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A short topological proof for the symmetry of 2 point homogeneous spaces

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Two point homogeneous spaces have been studied by G. Birkhoff [2], H. Busemann [3], J. Tits [10], and H.C. Wang [11]. Tits and Wang gave a classification of these spaces; it turned out, just from this list, that these spaces were symmetric.

We say that a connected locally compact metric space (M, d) is two point homogeneous if for every pair of points (p_1, q_1) and (p_2, q_2) with $d(p_1, q_1) = d(p_2, q_2)$, there is an isometry T of M such that $T(p_1) = p_2$ and $T(q_1) = q_2$. If M satisifies this condition, then the transitive isometry group of M is a Lie group; furthermore, the isotropy group J_{x_0} describes a sphere as an indicatrix at any tangent space. Consequently M^n is a Riemannian manifold and we work in that class henceforth.

It was a longstanding problem to find a direct proof that M^n is symmetric. In the non compact case, S. Helgason [6] found an elegant direct proof, but his method did not work in the compact case. Other authors have given proofs using methods which were heavily group theoretic; see for example M. Matsumoto [7], T. Nagano [8], and J. Wolf [12]. Recently, Q.-S. Chi [5] proved the so called Ossermann conjecture if n is odd or if $n \equiv 2 \bmod 4$; this also gives a direct proof in these cases. Chi uses vector bundle theory; this is also our approach in this paper. However, there does not exist a simple proof in the literature so far covering all cases.

In this brief note, we give a simple topological proof for this symmetry. The main tool is the following; purely topological, Theorem 1.

Let $S_0^{n-1} \subset \mathbf{R}^n$ be the unit sphere in Euclidean space around the origin 0. We identify the tangent space $T_p S_0^{n-1} = \{v \in \mathbf{R}^n : P \perp v\}$ to provide a natural isomorphism between the tangent spaces at P and the antipodal point $-P$.

Theorem 1 (a) *Let $X(P)$ be a continuous tangent vector field on $S_0^{n-1} \subset \mathbf{R}^n$. Then there exists an antipodal point pair $\pm P_1$ so that $X(P_1) = -X(-P_1)$.*
(b) *In general, let $X(P)$ be a continuous non-zero distribution on S_0^{n-1}. Then there*

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exists an antipodal point pair $\pm P_1$ *such that the subspaces* $X(P_1)$ *and* $X(-P_1)$ *are not independent, i.e.* $\dim(X(P_1) \cap X(-P_1)) > 0$.

Proof. (a) Let $X_{\mathrm{sym}}(P) = \frac{1}{2}(X(P) + X(-P))$ be the symmetric part of X; we must prove there exists P such that $X_{\mathrm{sym}}(P) = 0$. Suppose the contrary and let

$$f(P) = X_{\mathrm{sym}}(P)/|X_{\mathrm{sym}}(P)| : S^{n-1} \to S^{n-1}$$

be the normalized vector field. Consider the map

$$f_* : H_{n-1}(S^{n-1}, \mathbf{Z}_2) = \mathbf{Z}_2 \to H_{n-1}(S^{n-1}, \mathbf{Z}_2) = \mathbf{Z}_2 \,.$$

Since $f(P) \perp P$, so $f_\varepsilon(P) = \sin(\varepsilon) f(P) + \cos(\varepsilon) P$ is a smooth homotopy from f to the identity map. Therefore $f_*(1) = 1$. Let $RP = S/\{\pm 1\}$ be the real projective space and let $\pi : S \to RP$ be the natural projection. Since for

$$\pi_* : H_{n-1}(S^{n-1}, \mathbf{Z}_2) = \mathbf{Z}_2 \to H_{n-1}(RP^{n-1}, \mathbf{Z}_2) = \mathbf{Z}_2$$

we have $\pi_*(1) = 0$, furthermore $f_* = \tilde{f}_* \circ \pi_*$ therefore $f_*(1) = 0$. This contradiction completes the proof of (a). (Using $\mathbf{Z}$-homologies, the following alternative proof can be given: By the homotopy f_ε, $\deg f = 1$ and by the symmetry $f(P) = f(-P)$, $\deg f$ is even.)

