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The heat equation and harmonic maps of complete manifolds

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Table of contents

0 Introduction 1
1 Mean-value inequalities 4
2 Gradient estimates 11
3 Short time solutions of the parabolic harmonic map equation 17
4 Harmonic maps: $K = 0$ and $\kappa = 0$ 23
5 Harmonic maps: $\lambda(M) > 0$ and $\kappa = 0$ 28
6 Harmonic maps: Small total energy 39
References 45

0 Introduction

The primary goal of this article is to understand when one can solve the harmonic map equation between two complete noncompact manifolds. Specifically, we would like to obtain sufficient conditions to ensure that a C^1 map, h, with bounded energy density between two complete noncompact manifolds can be deformed smoothly to a smooth harmonic map by solving the parabolic equation for harmonic maps. The assumption on the energy density of h is natural for most of our applications. In fact, it is known that even the standard scalar heat equation on $\mathbf{R}^m$ does not have unique solution if the initial data grows too rapidly as a function of $\mathbf{R}^m$.

Let the metrics of two complete noncompact manifolds M^m and N^n be given locally by $ds_M^2 = g_{ij}dx^i dx^j$ and $ds_N^2 = h_{\alpha\beta}du^\alpha du^\beta$, respectively, then the energy density function of u is defined by

$$e(u)(x) = g^{ij}(x)\frac{\partial u^\alpha}{\partial x^i}(x)\frac{\partial u^\beta}{\partial x^j}(x)h_{\alpha\beta}(u(x)) ,$$

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and the total energy of u is given by

$$E(u) = \int_M e(u)(x)\,dx\ .$$

The harmonic map equation from M into N, which is the Euler-Lagrange equation for critical points of the total energy functional, can be written as

$$\Delta u^\alpha(x) = -\Gamma^\alpha_{\beta\gamma}(u(x))\frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\gamma}{\partial x^j}(x)\,g^{ij}(x)$$

for all α, where the $\{\Gamma^\alpha_{\beta\gamma}\}$ are the Christoffel symbols of N. The corresponding parabolic equation defined on $M \times [0, \infty)$, with initial condition h is given by

$$\left(\Delta - \frac{\partial}{\partial t}\right)u^\alpha(x, t) = -\Gamma^\alpha_{\beta\gamma}(u(x, t))\frac{\partial u^\beta}{\partial x^i}(x, t)\frac{\partial u^\gamma}{\partial x^j}(x, t)\,g^{ij}(x)$$

$$u(x, 0) = h(x)\ . \tag{0.1}$$

When M and N are compact without boundary and N has nonpositive sectional curvature, Eells-Sampson in [E-S] proved that any C^1 map from M into N can be deformed to a harmonic map by solving (0.1). The analogous version for compact manifolds with boundary was proved by Hamilton in [Hm]. Mitteau in [M] also proved that for compact manifolds M and N without boundary, there exists a constant $\varepsilon > 0$ depending on M and N such that any C^1 harmonic map from M to N, with energy density bounded above by ε, can be smoothly deformed via (0.1) to a smooth harmonic map. We would like to point out that these theorems can also be proven by considering the elliptic harmonic maps equation directly using techniques in calculus of variations. The reader should refer to the article of Schoen in [S] in regard to this approach. For other related topics concerning harmonic maps we would like to point out the survey articles of [E-L1], [E-L2], [J1], and [J2].

In [S-Y1], Schoen-Yau proved that if M is complete and noncompact and N has nonpositive sectional curvature, then any C^1 map from M into N with finite total energy can be deformed to a harmonic map with finite total energy. Jost-Yau in their articles [J-Y1] and [J-Y 2] applied the above results to study the rigidity of locally symmetric spaces with finite volume. Recently, Liao and the second author gave another proof of Schoen-Yau's theorem by solving the parabolic equation (0.1) in [Lo-T].

To this date, not much is known for a general noncompact manifold M, if we do not impose the finite total energy condition on the initial map. Struwe and Chen-Struwe in [St] and [C-St] studied equations (0.1) for the case when M is $\mathbf{R}^m$ and the initial map is C^1 with bounded energy density. A year ago, Choi-Treibergs in [C-T1] gave a 1-parameter family of examples of harmonic maps from the hyperbolic 2-space $\mathbf{H}^2$ into itself which have the property that they map the geometric boundary of $\mathbf{H}^2$ into itself. In [C-T2], they also constructed examples of harmonic maps from higher dimensional manifolds which are diffeomorphic to $\mathbf{R}^n$ into $\mathbf{H}^n$. These harmonic maps were obtained as the Gauss map of constant mean-curvature hypersurfaces of Minkowski space. Aviles-Choi-Micallef in [A-C-M] also proved that any continuous map from the geometric boundary of a strongly negatively curved Hadamard manifold into bounded convex ball can be extended uniquely to a harmonic map. Recently, Shi-Yau in [Sh-Y] solved (0.1), using a different argument than ours, for the case when M is noncompact with

nonnegative Ricci curvature, N is compact with nonpositive curvature, and the initial map has bounded energy density. This result is also a consequence of our program which will be discussed in §4. We would also like to point out that some of these theorems were reproved by Kendall [K] using probabilistic approach.

