frac{\partial H}{\partial \psi} = \frac{1}{2\lambda}(\psi + 4r(w - 1)) \qquad \dot{\psi} = -\frac{\partial H}{\partial r} = -4\lambda r\phi - \frac{2}{\lambda}(w - 1)\psi \ . \end{aligned} \tag{6.4a–d}$$

Furthermore, the subspaces of self-dual and anti-self-dual solutions are neatly described by

$$M_{\text{ASD}} = \{\phi = \psi = 0\} \quad \text{and} \quad M_{\text{SD}} = \{\phi^+ = \psi^+ = 0\} \,. \tag{6.5}$$

It follows immediately that M_{SD} and M_{ASD} are invariant submanifolds that are graphs over the (w, r) plane, and that $H \equiv 0$ on each.

In the next section we will investigate the global behavior of the flow lines emanating from the point $p = (1, 1, 0, 0)$ in T^*W. These correspond to the motions of a ball that starts at rest at the point $(1, 1)$ in the (r, w) plane—the top of the mountain in Fig. 3. These flow lines are difficult to study because they take infinitely long to emerge from a neighborhood of p. Hence we first examine the flow in a neighborhood of p when $\lambda \equiv 1$.

Linearizing the flow (6.3) at p with $\lambda \equiv 1$ and diagonalizing, one sees that the tangent space of the phase space at p splits into a two dimensional unstable subspace E_- and a two dimensional stable subspace E_+:

$$T_p(T^*W) = E_- \oplus E_+ \quad \text{where} \begin{cases} E_- = \text{span}\{(1, 1, 0, 0), (2, -1, 16, -16)\} \\ E_+ = \text{span}\{(1, 1, 0, 0), (1, 1, -4, -8)\} \,. \end{cases} \tag{6.6}$$

By the Stable Manifold Theorem ([AR]) the unstable manifold U of p (i.e. the set of flow lines that converge to p at $t \to -\infty$) is a smooth immersed submanifold with $T_p U = E_-$. The next proposition provides a way of parameterizing those flow lines in U whose projections into the (w, r) plane emerge from $(1, 1)$ and stay in the set V of Fig. 5.

Let $\gamma(\varepsilon, \theta)$ be the portion of the circle $\left(1 + \varepsilon\cos\left(\frac{5\pi}{4} + \theta\right), 1 + \varepsilon\sin\left(\frac{5\pi}{4} + \theta\right)\right)$ around $(1, 1)$ in the (w, r) plane that lies below both the line $w = r$ and the flow curve ξ of Fig. 4 (thus, for small ε, θ varies from 0 to approximately $\frac{5\pi}{6}$).

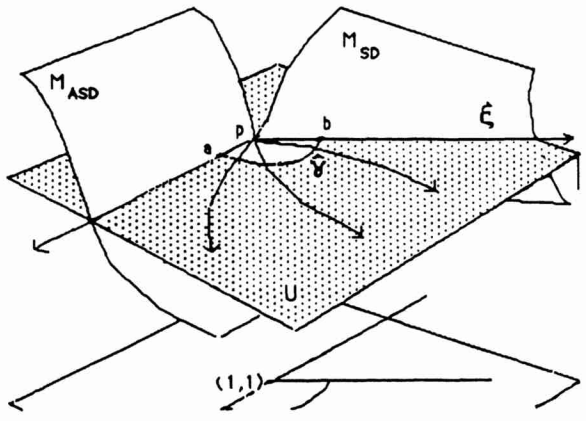

Fig. 6

Proposition 6.1 *Suppose that* $\lambda \equiv 1$ *for* $t \leqq 0$. *Then for any sufficiently small* $\varepsilon > 0$ *there is a curve* $\hat{\gamma} = \hat{\gamma}(\theta)$ *in* U *which projects to the arc* $\gamma(\varepsilon, \theta)$. *Then*
(a) $\phi = \psi = 0$ *at the endpoint* $a = \hat{\gamma}(0)$, *and* $\phi > 0, \psi < 0$ *at all other points of* $\hat{\gamma}$, *and*
(b) $\phi^+ = \psi^+ = 0$ *at the other endpoint of* $\hat{\gamma}$, *and* $\phi^+ < 0, \psi^+ < 0$ *at all other points of* $\hat{\gamma}$.