(b) Take an arbitrary continuous vector field v on S tangent to X such that v vanishes only at one point, say at P_0. There exists such v, because $D = S \backslash P_0$ is contractible and X is trivial over D. Choose a non-zero section w to X on D and let $\Phi : S \to \mathbf{R}$ be a continuous function satisfying $\Phi^{-1}(0) = P_0$. Then $v(P) = \Phi(P) w(P)$ is a suitable vector field. Since $v(-P_0) \neq 0$; there exists a point-pair $\pm P_1 (\neq \pm P_0)$ such that $v(P_1) = -v(-P_1) \neq 0$, i.e. the subspaces $X(P_1)$ and $X(-P_1)$ are not independent. □

Corollary 1 *Let* $A_P(\cdot)$ *be a continuous operator field on* S_0^{n-1} *(i.e.* $P \in S_0^{n-1}$ *and* $A_P(\cdot) \colon T_P(S_0^{n-1}) \to T_P(S_0^{n-1})$*) such that*

(1) *it is self adjoint:* $A_P^* = A_P$ *and "skew" in the sense:* $A_{-P} = -A_P$ *furthermore;*

(2) *the eigenvalues* $\lambda_1(P) \leqq \lambda_2(P) \leqq \ldots \leqq \lambda_{n-1}(P)$ *of* A_P *are constant along* S_0^{n-1}*. Then* $A_P \equiv 0$ *everywhere on* S_0^{n-1}.

Proof. We have to prove that every λ is zero. Assume the contrary and let λ be a non-zero eigenvalue. Let $V(\lambda, P)$ be the corresponding continuous eigensubspace distribution on S_0^{n-1}. Since $A_{-P} = -A_P$, so $V(\lambda, P) = V(-\lambda, -P)$. Therefore $V(\lambda, P) \perp V(\lambda, -P)$ for all $P \in S_0^{n-1}$. This is impossible because of Theorem 1(b) which proves the statement completely. □

Remark. Property (2) can be replaced by much more weaker assumptions, for instance, by rank A_P = constant or by assuming that the multiplicity of the highest eigenvalue $\lambda_{n-1}(P)$ is constant.

The symmetry of a 2 point homogeneous spaces directly follows from Corollary 1.

Theorem 2 *The locally 2 point homogeneous spaces are locally symmetric.*

Proof. Apply Corollary 1 to the field $A_X(\cdot) := (\nabla_X R)(\cdot, X)X$ where R is the curvature, ∇ is the covariant derivative and X is a unit vector. Fix a point $p \in M^n$ and consider the unit sphere $S_p^{n-1} \subset T_p(M^n)$. By the Bianchi identities, the $A_X(\cdot)$:

$T_X(S_p^{n-1}) \to T_X(S_p^{n-1})$; $X \in S_p^{n-1}$; is a self-adjoint operator field on S_p^{n-1} satisfying the "skew" property $A_{-X} = -A_X$ as well. Since A_X is invariant under the transitive action of the isotropy group J_p on S_p^{n-1}, it has constant eigenvalues. Therefore $A_X = 0$; $(\nabla_X R)(\cdot, X)X = 0$ at any point p which implies the local symmetry $\nabla R = 0$ easily (see [1, Proposition 2.35]). □

Let M^n be a globally 2 point homogeneous space. Since M^n is complete, its universal covering space $\tilde{M}^n$ is globally symmetric. The global symmetry of the base space M^n directly follows from the next stronger statement, where we show that a non-simply-connected globally 2 point homogeneous space has to be a real projective space $\mathbf{R}P^n$ with the elliptic metric.