In the first two sections of this paper, we will deal with the heat equation for the Laplacian on functions. These estimates are not only essential for the purpose of solving (0.1), they are also interesting in their own rights. In §1, we will first prove two versions of the mean-value inequality for nonnegative subsolutions of the heat equation. The first version (Theorem 1.1.) can be viewed as the parabolic analogue of the mean-value inequality for nonnegative subharmonic functions derived in [L-S] by Schoen and the first author. The main point of this mean-value inequality is that the dependency of the constant is explicit in terms of the lower bound of the Ricci curvature and the radius of the ball alone. In the second section, we will derive estimates for solutions of the homogeneous and inhomogeneous heat equations for functions on a complete manifold.

The third section §3 is devoted to the question of short time existence of solutions for (0.1). We will prove that if M is noncompact with Ricci curvature bounded from below and N is an arbitrary noncompact manifold, then (0.1) can be solved on $M \times [0, \varepsilon)$ for any initial map h with bounded energy density and $h(M)$ being bounded in N.

In §4, §5, and §6 we will consider the questions of long time existence and of convergence to a harmonic map as time approaches infinity. Theorems 4.1, and 4.1′ are the noncompact, bounded energy density anologue of the long time existence theorem of Eells-Sampson. Both theorems deal with the case when M is noncompact with Ricci curvature bounded from below, N is noncompact with nonpositive sectional curvature, and the initial map h is of bounded energy density. Under these assumptions, these theorems assert that there is a unique solution of (0.1) on $M \times [0, \infty)$. In the event if the Ricci curvature of M is nonnegative, then the solution of the parabolic harmonic map equation obtained by Theorem 4.1′ in fact converges to a harmonic map (see Theorem 4.3) provided that there exists a point $p \in M$ such that some sequence $u(p, t_\nu)$ is convergent in N.

In §5, we consider the situation which is basically opposite to the assumption that M has nonnegative Ricci curvature. We will assume that the Ricci curvature of M is bounded from below by $-K \leqq 0$, the volume of the unit ball centered at any point in M is uniformly bounded away from 0, and the lower bound of the spectrum $\lambda(M)$ is positive. If the target manifold N has nonpositive sectional curvature, then for any initial map with bounded energy density and tension field in $L^p \cap L^\infty$ for some $p > 2$, it can be deformed to a harmonic map through (0.1). Applying this general existence theorem to hyperbolic spaces, we will prove in Theorem 5.4 that given any C^3 map from the infinity sphere $\mathbf{S}^{m-1}$ of $\mathbf{H}^m$ to the infinity sphere $\mathbf{S}^{n-1}$ of $\mathbf{H}^n$ whose energy density is nonvanishing, there exists a harmonic from $\mathbf{H}^m$ to $\mathbf{H}^n$ which realizes the given boundary map. In Theorem 5.5, we prove that if in addition $m = n = 2$ and the boundary map is a diffeomorphism, then the harmonic map is in fact a diffeomorphism. Recently, K. Akutagawa [A] has independently proved the case $m = n = 2$ but assuming the boundary map is C^4 instead. His argument used the fact that harmonic maps are conformally invariant when the domain manifold is of dimension 2.

In Theorem 6.1, we will prove a noncompact version of Mitteau's theorem. We assume that M is noncompact with Ricci curvature bounded from below by $-K \leqq 0$ and N is complete with section curvature bounded from above by $\kappa \geqq 0$.

Suppose that the volume of unit balls in M is uniformly bounded from below by $b > 0$. Then for any $\alpha > 0$, there exists a constant $\varepsilon > 0$ depending only on m, K, κ, b, and α, such that for any C^1 map h which has its energy density bounded from above by α, its total energy bounded from above by ε, and $h(M)$ is bounded, then (0.1) has a solution on $M \times [0, \infty)$. Moreover, if there exists a point $p \in M$ and a sequence $t_v \to \infty$ such that $u(p, t_v)$ converges in N, then the sequence of maps $u(\cdot, t_v)$ converges uniformly on compact subsets to a harmonic map. The assumption that $h(M)$ is bounded in N can be deleted if N satisfy a retraction assumption which we refer to as condition (A). Theorem 6.5 states that if M has nonnegative Ricci curvature and has volume of geodesic balls grows like $V_p(r) \geqq C_5 r^2$, then there exists a constant $\varepsilon > 0$ depending only on κ, C_5, and α, such that any harmonic map whose energy density is bounded from above by α and whose total energy is bounded from above by ε must be a constant map. Finally, we will give an example of a nonconstant harmonic map which has small total energy satisfying the hypotheses of Theorem 6.1.