Proof. First consider the (non-invariant) subspaces

$$M_\pm = \{\phi^+ + \psi = 0\} \quad \text{and} \quad M_\mp = \{\phi = \psi^+ = 0\}$$

of $T^* W$; these consist of solutions of Eqs. (5.3) with mixed signs. It is straightforward to verify that they are submanifolds of $T^* W$ with $T_p M_\pm = \text{span}\{(1, 0, 4, 0), (0, 1, -8, 0)\}$ and $T_p M_\mp = \text{span}\{(1, 0, 0, -8), (0, 1, 0, 0)\}$, and are therefore both transverse to U at p. Thus in a sufficiently small neighborhood of p we have $M_\pm \cap U = M_\mp \cap U = \{p\}$.

The unstable manifold U is transverse to the projection π from $T^* W$ to the (w, r) plane, so is locally a graph over the (w, r) plane. Thus for sufficiently small ε we can lift the arc $\gamma(\varepsilon, \theta)$ (defined above) to U and have $\hat{\gamma} \cap M_\pm = \hat{\gamma} \cap M_\mp = \varnothing$. Note also that H is conserved along the flow (since $\lambda \equiv 1$) and $H(p) = 0$, so $H \equiv 0$ on U.

Now consider the zeros of ϕ and ψ along $\hat{\gamma}$. If $\phi(q) = 0$ at some $q \in \hat{\gamma}$ then by (6.3) we have $\psi(q)\psi^+(q) = 4\lambda H(q) = 0$. But if $\psi^+(q) = 0$ then q lies on $M_\mp \cap U$, contradicting the above. The remaining alternative, $\phi(q) = 0$, means that $q \in M_{\text{ASD}} \cap U$, i.e. q is an unstable point for p in the ASD flow. The results of §4 then imply that q must be the endpoint a of $\hat{\gamma}$ above the line $r = w$. A similar argument holds for ψ. Thus the sign of both ϕ and ψ is constant along $\hat{\gamma} - \{a\}$.

Finally, we can determine the sign of ϕ and ψ be examining their values at the endpoint $b = \hat{\gamma} \cap \xi$. Since ξ is a stable curve for p in the ASD flow (see Fig. 4) it is an unstable curve for the SD flow. Hence ξ lies in $M_{\text{SD}} \cap U$. The endpoint b therefore lies on M_{SD} over a point $\pi(b)$ at which $w > 1$ and $r < 1$ (ε is small and the initial tangent to the curve $M_{\text{SD}} \cap U$ is $(1, -\frac{1}{2}, 8, -8)$). Hence at b—and therefore on $\hat{\gamma} - \{a\}$—we have $\phi = \phi^+ - 4(r^2 - w) = -4(r^2 - w) > 0$ and $\psi = \psi^+ - 8r(w-1) = -8r(w-1) < 0$.

The argument for ϕ^+ and ψ^+ is analogous: both vanish at $b \in M_{\text{SD}}$ and have no other zeros on $\hat{\gamma}$, and one verifies that $\phi^+(a) < 0$ and $\psi^+(a) < 0$. □

7 The global behavior of solutions

Fix a positive function $\lambda(t)$ with $\lambda \equiv 1$ for $t \leqq 0$, $\frac{1}{2} \leqq \lambda \leqq 1$ and $\dot{\lambda} \leqq 0$ for all t. From §5 we know that there is an ASD path $z(t)$, unique up to time translation, from (1, 1) to (0, 0). Let $\hat{\alpha}$ be the portion of the arc $\hat{\gamma} \subset U$ (defined in Proposition 6.1) between $a = \hat{\gamma}(0)$ and $c = \hat{\gamma}\left(\frac{\pi}{4}\right)$. Write $\pi(c) = (1, r_c)$ and let V be the region in the (w, r) plane bounded by $\alpha = \pi(\hat{\alpha})$, the line segments $I_2 = \{(1, r) \mid 0 \leqq r \leqq r_c\}$ and $I_3 = \{(w, 0) \mid 0 \leqq w \leqq 1\}$, and the ASD curve $z(t)$ (see Figs. 5 and 7). Our goal in this section is to study the global behavior of the trajectories from $\hat{\alpha}$ as they pass through $\pi^{-1}(V)$.

