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A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras

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Summary. Some generalizations of the Lusztig-Lascoux-Schützenberger operators for affine Hecke algebras are considered. As corollaries we obtain Lusztig's isomorphisms from affine Hecke algebras to their degenerate versions, a "natural" interpretation of the Dunkl operators and a new class of differential-difference operators generalizing Dunkl's ones and the Knizhnik-Zamolodchikov operators from the two dimensional conformal field theory.

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Introduction

The first aim of this paper is to consider natural vector versions of the Lusztig operators [Lu3] and the Lascoux-Schützenberger operators [LS1, LS2] and calculate (in the scalar case) the representations of the corresponding affine Hecke algebras in which these operators act. The key point of this calculation is equivalent to some form of the main theorem from [Ka] (we give a new more simple proof of it.) As corollaries one obtains Lusztig's isomorphisms between affine Hecke algebras and their degenerate (graded) versions [Lu1] and a natural construction of the Dunkl differential-difference operators [Du, He1] together with their trigonometric counterparts close to Heckman's operators [He2]. The second aim is a unification of the Dunkl and the Knizhnik-Zamolodchikov operators from [Ch1, Ch 2] taking the vector analogue of the Lusztig operators as a basis. Given a root system $\Sigma \subset \mathbf{R}^n$ and a representation of the corresponding Weyl group

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$W \subset \mathrm{Aut}(\mathbf{R}^n)$, we define a new commutative family of differential-difference operators generalizing both the Knizhnik-Zamolodchikov and the Dunkl operators.

Several preliminary points on affine Hecke algebras and the intertwining operators are worth mentioning. The intertwiners play a very important role in the theory of p-adic representations of unramified principal series. The latter are directly connected with the representations of the corresponding p-adic affine Hecke algebra $\mathscr{H}$ which are induced from characters of the so-called Bernstein-Zelevinsky commutative subalgebra $\mathscr{Y} \subset \mathscr{H}$. Here $\mathscr{H}$ depends on a parameter q which is a power of prime p. However it is quite natural to assume q to be an arbitrary complex number, because the defining relations for $\mathscr{H}$ depends on q algebraically. In several papers (see [Ma, Ka, Ro]) explicit formulas were used for the intertwining operators between the representations induced from conjugated characters with respect to a natural action of W on $\mathscr{Y}$. For example, they were useful to Rogawski in making more lucid the Zelevinsky theorems on p-adic representations of GL_n [Ze]. These intertwiners can be considered as elements of $\mathscr{H}$ satisfying the Coxeter relations of W. The last fact was not formulated in the above papers, but follows directly from them (see [Lu1] and e.g. [Ch 3], where the case $W = \mathfrak{S}_{n+1}$ was considered).

As a consequence one gets an isomorphism π between $\mathscr{Y}[W]$ (the semi-direct product of $\mathscr{Y}$ and $\mathbf{C}[W]$) and $\mathscr{H}$ after some localization of $\mathscr{Y}$. This isomorphism is useless for the most interesting (special) representations of $\mathscr{H}$ because of this localization. Nevertheless, it can be applied to obtain a certain map without denominators from $\mathscr{H}$ to its degeneration $\mathscr{H}'$.

The relations for $\mathscr{H}'$ in the case $W = \mathfrak{S}_{n+1}$ were found for the first time by Murphy (see [Mu]). She defined a commutative subalgebra in $\mathbf{C}[\mathfrak{S}_{n+1}]$, closely connected with the so-called Young's bases for $\mathfrak{S}_{n+1}$, and calculated the cross-relations between its generators and the adjacent transpositions. It was shown in [Dr] (see also [Ch 4]) that her subalgebra is the image of the counterpart $\mathscr{Y}' \subset \mathscr{H}'$ of $\mathscr{Y}$ with respect to a canonical surjection $\mathscr{H}' \to \mathbf{C}[\mathfrak{S}_{n+1}]$. Drinfeld defined $\mathscr{H}'$ for $W = \mathfrak{S}_{n+1}$ as a certain limit of $\mathscr{H}$, when $q \to 1$.

Drinfeld's construction can be extended naturally to arbitrary Weyl groups W. Lusztig gave the general definition of $\mathscr{H}'$ (which he called the graded affine Hecke algebra) in papers [Lu1, Lu2]. The analogues of the above intertwiners can be easily calculated for $\mathscr{H}'$ and coincide with the formulas from the paper [Ch 5] devoted to the W-invariant quantum R-matrices. By the way, the Matsumoto-Rogawski formulas for the intertwiners are closely connected with the basic trigonometric R-matrix (in Jimbo's form).

In [Ch 3, Ch 4] and some other papers it was shown by means of the technique of intertwiners that the classification of the irreducible representations, the theory of Young bases, the character formulas and some other points are quite parallel for $\mathscr{H}'$ and $\mathscr{H}$, when $W = \mathfrak{S}_{n+1}$ and q is generic. It is now possible to explain this coincidence *a priori*.

After some localization we get an isomorphism π': $\mathscr{H}'_{\mathrm{loc}} \xrightarrow{\sim} \mathscr{Y}'_{\mathrm{loc}}[W]$ in the same manner as π. The semi-direct product $\mathscr{Y}'[W]$ of $\mathscr{Y}'$ and $\mathbf{C}[W]$ can be identified with $\mathscr{Y}[W]$ after a suitable completion of $\mathscr{Y}$ and $\mathscr{Y}'$. Then the composition map $\tilde{\pi} = \pi \circ (\pi')^{-1}$ will be an isomorphism between $\mathscr{H}$, $\mathscr{H}'$ both localized and completed. It follows from [Lu1] that the completion (without any localization) is enough to define $\tilde{\pi}$. This completion is compatible with the category of finite-dimensional representations.