Acknowledgement. The first author would like to thank R. Schoen for many valuable discussions concerning Theorem 5.4.

1 Mean-value inequalities

In this section we will prove two versions of the mean-value inequality for positive subsolutions of the heat equation on a manifold. The first (Theorem 1.1) is a parabolic version of the mean-value inequality for positive subharmonic functions derived in [L-S], while the second (Theorem 1.2) is a slight modification of the first version with the initial data taken into account. We will also apply this to study the question of uniqueness of L^p solutions for the heat equation.

Theorem 1.1 *Let M^m be a complete noncompact Riemannian manifold with boundary. Let $x_0 \in M$ and $R > 0$ be such that the geodesic ball $B(2R) = B_{x_0}(2R)$, centered at x_0 of radius $2R$, does not intersect the boundary of M. Suppose g is a nonnegative subsolution of the heat equation on $B(2R) \times [0, T]$ for some $0 < T \leqq \frac{1}{4}R^2$. If $-K \leqq 0$ is the lower bound of the Ricci curvature of $B(2R)$, then for $p > 0$, there exists positive constants C_{14} and C_{15} depending only on m and p such that for any $0 < \tau < T$, $0 < \delta < \frac{1}{2}$, and $0 < \eta < \frac{1}{2}$,*

$$\sup_{B((1-\delta)R) \times [\tau, T]} g^p \leqq C_{14} \frac{V(K, 2R)}{V(R)} (R\sqrt{K} \coth(R\sqrt{K}) + 1) \exp(C_{15} K T)$$

$$\times \left(\frac{1}{\delta R} + \frac{1}{\sqrt{\eta \tau}} \right)^{m+2} \int_{(1-\eta)\tau}^{T} ds \int_{B(R)} g^p(y, s)\, dy ,$$

where $V(K, r)$ is the volume of the geodesic ball of radius r in the model space of constant sectional curvature $-\dfrac{K}{m-1}$, and $V(r)$ is the volume of the geodesic ball of radius r centered at x_0.

Proof. For any $0 < \delta, \eta < \frac{1}{4}$, let $\tilde{\phi}$ and $\tilde{\psi}$ be two smooth functions on $[0, \infty)$ so that $0 \leqq \tilde{\phi}, \tilde{\psi} \leqq 1$ and

$$\tilde{\phi}(s) = \begin{cases} 1, & 0 \leqq s \leqq 1 - \delta; \\ 0, & s \geqq 1, \end{cases}$$

$$\tilde{\psi}(s) = \begin{cases} 0, & 0 \leqq s \leqq 1 - 2\eta; \\ 1, & s \geqq 1 - \eta. \end{cases}$$

We can choose $\tilde{\phi}$ and $\tilde{\psi}$ such that,

$$-\frac{C_1}{\delta} \leqq \frac{\tilde{\phi}'}{\tilde{\phi}^{1/2}} \leqq 0\,,$$

$$\tilde{\phi}'' \geqq -\frac{C_1}{\delta^2}\,,$$

and

$$0 \leqq \tilde{\psi}' \leqq \frac{C_1}{\eta}\,,$$

for some constant $C_1 > 0$ which is independent of δ and η.

Let $r(x)$ denote the distance function from x_0, we defined $\phi(x) = \tilde{\phi}\left(\frac{r(x)}{R}\right)$, and $\psi(t) = \tilde{\psi}\left(\frac{t}{\tau}\right)$. Since g is a nonnegative subsolution of the heat equation, for $p > 1$, we have

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)(\phi(x)\psi(t)g^p(x,t)) &\geqq (\Delta\phi)\psi g^p + 2p\psi g^{p-1}\langle \nabla\phi, \nabla g\rangle \\ &\quad + p(p-1)\phi\psi g^{p-2}|\nabla g|^2 - \phi\psi' g^p \\ &\geqq (\Delta\phi)\psi g^p - \frac{p}{p-1}\psi g^p \frac{|\nabla\phi|^2}{\phi} - \phi\psi' g^p\,.\end{aligned}$$

On the other hand, a standard comparison theorem in Riemannian geometry asserts that

$$\left(\Delta\phi - \frac{p}{p-1}\frac{|\nabla\phi|^2}{\phi}\right) \geqq -C_2\left(\frac{R\sqrt{K}\coth(R\sqrt{K}) + 1}{\delta^2 R^2}\right), \tag{1.1}$$

for some constant C_2 depending only on p and m.