Lemma 7.1 *Let* $x(t)$ *be the flow with initial point* $x_0 \in \hat{\alpha} - \{a\}$. *The projection* $\pi(x(t))$ *either exits* V *through* $I_2 \cap I_3$ *at some finite time* T, *or remains in* V *for all*

time and converges to $(0, 0)$ *as* $t \to \infty$. *Furthermore,* $\phi > 0$ *and* $\psi < 0$ *at each point of the flow with* $\pi(x(t)) \in V$.

Proof. By Proposition 6.1 the flow from any such point x_0 begins with $\phi > 0$ and $\psi < 0$. Write $\phi(t)$ for $\phi(x(t))$, etc., and let

$$T = \sup\{t \mid \psi(t) < 0 \quad \text{and} \quad \pi(x(t)) \in V\} .$$

Then by (6.4) we have $\dot{\phi} \geqq 2\lambda\phi \geqq \phi$ for $0 \leqq t \leqq T$, so

$$\phi(t) \geqq \phi_0 e^t \geqq 0 \quad \forall t \in [0, T] . \tag{7.1}$$

Putting this into the last equation in (6.4) gives

$$\dot{\psi} = -4\lambda r\phi(t) - \frac{2}{\lambda}(w - 1)\psi(t) \leqq -4\lambda\phi_0 r(t) e^t \quad \forall t \in [0, T] . \tag{7.2}$$

This implies that ψ remains negative as long as $\pi(x(t)) \in V$. But then T is simply $\sup\{t \mid \pi(x(t)) \in V\}$, so (7.1) shows that ϕ remains positive as long as $\pi(x(t)) \in V$.

We can now compare the expressions for $\dot{w}$ and $\dot{r}$ in (6.4) with the corresponding expressions (5.3) for the ASD flow. The inequalities

$$\dot{w} = \lambda(\phi + 2(r^2 - w)) \geqq 2\lambda(r^2 - w) \quad \text{and}$$

$$\dot{r} = \frac{1}{2\lambda}(\psi + 4r(w - 1)) \leqq \frac{2r}{\lambda}(w - 1) \leqq 0$$

show that at each point where $\dot{w} < 0$ the slope $\dot{r}/\dot{w}$ of $\pi(x(t))$ is greater than the slope of the ASD flow (5.3) at each point in V. Hence $\pi(x(t))$ cannot exit V through $z(t)$. The first statement of the lemma then follows from the proof of Lemma 5.1. □

We next focus attention on those trajectories from $\hat{\alpha}$ that pass near $w = 0, r = 0$. The linearization of (4.14) or (6.4) at $(0, 0, 0, 0)$ with λ constant can be viewed as the equations of motions for two uncoupled harmonic oscillators; one aligned along the x-axis, the other along the r-axis. Each has its Hamiltonian; this gives two natural constants of motion for the linearized flow. The key to analyzing the flow curves near $(0, 0)$ is to write the Hamiltonian (6.3) as

$$H = \frac{\lambda}{2} H_1 + \frac{1}{4\lambda} H_2 \tag{7.3}$$

where

$$H_1 = \phi\phi^+ = \phi[\phi + 4(r^2 - w)] \quad \text{and} \quad H_2 = \psi\psi^+ = \psi[\psi + 8r(w - 1)] . \tag{7.4}$$

For fixed λ, H_1 and H_2 are constant multiples of the Hamiltonians of the linearized system at the origin. The next lemma shows that the sign of H_1 and H_2 is constant along the trajectories through $\hat{\alpha}$ and that the energy increases when λ decreases.

Lemma 7.2 *Let* $x(t)$ *be a trajectory of* (6.4) *passing through* $\hat{\alpha} - \{a\}$. *Then* $H_1 < 0$, $H_2 > 0$, $H \geqq 0$ *and* $|H_1| \leqq 2H_2$ *at each point along* $x(t)$ *with* $\pi(x(t)) \in V$.