Theorem 3 (A) *The universal covering space $\tilde{M}^n$ is either the euclidean space $\mathbf{R}^n$ or it is a rank 1-symmetric space.*
(B) *If the covering map $\rho: \tilde{M}^n \to M^n$ is non-trivial, then the base space M^n is the real projective space $\mathbf{R}P^n$ with elliptic metric and $\tilde{M}^n$ is the sphere S^n with round metric.*

Proof. (A) Let $\tilde{M}^n = \tilde{M}_0 \times \tilde{M}_1 \times \ldots \times \tilde{M}_k$ be the De Rham decomposition, where $\tilde{M}_0$ is the euclidean component. Replacing each non-compact component by its compact dual, the rank and the isotropy group remain unaltered; i.e. this new space $\tilde{M}^{*n} = \tilde{M}_0 \times \tilde{M}_1^* \times \ldots \times \tilde{M}_k^*$ is two point homogeneous as well. If there is a non-trivial compact component $\tilde{M}_1^* \times \ldots \times \tilde{M}_k^*$ then choose a closed geodesics γ on a maximal torus determined by the rank of $\tilde{M}_1^* \times \ldots \times \tilde{M}_k^*$. By the existence of such a γ, all geodesics have to be closed on the covering space $\tilde{M}^{*n}$. Therefore the rank of $\tilde{M}^{*n}$ must be 1; and consequently it is compact and irreducible; because in the opposite case there exist also non-closed geodesics on a maximal torus determined by the rank. This completes the proof of (A).

(B) If the covering $\rho: \tilde{M}^n \to M^n$ is non-trivial then choose two points $\tilde{p} \neq \tilde{q}$ on $\tilde{M}^n$ satisfying $\rho(\tilde{p}) = \rho(\tilde{q}) = P$ and connect these by a geodesics $\tilde{\gamma}(s)$, where $\tilde{\gamma}(0) = \tilde{p}$ and $\gamma(\tilde{L}) = \tilde{q}$. The projected geodesics $\gamma(s) = \rho(\tilde{\gamma}(s))$; $0 \leqq s \leqq \tilde{L}$; is closed, since $\gamma(0) = \gamma(\tilde{L})$. We show that it is a smooth simply closed geodesics.

Let Φ_s be a continuous family of global isometries on M^n satisfying $\Phi_s(\gamma(0)) = \gamma(s)$; $\Phi_{s*}(\dot{\gamma}(0)) = \dot{\gamma}(s)$.

The isometries Φ_s map the geodesics γ onto itself, more precisely $\Phi_s(\gamma(t)) = \gamma(t+s)$ is satisfied. If $Q = \gamma(t_0) = \gamma(t_1)$ is an intersection point on γ then

$$\gamma(t_0 + s) = \Phi_s(\gamma(t_0)) = \Phi_s(\gamma(t_1)) = \gamma(t_1 + s)$$

and so $\dot{\gamma}(t_0) = \dot{\gamma}(t_1)$. This exactly means that γ is a simply closed geodesics. By the two point homogeneous property, each geodesics is simply closed with the same length, say ℓ on M^n. Furthermore if $\tilde{\gamma}(s)$ is an arbitrary geodesics on the universal covering space $\tilde{M}^n$ then for any s the points $\tilde{\gamma}(s)$ and $\tilde{\gamma}(s+\ell)$ are mapped to the same point of M^n by ρ. This means that the points of a geodesic sphere with radius ℓ are mapped to the same point of M^n. Since ρ is immersion, the manifold $\tilde{M}^n$ has to be a compact space; which has simply closed geodesics with the same length, say L; and a geodesic sphere $S_{p;\ell}$ with the centre p and radius ℓ has to be 0-dimensional cut locus $C(p)$ regarding the point $p \in \tilde{M}^n$. The cut value on $\tilde{M}^n$ is $L/2$ regarding every point and every direction, therefore $\ell = L/2$ and the cut locus $C(p) = S_{p;\ell}$

contains only one point. This means, that the space $\tilde{M}^n$ is a sphere topologically, as it is the compactification of an open ball by 1 point. By the symmetry of $\tilde{M}^n$ and by using geodesics variations we get that the Jacobi fields $Y(t)$ satisfying $Y(0) = 0$; $|Y'(0)| = 1$ are of the form $(L/2\pi)(\sin 2\pi t/L)\,X(t)$ on such spaces, where $X(t)$ is a parallel unit vector field along a geodesics. Therefore the space $\tilde{M}^n$ is a round sphere with the constant sectional curvature $(2\pi/L)^2$ and the base space M^n is the real projective space $\mathbf{R}P^n$ with the elliptic metric. This completes the proof of (B). □

Theorem 3(B) is proved for compact rank 1-symmetric spaces also in [4, pages 72].

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