If $H(x, y, t)$ is the heat kernel of M, then for $x \in B((1-2\delta)R)$ and $\tau < t < T$, there exists a constant C_3 depending only on m and p so that,

$$\begin{aligned} g^p(x,t) &= g^p(x,t)\phi(x)\psi(t) \\ &= -\int_0^t ds \int_M H(x, y, t-s)\left(\Delta - \frac{\partial}{\partial t}\right)(g^p(y,s)\phi(y)\psi(s))\,dy \end{aligned}$$

$$(1.2)\quad = -\int_{(1-2\eta)\tau}^{t} ds \int_{B(R)-B((1-\delta)R)} H(x,y,t-s)\left(\Delta - \frac{\partial}{\partial t}\right)(g^p(y,s)\phi(y)\psi(s))\,dy$$

$$\leqq C_3\left\{\left(\frac{R\sqrt{K}\coth(R\sqrt{K})+1}{\delta^2 R^2}\right)\int_{(1-2\eta)\tau}^{T} ds \int_{B(R)-B((1-\delta)R)} H(x,y,t-s)\right.$$

$$\left. g^p(y,s)\,dy + \frac{1}{\eta\tau}\int_{(1-2\eta)\tau}^{(1-\eta)\tau} ds \int_{B(R)} H(x,y,t-s) g^p(y,s)\,dy\right\}.$$

Here we have used (1.1) and the definitions of ϕ and ψ.

In [L-Y] (see Theorem 3.3), Li-Yau derived local estimates on fundamental solutions of parabolic Schrödinger equations. By setting the potential to be identically zero and the sets $S_1 = B_y(\sqrt{t-s})$ and $S_2 = B_x(\sqrt{t-s})$, for any $x, y \in B(R)$ and $t - s \leqq \frac{1}{4}R^2$, we have

$$H(x,y,t-s) \leqq C_4 V_x^{-1/2}(\sqrt{t-s})\, V_y^{-1/2}(\sqrt{t-s})$$
$$\exp\left(-C_5\frac{r^2(x,y)}{t-s} + C_6 K(t-s)\right),$$

where C_4, C_5 and C_6 are constants depending only on m, $V_x(\sqrt{t-s})$ is the volume of the geodesic ball of radius $\sqrt{t-s}$ centered at x, and $r(x,y)$ is the distance between x and y.

On the other hand by the volume comparison theorem, for $x \in B(R)$ and $0 < s < t \leqq T \leqq \frac{1}{4}R^2$,

$$\frac{V_{x_0}(R)}{V_x(\sqrt{t-s})} \leqq \frac{V_x(2R)}{V_x(\sqrt{t-s})} \leqq \frac{V(K,2R)}{V(K,\sqrt{t-s})} \leqq C_7(t-s)^{-m/2} V(K,2R),$$

for some constant C_7 depending only on m. Hence

$$(1.3)\quad H(x,y,t-s) \leqq C_8\frac{V(K,2R)}{V(r)}(t-s)^{-m/2}\exp\left(-C_5\frac{r^2(x,y)}{t-s} + C_6K(t-s)\right)$$

$$\leqq C_9(\delta R)^{-m}\frac{V(K,2R)}{V(R)}\exp(C_6K(t-s)),$$

for $x \in B((1-2\delta)R)$, $y \in B(R) - B((1-\delta)R)$ and $0 < s \leqq t \leqq T \leqq \frac{1}{4}R^2$, where C_8 and C_9 are constants depending only on m.

Also for $t > \tau$, and $s \leqq (1-\eta)\tau$, we can find a constant C_{10} depending only on m so that

$$(1.4)\qquad H(x,y,t-s) \leqq C_{10}(\eta\tau)^{-m/2}\exp(C_6K(t-s))\,.$$

Combining (1.2)–(1.4), we conclude that

$$(1.5)\quad \sup_{B((1-2\delta)R)\times[\tau,T]} g^p \leqq C_{11}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\exp(C_{12}KT)$$

$$\times\left(\frac{1}{\delta R}+\frac{1}{\sqrt{\eta\tau}}\right)^{m+2}\int_{(1-2\eta)\tau}^{T} ds \int_{B(R)} g^p(y,s)\,dy\,,$$

for $0 < \tau < T \leqq \frac{1}{4}R^2$, $0 < \delta, \eta \leqq \frac{1}{4}$ and $p > 1$, where C_{11} and C_{12} depend only on m and p. This completes the proof of the theorem for the case when $p > 1$ if we replace 2δ by δ and 2η by η.

In order to prove the case when $p \leqq 1$, we can proceed as in [L-S]. Take $p = 2$ in (1.5), we have

$$(1.6)\quad \sup_{B((1-\delta)R)\times[\tau,\,T]} g^2 \leqq C_{11}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\exp(C_{12}KT) \times\left(\frac{1}{\delta R}+\frac{1}{\sqrt{\eta\tau}}\right)^{m+2}\int_{(1-\eta)\tau}^{T} ds\int_{B(R)} g^2(y,s)\,dy\,,$$

for $0 < \delta, \eta < \frac{1}{2}$, and $T \leqq \frac{1}{4}R^2$.

For $k \geqq 0$ and $l \geqq 1$, let

$$M(k,l) = \sup_{B((1-\delta_k)R)\times[\tau_l,\,T]} g^2\,,$$

where $\delta_k = 2^{-k}\delta$, and $\tau_l = (1-\eta\sum_{i=1}^{l}2^{-i})\tau$.