Proof. From (7.4) and (6.4) one finds that

$$\dot{H}_1 = \frac{8r}{\lambda}\left[2r(w-1)\phi - (r^2 - w)\psi \right]. \tag{7.5}$$

The previous lemma and the fact that $\pi(x(t)) \in V$ imply that both of these terms are non-positive, so $\dot{H}_1 \leqq 0$. But $H_1 = \phi\phi^+ < 0$ on $\hat{\alpha} - \{a\}$ by Proposition 6.1, so $H_1(t) < 0$ up until the time that the trajectory leaves V. The proof that $H_2 > 0$ is similar. Finally, differentiating (7.3) and using (6.4) gives

$$\dot{H} = \lambda^{-1}\dot{\lambda}\left\{ \frac{\lambda}{2} H_1 - \frac{1}{4\lambda} H_2 \right\}. \tag{7.6}$$

This shows, first of all, that H is constant for $t \leqq 0$ ($\lambda \equiv 1$ for $t \leqq 0$). But $x(t) \to p = (1, 1, 0, 0)$ as $t \to -\infty$, so $H(0) = H(p) = 0$. Integrating (7.6) then gives $H \geqq 0$. But then $0 \leqq \frac{2}{\lambda} H = H_1 + H_2/2\lambda^2$, so $|H_1| = -H_1 \leqq 2H_2$. □

The next step is to get a bound on the time T when the trajectory exists V. For trajectories close to the ASD curve this can be done by estimating the integrals of the system (6.4).

Trajectories near the ASD one are nicely parameterized by the angle θ introduced in Proposition 6.1; $\theta = 0$ is the ASD curve and each $\theta > 0$ designates a nearby trajectory that satisfies the hypotheses of Lemma 7.1. Fix w_0 with $0 < w_0 \leqq \frac{1}{8}$. The ASD curve flows to the origin, so for all nearby trajectories $\pi(x(t))$ crosses the line $w = w_0$ at some point (w_0, r_0); after a time translation we can assume that this occurs at $t = 0$. We will denote this family of trajectories by $x_\theta(t)$. By Lemma 5.1 the height $\bar{r}_0$ where the ASD curve $z(t) = \pi(x_0(t))$ crosses $w = w_0$ satisfies $0 < \bar{r}_0 \leqq w_0$. We also know that the functions ϕ, ψ, H_1, and H_2 vanish along $x_0(t)$. Hence we can take $\theta > 0$ small enough that

$$\begin{aligned}
&\text{(a)} \quad \frac{\bar{r}_0}{2} < r_0 \leqq w_0 \leqq \frac{1}{8} \\
&\text{(b)} \quad |H_1(0)| < 4(w_0 - \bar{r}_0^2)^2 \\
&\text{(c)} \quad H_2(0) < (\bar{r}_0)^2 \\
&\text{(d)} \quad H_2(0) < w_0^2 10^{-11}.
\end{aligned} \tag{7.7}$$

The first three of these conditions allow us to bound ϕ and ψ in terms of H_1 and H_2, as follows. We first solve the quadratic Eqs. (7.4) (each has two solutions, but only one vanishes along the ASD curve):

$$\begin{cases}
\phi = 2(w - r^2)\left[1 - \sqrt{1 + \dfrac{H_1}{4(w - r^2)^2}} \right] \\
\psi = 4r(1 - w)\left[1 - \sqrt{1 + \dfrac{H_2}{16r^2(1 - w)^2}} \right].
\end{cases} \tag{7.8}$$

We can then take $z = H_1/4(w - r^2)^2$ in the first equation and $z = H_2/16r^2(1 - w)^2$ in the second. Conditions (7.7) imply that $|z| \leqq 1$ for $t = 0$ and that $w_0(1 - w_0) \leqq w_0 - r_0^2 \leqq w_0$. Since $|1 - \sqrt{1 + z}|$ lies between $\frac{|z|}{4}$ and $|z|$ for $|z| \leqq 1$ we obtain

$$\frac{|H_1(0)|}{8w_0} \leqq \phi_0 \leqq \frac{4|H_1(0)|}{7w_0} \quad \text{and} \quad \frac{H_2(0)}{16r_0} \leqq |\psi_0| \leqq \frac{2H_2(0)}{7r_0} . \tag{7.9}$$