Inequality (1.6) implies that

$$M(k,l) \leqq C_{11}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\exp(C_{12}KT) \times\left(\frac{2^{k+1}}{\delta R}+\frac{2^{(l+1)/2}}{\sqrt{\eta\tau}}\right)^{m+2}(M(k+1,l+1))^{\lambda}\int_{(1-\eta)\tau}^{T} ds\int_{B(R)} g^p(y,s)\,dy\,,$$

where $\lambda = 1 - \dfrac{p}{2}$.

Let $\tilde{M}(k) = M(k, 2k+1)$, then

$$\tilde{M}(k) \leqq A(2^{m+2})^{k+1}\cdot(\tilde{M}(k+1))^{\lambda}\,,$$

where

$$A = C_{11}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\exp(C_{12}KT)\left(\frac{1}{\delta R}+\frac{1}{\sqrt{\eta\tau}}\right)^{m+2} \times\int_{(1-\eta)\tau}^{T} ds\int_{B(R)} g^p(y,s)\,dy\,.$$

Iterating as in [L-S], we get

$$\tilde{M}(0) \leqq C_{13}A^{1/(1-\lambda)}\,,$$

where C_{13} is a constant depending only on m and p. From this, it is easy to conclude that the theorem is also true for $0 < p \leqq 1$.

Using a similar argument, taking the initial data into consideration, we can obtain another form of the mean-value inequality for positive subsolutions of the heat equation.

Theorem 1.2 *With the same notations and the same assumptions as in Theorem* 1.1, *for any* $p > 0$ *and* $\varepsilon > 0$, *there exists positive constants* C_{16} *and* C_{17} *depending only on* m *and* p *with* C_{16} *depending also on* ε, *such that for any* $0 < \delta < \frac{1}{2}$,

$$\sup_{B((1-\delta)R)\times[0,T]} g^p \leqq C_{16}\left\{\frac{1}{(\delta R)^{m+2}}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\right.$$
$$\left.\exp(C_{17}KT)\right\}\times\int_0^T ds\int_{B(R)} g^p(y,s)\,dy + ((1+\varepsilon)S(R))^p,$$

where $S(r) = \sup_{B(r)} g(\cdot, 0)$.

Proof. Let ϕ be a cut-off function with support in $B(R)$, then for $p > 1$ and for $x \in B(R)$ and $0 < t < T$, we have

$$g^p(x,t) = g^p(x,t)\phi(x)$$
$$= -\int_0^t ds\int_M H(x,y,t-s)\left(\Delta - \frac{\partial}{\partial s}\right)(g^p(y,s)\phi(y))\,dy$$
$$+ \int_M H(x,y,t)g^p(y,0)\phi(y)\,dy\,.$$

Hence, using a similar argument as before, for $p > 1$, $T \leqq \frac{1}{4}R^2$ and $0 < \delta < \frac{1}{2}$, we have

$$\sup_{B((1-\delta)R)\times[0,T]} g^p \leqq \frac{C_{18}}{(\delta R)^{m+2}}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\exp(C_{19}KT)$$
$$\times\int_0^T ds\int_{B(R)} g^p(y,s)\,dy + (S(R))^p,$$

for some constants C_{18} and C_{19} depending only on m and p, where we have used the fact that $H(x,y,t)$ is positive and $\int_M H(x,y,t)\,dy = 1$.

Now consider the case when $0 < p \leqq 1$. For $k \geqq 0$, let $\delta_k = 2^{-k}\delta$, and let $M(k)$ to be the supremum of g^2 on $B((1-\delta_k)R)\times[0,T]$. If $\lambda = 1 - \frac{p}{2}$, then

$$M(k) \leqq A2^{(m+2)(k+1)}(M(k+1))^\lambda + (S(R))^2\,, \tag{1.6}$$

for all $k \geqq 0$, where

$$A = \frac{C_{18}}{(\delta R)^{m+2}}\frac{V(K,2R)}{V(R)}(R\sqrt{K}\coth(R\sqrt{K})+1)\exp(C_{19}KT)\int_0^T ds\int_{B(R)} g^p(y,s)\,dy\,.$$

If $M(0) \leqq ((1+\varepsilon)S(R))^2$, then

$$\sup_{B((1-\delta)R)\times[0,T]} g^p \leqq ((1+\varepsilon)S(R))^p\,. \tag{1.7}$$

and the theorem is proved. We may assume that $M(0) > ((1+\varepsilon)S(R))^2$. Since $M(k)$ is nondecreasing in k, so $M(k) > ((1+\varepsilon)S(R))^2$ for all k. By (1.6), we have

$$M(k) \leqq \frac{(1+\varepsilon)^2}{\varepsilon(2+\varepsilon)}A2^{(m+2)(k+1)}(M(k+1))^\lambda\,.$$

Iterating as in the proof of Theorem 1.1 and taking into account of (1.7), the result follows.