Combining these with the inequality $|H_1| \leqq 2H_2$ of Lemma 7.2 yields

$$\phi_0 \leqq 19|\psi_0| . \tag{7.10}$$

Lemma 7.3 *Let $x(t) = (w(t), r(t), \phi(t), \psi(t))$ be a trajectory satisfying* (7.7) *and λ as described in the first sentence of this section. Then $\pi(x(t))$ crosses the w-axis at a finite time T satisfying* (7.18) *below. Furthermore, $\dot{r}(t) < 0$ for $0 \leqq t \leqq T$ and $w(T) \leqq 2w_0$.*

Proof. Let $V(2w_0)$ be the region in the (w, r) plane bounded by the w-axis, the line $w = 2w_0$, and the projected ASD curve $\pi(x_0(t))$, and let S be the largest t such that $\pi(x(t)) \in V(2w_0)$ for $0 \leqq t \leqq S$. Set $u_0 = (1 - 2w_0)$, so $\frac{3}{4} \leqq u_0 \leqq 1$ and $u_0 \leqq 1 - w \leqq 1$ in $V(2w_0)$. We will use the results of the previous two lemmas and the inequalities (7.7)–(7.10) to obtain bounds on the solutions of the system (6.4) for $0 \leqq t \leqq S$. For example, Eq. (6.4d) and the fact that $\phi > 0$ and $\psi < 0$ show that $-\dot{\psi} \geqq 2u_0|\psi|$ on $V(2w_0)$, so

$$-\psi(t) = |\psi(t)| \geqq |\psi_0| e^{2u_0 t} \tag{7.11}$$

Putting this into (6.4b) then shows that $\dot{r} \leqq -\frac{1}{2}|\psi_0| e^{2u_0 t} - 2u_0 r$ for $0 \leqq t \leqq S$. Integrating again yields

$$r(t) \leqq (r_0 + \tfrac{1}{6}|\psi_0|) e^{-2u_0 t} - \tfrac{1}{6}|\psi_0| e^{2u_0 t} \leqq r_0 e^{-2u_0 t} . \tag{7.12}$$

Now set $\xi = \phi^2 + \psi^2$. Equations (6.4) imply that $\dot{\xi} \leqq 10\xi$ whenever $0 \leqq t \leqq S$. Since $\xi_0 \leqq 400\psi_0^2$ by (7.10) we obtain, in particular,

$$|\psi(t)| \leqq 20|\psi_0| e^{5t} . \tag{7.13}$$

Turning next to Eq. (6.4c), we find that $\dot{\phi} \leqq 2\phi + 4r|\psi|$. Using (7.9), (7.12) and (7.13) we therefore have $\dot{\phi} - 2\phi \leqq 23H_2(0) e^{(5-2u_0)t}$, and hence

$$\phi(t) \leqq \phi_0 e^{2t} + 23H_2(0) e^{(5-2u_0)t} . \tag{7.14}$$

Equation (6.4d) gives $|\dot{\psi}| - 4|\psi| \leqq 4r\phi$, which can now be estimated using (7.12) and (7.14) and the inequality $r_0\phi_0 \leqq \frac{8}{7}H_2(0)$ from (7.9). This yields an improved bound on $|\psi|$:

$$|\psi(t)| \leqq [|\psi_0| + 8H_2(0)] e^{4t} . \tag{7.15}$$

This in turn can be used in the inequality $\dot{\phi} - 2\psi \leqq 4r|\psi|$ from (6.4c). Integrating and combining the result with Eq. (7.1) gives the improved bound on ϕ

$$\phi_0 e^t \leqq \phi(t) \leqq \phi_0 e^{2t} + \frac{3}{2w_0} H_2(0) e^{(2+4w_0)t} \leqq \frac{3}{w_0} H_2(0) e^{(2+4w_0)t} . \tag{7.16}$$

Finally, from Eq. (6.4a) we get $\dot{w} + w \leqq \phi + 2r^2 \leqq \frac{3}{w_0} H_2(0) e^{(2+4w_0)t} + 2r_0^2 e^{(-4u_0)t}$. Integrating one last time yields

$$w(t) \leqq w_0 \left[\frac{9}{8} + \left(\frac{H_2(0)}{w_0^2} \right) e^{(2+4w_0)t} \right]. \tag{7.17}$$

These bounds are valid for $0 \leqq t \leqq S$. If the trajectory crosses the w-axis at a time $T \leqq S$ then setting $r(T) = 0$ in (7.12) and using (7.9) and (7.7c) gives the bound

$$e^{4u_0 T} \leqq \left[\frac{100 w_0^2}{H_2(0)} \right]. \tag{7.18}$$

But for this T Eqs. (7.17) and (7.7a, d) show that

$$\left[\frac{H_2(0)}{w_0^2} \right] e^{(2+4w_0)t} \leqq 100 \left[\frac{H_2(0)}{100 w_0^2} \right]^{1/6} \leqq \frac{7}{8},$$

so $w(t) \leqq 2w_0$ at each $t \in [0, T]$. Thus $\pi(x(t))$ remains in $V(2w_0)$ for $0 \leqq t \leqq T$, so reaches $r = 0$ at some time $\tau \leqq T$ satisfying (7.18). □

Proposition 7.4 *Suppose that λ is as described in the first sentence of this section, and that there is a $\lambda_0 < 1$ such that $\frac{1}{2} \leqq \lambda \leqq \lambda_0$ for all sufficiently large t. Then there is a small $\theta_0 > 0$ such that for any $0 < \theta \leqq \theta_0$ the projection $\pi(x(t))$ of the trajectory $x(t) = x_\theta(t)$ through $\gamma(\theta)$ crosses the w-axis at a point where $\dot{w} < 0$ and $\dot{r} < 0$.*

Proof. Let $x(t)$ be a trajectory as in Lemma 7.3. From Eq. (6.4a) we have $\dot{w} \geqq -2\lambda_0 w$ for $t \geqq 0$ and therefore

$$w(t) \geqq w_0 e^{-2\lambda_0 t}. \tag{7.19}$$

Now evaluate (6.4a) at the time T when $r(T) = 0$, where $\dot{r}(T) < 0$ by the previous lemma. By making use of (7.16), (7.19), and (7.18) we obtain

$$\lambda^{-1} \dot{w}(T) = \phi(T) - w(T) \leqq w_0 e^{-2\lambda_0 T} \left[300 \left(\frac{H_2(0)}{100 w_0^2} \right)^{\alpha} - 1 \right],$$

where $\alpha = (1 - \lambda_0 - 4w_0)/2u_0$. Now fix a w_0 small enough that $1 - \lambda_0 > 4w_0$, so the exponent α is positive. Because $H_2(0) \to 0$ as $\theta \to 0$, we can then choose θ_0 sufficiently small that $\dot{w}(T) < 0$. □

8 The existence of non-minimal Yang-Mills fields

The results of the previous several sections give us enough information about the behavior of the solutions of the system (4.14) to complete the program described at the end of Sect. 4, that is, to prove the existence of trajectories going between the mountaintops in Fig. 4.

Proposition 8.1 *There exists a family of functions $\lambda(t)$ on $S^3 \times \mathbb{R}$ each of which admits a solution $x(t)$ of the system (6.4) that starts at $x(-\infty) = p = (1, 1, 0, 0)$ and flows to $x(\infty) = \bar{p} = (\frac{1}{2}, -1, 0, 0)$. These functions have $\frac{1}{2} \leqq \lambda \leqq 1$ everywhere, with $\lambda \equiv 1$ off a compact set and with $\inf \lambda$ arbitrarily close to 1.*