Now we apply Theorem 1.2 to prove some uniqueness results for L^p solutions of the heat equation on manifolds.

Theorem 1.3 *Let M^m be a complete Riemannian manifold satisfying either one of the following conditions:*

(a) M has nonnegative Ricci curvature; or

(b) The Ricci curvature of M is bounded below by $-K \leqq 0$ and $\inf_M V_x(1) > 0$ where $V_x(1)$ is the volume of the geodesic ball of radius 1 with center at x.

If $g(x, t)$ is a nonnegative subsolution of the heat equation on $M \times [0, T]$ such that $g \in L^p(M \times [0, T])$ for some $p > 0$, then

$$\sup_{M \times [0, T]} g \leqq \sup_{x \in M} g(x, 0) .$$

In particular, any $L^p(M \times [0, T])$ solution of the heat equation is uniquely determined by its initial data.

Proof. First we remark that for $p \geqq 1$, the theorem was proved by the first author in [L]. In fact, if $p > 1$, then we do not need any curvature assumption on M.

Now let g be a nonnegative subsolution of the heat equation. If M satisfies (a), then by Theorem 1.2, for a fixed point $x_0 \in M$ and for any $\varepsilon > 0$, there exists a constant C_{20} depending only on m, p and ε, such that for all $R > \sqrt{T}$,

$$\sup_{B_{x_0}(1/2R) \times [0, T]} g^p \leqq \frac{C_{20}}{R^2 V_{x_0}(R)} \int_0^T ds \int_{B_{x_0}(R)} g^p(y, s)\, dy + ((1 + \varepsilon)S)^p ,$$

where $S = \sup_{x \in M} g(x, 0)$. A theorem of Yau in [Y1] shows that

$$V_{x_0}(R) \geqq C_{21} R ,$$

for some constant $C_{21} > 0$. By letting $R \to \infty$ and then $\varepsilon \to 0$, we conclude that

$$\sup_{M \times [0, T]} g \leqq \sup_{x \in M} g(x, 0) .$$

Suppose M satisfies (b). By the semigroup property of $g(x, t)$, we may assume that $T < \frac{1}{4}$. Taking $R = 1$ in Theorem 1.2, for any $\varepsilon > 0$ there exists a constant $C_{22} > 0$, not depending on x and t, such that

$$g^p(x, t) \leqq C_{22} \int_0^T ds \int_{B_x(1)} g^p(y, s)\, dy + ((1 + \varepsilon)S)^p ,$$

for all $x \in M$ and $0 \leqq t \leqq T$. Here we have used the fact that $\inf_M V_x(1) > 0$. Since $g \in L^p(M \times [0, T])$, we conclude that

$$\limsup_{x \to \infty} g(x, t) \leqq (1 + \varepsilon) \sup_{x \in M} g(x, 0)$$

uniformly on $t \in [0, T]$. By the maximum principle and by letting $\varepsilon \to 0$, the result follows.

Theorem 1.4 *Let M^m be a complete noncompact Riemannian manifold. There exists a constant $\delta(m) > 0$ depending only on m, such that if for some point $x_0 \in M$, the Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geqq -\delta(m) r_0^{-2}(x)$$

whenever the distance from x to x_0, $r_0(x)$, is sufficiently large, then any nonnegative subsolution $g \in L^p(M \times [0, T])$, for some $p > 0$, of the heat equation on $M \times [0, T]$ must satisfy

$$\sup_{M \times [0, T]} g \leqq \sup_{x \in M} g(x, 0) .$$

Proof. Let $\beta > \dfrac{2}{2^{-m} - 1} > 1$ be a fixed number. For any positive integer k, let us define

$$R_k = 1 + 2\beta + \beta^2 + \ldots + 2\beta^{k-1} + \beta^k = 2 \sum_{i=0}^{k} \beta^i - 1 - \beta^k .$$

By the proof of Theorem 2.5 in [L-S], there exists $\delta(m) > 0$ depending only on m, such that if the Ricci curvature of M satisfies the condition of the theorem then there exists a constant $C_{23} > 0$ depending only on m so that for all $x \in \partial B_{x_0}(R_k)$

$$V_x\left(\frac{\beta^k}{4}\right) \geqq C_{23} \left(\frac{\beta^m}{(\beta + 2)^m - \beta^m}\right)^k V_{x_0}(1) \tag{1.8}$$

for all k.