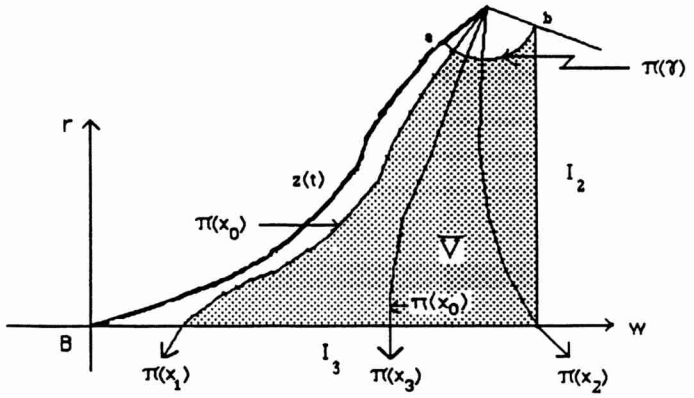

Fig. 7

Proof. Fix $\lambda(t)$ with $\lambda \equiv 1$ for $t \leqq 0$ and $\lambda \equiv \lambda_0 < 1$ for $t \geqq T$. For each sufficiently small ε, Proposition 6.1 produces an arc $\hat{\gamma}(\varepsilon, \theta)$ on the unstable manifold of $p = (1, 1, 0, 0)$. Let $\hat{\gamma}_t$ be the image of $\hat{\gamma}$ under the flow for time t. By making ε smaller we may assume that $\hat{\gamma}_T$ lies in the ball of radius $\frac{1}{2}$ around p. Fix this ε and write $\hat{\gamma}(\theta)$ for $\hat{\gamma}(\varepsilon, \theta)$ and $\gamma(\theta)$ for $\pi(\hat{\gamma}(\varepsilon, \theta))$. Let $a = \hat{\gamma}(0) = \hat{\gamma} \cap M_{\text{ASD}}$ and $c = \hat{\gamma}(\theta_1) = \hat{\gamma} \cap M_{\text{SD}}$ be the endpoints of $\hat{\gamma}$.

According to Proposition 7.4, there is a $0 < \theta_0 < \dfrac{\pi}{4}$ such that the flow through $\hat{\gamma}(\theta_0)$ projects to a curve $\pi(x_0(t))$ that exits the region V (defined at the beginning on §7) at a point $(w_0, 0)$ where $\dot{w} < 0$ and $\dot{r} < 0$.

Let $\bar{V}$ be the region in the (w, r) plane bounded by the w-axis, this curve $\pi(x_0(t))$, the arc $\gamma(\theta)$), and the vertical line through $\pi(b)$ (this is a slight enlargement of V, see Fig. 7). We will consider the projection of the forward flow from $\hat{\gamma}(\theta)$ in $\bar{V}$ for $\theta_0 \leqq \theta \leqq \theta_1$.

First observe that at the endpoint b of $\hat{\gamma}$ we have $w > 1, r^2 - w < 0$, and (c.f. (6.2.))

$$0 = \phi^+ = \phi + 4(r^2 - w) < \phi + 2(r^2 - w)$$

$$0 = \psi^+ = \psi + 8r(w - 1) < \psi + 4r(w - 1)\,.$$

It follows from (6.4) that $\dot{w} > 0$ and $\dot{r} < 0$ at b, so the tangent vector to $\pi(x(t))$ at $\pi(b)$ points out of the region $\bar{V}$.

Next observe that Eq. (6.4b) and the inequality $\psi < 0$ of Lemma 7.1 show that $\dot{r} < 0$ whenever $\pi(x(t)) \in \bar{V}$ with $w < 1$. Thus r is decreasing along the trajectory through this region. But $\gamma(\theta_0)$ lies on the circle of radius ε around $(1, 1)$ with $\theta_0 < \dfrac{\pi}{4}$. We conclude that no trajectory that starts on $\gamma(\theta)$, $\theta_0 \leqq \theta \leqq \theta_1$, can exit $\bar{V}$ through the point $\gamma(\theta_0)$.

The vector field (6.4) has no zeros in $\pi^{-1}(\bar{V})$. Hence the exit map is a continuous map from the connected arc $\gamma(\theta)$, $\theta_0 \leqq \theta \leqq \theta_1$, into $\partial \bar{V}$. Furthermore, as we just observed, the image of this exit map misses the point $\gamma(\theta_0)$. Since the image of a connected set is connected, there must be a θ_2 between θ_0 and θ_1 such that the

projection of the trajectory from $\hat{\gamma}(\varepsilon, \theta_2)$ exits $\bar{V}$ at its lower right corner. At this point the trajectory has slope $\dot{r}/\dot{w} \leqq 0$.

For each θ let $\Phi(\theta)$ be the "exit angle" for the trajectory from $\hat{\gamma}(\theta)$, that is, the angle between the positive w-axis and the tangent vector $\pi(\dot{x}(t))$ at the point where the projected trajectory $\pi(x(t))$ first exits $\bar{V}$. This is a continuous function of θ, and from above we have $\Phi(\theta_0) < 90°$ and $\Phi(\theta_2) \geqq 90°$. Hence there is a θ_3 between θ_0 and θ_2 for which $\Phi(\theta_3) = 90°$. Denote this trajectory by $x_3(t)$ and let T_3 be the time when it crosses the w-axis. The choice of ε made at the beginning of this proof insures that $\lambda(t) \equiv \lambda_0$ for all t near T_3.