Let the lower bound of the Ricci curvature of $B_x(r)$ be $-K(x, r) \leqq 0$. Let $g \in L^p(M \times [0, T])$ be a nonnegative subsolution of the heat equation on $M \times [0, T]$. By Theorem 1.2, given $\varepsilon > 0$, for all $x \in \partial B_{x_0}(R_k)$, $k > 0$, and $0 \leqq t \leqq T$, we have

$$g^p(x, t) \leqq C_{24} \frac{1}{\beta^{k(m+2)}} \times \frac{V_x(K(x, \frac{1}{2}\beta^k), \frac{1}{2}\beta^k)}{V_x(\frac{1}{4}\beta^k)}$$

$$\times \left(\frac{1}{2}\beta^k \sqrt{K\left(x, \frac{1}{2}\beta^k\right)} \coth\left(\frac{1}{2}\beta^k \sqrt{K\left(x, \frac{1}{2}\beta^k\right)}\right) + 1\right) + ((1 + \varepsilon)S)^p \tag{1.9}$$

for some constant C_{24} depending only on $\int_0^T \int_M g^p$, m, p, K, T, and ε, where $S = \sup_{x \in M} g(x, 0)$.

By the assumption on the Ricci curvature, for k sufficiently large, we have

$$K\left(x, \frac{1}{2}\beta^k\right) \leqq C_{25} \left(2 \sum_{i=0}^{k} \beta^i - 1 - \beta^k\right)^{-2} \leqq (C_{26}\beta^k)^{-2} \tag{1.10}$$

for some constants C_{25}, $C_{26} > 0$ depending only on m, and

$$V_x\left(K\left(x, \frac{1}{2}\beta^k\right), \frac{1}{2}\beta^k\right) \leqq C_{27} \int_0^{1/2\beta^k} \left(C_{26}\beta^k \sinh\left(\frac{t}{C_{26}\beta^k}\right)\right)^{m-1} dt$$

$$\leqq C_{28}\beta^{mk} \tag{1.11}$$

for some C_{27} and C_{28} depending only on m. From (1.8)–(1.11), it is easy to see that

$$\lim_{k\to\infty} \sup_{\partial B_{x_0}(R_k)\times[0,\,T]} g \leqq (1+\varepsilon)S\,.$$

The theorem then follows from the maximum principle and by letting $\varepsilon \to 0$.

2 Gradient estimates

In this section, we will derive various estimates for solutions of the homogeneous and the inhomogeneous heat equations on a complete manifold. We will also estimate the rate of convergence of any solution of the heat equation to the initial data as $t \to 0$. Other than their application to harmonic maps, these results are also interesting in their own right.

Lemma 2.1 *Let M^m be a complete noncompact Riemannian manifold with Ricci curvature bounded from below by $-K$. Let $H(x, y, t)$ be the heat kernel of M. For any $T > 0$, there exists a constant $C_1(m, K, T)$ so that for $0 < t < T$ and $p, q \in M$,*

$$\int_M |H(p, y, t) - H(q, y, t)|\, dy \leqq C_1\left(\frac{r(p, q)}{\sqrt{t}}\right),$$

where $r(p, q)$ is the distance between p and q. When $K = 0$, the constant C_1 can be taken to depend only on m alone.

Proof. Let $1 < \alpha < 4$ be fixed. For $p, q \in M$ with $p \neq q$ and $\delta > 0$ let us consider

$$\begin{aligned}(2.1)\qquad \int_M |H(p, y, t) - H(q, y, t)|dy &\leqq \int_M |H(q, y, (1+\delta)t) - H(q, y, t)|\, dy \\ &\quad + \int_M |H(p, y, t) - H(q, y, (1+\delta)t|\, dy \\ &= I + II\,.\end{aligned}$$

To estimate I, we use the Harnack inequality of [L-Y] which asserts that

$$H(q, y, t) \leqq (1+\delta)^{m\alpha/2} \exp(A\delta t) H(q, y, (1+\delta)t)\,,$$

where $A > 0$ is a constant depending only on m, K and α. Hence

$$\begin{aligned} I &\leqq \int_M |(1+\delta)^{m\alpha/2} \exp(A\delta t) H(q, y, (1+\delta)t) - H(q, y, t)|\, dy \\ &\quad + \int_M |\{(1+\delta)^{m\alpha/2} \exp(A\delta t) - 1\} H(q, y, (1+\delta)t)|\, dy \\ &= \int_M \{(1+\delta)^{m\alpha/2} \exp(A\delta t) H(q, y, (1+\delta)t) - H(q, y, t)\}\, dy \\ &\quad + \{(1+\delta)^{m\alpha/2} \exp(A\delta t) - t\} \int_M H(q, y, (1+\delta)t)\, dy\,. \end{aligned}$$

Using the facts that $H(x, y, t) > 0$ and $\int_M H(x, y, t)\, dy = 1$ for all x and t (see [Y2] and [K-L]), this inequality can be written as

$$(2.2)\qquad I \leqq 2((1+\delta)^{m\alpha/2} \exp(A\delta t) - 1)\,.$$

Similarly, by the Harnack inequality of [L-Y], we have

$$H(p, y, t) \leqq (1+\delta)^{m\alpha/2} \exp\left(A\delta t + \frac{\alpha}{4\delta t} r^2(p, q)\right) H(q, y, (1+\delta)t) .$$