Now reflect this solution across the w-axis. Specifically, we let $\lambda(t)$ be as above for $t \leqq T_3$, and extend it to a function on $\mathbb{R}$ that is symmetric about T_3; this is a smooth function since $\lambda \equiv \lambda_0$ near T_3. We then reflect the trajectory $x_3(t)$, $-\infty < t \leqq T_3$, across the plane $r = 0, p = 0$ in the phase space. This reflection preserves the Hamiltonian (4.13) and reverses the sign of the canonical form $dw \wedge dp + dr \wedge dq$. Hence when we also reflect time through $t = T_3$ the reflected trajectory satisfies the equations of motion (6.4). The union of $x_3(t)$ and its reflection is then a trajectory from $x(-\infty) = p$ to $x(\infty) = \bar{p}$. □

The final step is to apply Theorem 4.2 to these solutions to obtain Yang-Mills fields on $S^3 \times \mathbb{R}$ and on S^4 that, according to equation (4.16), have instanton number $k = 0$. The curvature formula (4.15) shows that these fields are not flat and not reducible (at almost every (w, r) the curvature spans the Lie algebra). Hence they are irreducible non-minimal Yang-Mills fields on the trivial bundle. This gives the two main results of this paper.

Theorem 8.2 *There exists a family of metrics on $S^3 \times \mathbb{R}$ each of which admits an irreducible non-minimal Yang-Mills field. These metrics have the form* (4.5) *with $\frac{1}{2} \leqq \lambda \leqq 1$ everywhere, with $\lambda \equiv 1$ off a compact set and with $\inf \lambda$ arbitrarily close to* 1.

Theorem 8.3 *There exists a family of metrics on S^4 each of which admits an irreducible non-minimal Yang-Mills field on the trivial* SU(2) *bundle. There are metrics in this family arbitrarily close to the standard metric.*

Again, because of the conformal invariance of the Yang-Mills equations these are actually theorems about conformal structures (rather than metrics) on $S^3 \times \mathbb{R}$ and S^4. Furthermore, in each case we have actually produced a one-parameter family of solutions; these differ by time translations on $S^3 \times \mathbb{R}$ and by the action of the corresponding one-parameter group of conformal transformations on S^4.

We conclude with some comments, without proof, on the behavior of these Yang-Mills fields as the metric approaches the standard metric on $S^3 \times \mathbb{R}$ or on S^4.

The trajectories obtained in Proposition 8.1 behave nicely as the function $\lambda(t)$ approaches $\lambda \equiv 1$ (this can be seen by integrating differential inequalities as in the proof of Lemma 7.3, and is seen directly by integrating the system (6.4) on a computer). Specifically, suppose that $\lambda_s(t)$, $0 \leqq s \leqq \varepsilon$, is a family of functions on $\mathbb{R}$ that satisfy the hypotheses of Proposition 8.1 and have $\lambda_0 = \inf \lambda(t) \to 1$ as $s \to 0$, and let $\{x_s(t)\}$ be the corresponding solutions. Then as $s \to 0$ the $\{x_s\}$, considered as (unparameterized) curves in the phase space, converge in the C^0 norm to the piecewise linear curve that first joins $p = (1, 1, 0, 0)$ to $q = (0, 0, 0, 0)$ and then joins q to $\bar{p} = (1, -1, 0, 0)$. Moreover, the action (4.12) of $\{x_s\}$ approaches 2. This means that, for metrics on $S^3 \times \mathbb{R}$ very close to the standard one, the solutions resemble an

instanton solution followed by an anti-instanton. Moreover, as $\lambda_0 \to 1$ the solution curves remain in a neighborhood of the origin for longer and longer times. Hence in the resulting Yang-Mills fields on $S^3 \times \mathbb{R}$ the instanton and anti-instanton parts become increasingly far apart. The corresponding family of Yang-Mills fields on the sphere S^4 has an instanton bubbling off at one pole p and an anti-instanton bubbling off at the opposite pole $\bar{p}$, with the connection approaching the trivial connection uniformly on compact sets in $S^4 - \{p, \bar{p}\}$.

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