Hence

$$\begin{aligned} II &\leqq \int_M |H(p, y, t) - (1+\delta)^{m\alpha/2} \exp(A\delta t) H(q, y, (1+\delta)t)|\, dy \\ &\quad + \int_M |((1+\delta)^{\alpha/4\delta t} r^2(p, q) - 1) H(q, y, (1+\delta)t)|\, dy \\ &\leqq 2\left((1+\delta)^{m\alpha/2} \exp\left(A\delta t + \frac{\alpha}{4\delta t} r^2(p, q)\right) - 1\right) . \end{aligned} \tag{2.3}$$

Combining (2.1)–(2.3), we have

$$\int_M |H(p, y, t) - H(q, y, t)|\, dy \leqq 4\left\{(1+\delta)^{m\alpha/2} \exp\left(A\delta t + \frac{\alpha}{4\delta t} r^2\right) - 1\right\}, \tag{2.4}$$

where $r = r(p, q)$. Let us first assume that $\dfrac{r}{\sqrt{t}} \leqq 1$. By setting $\delta = \dfrac{r\sqrt{\alpha}}{2\sqrt{t}}$, we have $\delta^2 \leqq \dfrac{\alpha}{4} < 1$, because $1 < \alpha < 4$. For $0 < t < T$,

$$\begin{aligned} (1+\delta)^{m\alpha/2} \exp\left(A\delta t + \frac{\alpha}{4\delta t} r^2\right) - 1 &\leqq (1+\delta)^{m\alpha/2} \exp((AT+1)\delta) - 1 \\ &\leqq C_2(m, \alpha, K, T) \times \delta \\ &= C_2(m, \alpha, K, T)\left(\frac{r}{\sqrt{t}}\right), \end{aligned}$$

where we have used the fact that $\delta \leqq 1$. Combining this with (2.4), we have

$$\int_M |H(p, y, t) - H(q, y, t)|\, dy \leqq C_3(m, \alpha, K, T)\left(\frac{r}{\sqrt{t}}\right) . \tag{2.5}$$

On the other hand if $\dfrac{r(p, q)}{\sqrt{t}} \geqq 1$, then

$$\int_M |H(p, y, t) - H(q, y, t)|\, dy \leqq 2 \leqq 2\left(\frac{r(p, q)}{\sqrt{t}}\right) .$$

The proof of the theorem is now complete. If $K = 0$, then the constant A in the proof can be taken to be 0, and the constant C_1 will depend only on m.

Theorem 2.2 *Let $f: M \times [0, T) \to \mathbf{R}$ be a bounded function, where M satisfies the hypotheses of Theorem* 2.1. *Let $H(x, y, t)$ be the heat kernel of M, and*

$$u(x, t) = -\int_0^t d\tau \int_M H(x, y, t-\tau) f(y, \tau)\, dy .$$

Then u is Lipschitz in the space variable. More precisely, for any $T > 0$, there exists a constant $C_4 = C_4(m, K, T)$ such that

$$\sup_{\substack{x, x' \in M,\ x \neq x' \\ \tau \in (0, t)}} \frac{|u(x, \tau) - u(x', \tau)|}{r(x, x')} \leqq C_4 \left(\sup_{M \times (0, t)} |f| \right) t^{1/2} . \tag{2.6}$$

When $K = 0$, the constant C_4 can be taken to depend on m alone.

Proof. By Theorem 2.1, for x and $x' \in M$, we have

$$\begin{aligned} |u(x, t) - u(x', t)| &\leqq \int_0^t d\tau \int_M |H(x, y, t - \tau) - H(x', y, t - \tau)|\, |f(y, \tau)|\, dy \\ &\leqq \left(\sup_{M \times (0, t)} |f| \right) \int_0^t d\tau \int_M |H(x, y, t - \tau) - H(x', y, t - \tau)|\, dy \\ &\leqq C_5(m, K, T) \left(\sup_{M \times (0, t)} |f| \right) \int_0^t \frac{r(x, x')}{\sqrt{t - \tau}}\, d\tau \\ &= 2C_5 r(x, x') \left(\sup_{M \times (0, t)} |f| \right) t^{1/2} . \end{aligned}$$

One can also obtain a similar estimate for the homogeneous heat equation.

Theorem 2.3 *Let M be a complete noncompact manifold as in Theorem 2.1. Suppose $u(x, t)$ is a solution of the heat equation on M with bounded initial data h. Then for any $T > 0$, there exists $C_6(m, K, T)$ such that for $0 < t < T$ and $x \in M$,*

$$|\nabla u(x, t)| \leqq C_6 \left(\sup_M |h| \right) t^{-1/2} .$$

When $K = 0$, the constant C_6 can be taken to depend on m alone.

Proof. For $x, x' \in M$ and $0 < t < T$, by Theorem 2.1, we have

$$\begin{aligned} |u(x, t) - u(x', t)| &\leqq \int_M |H(x, y, t) - H(x', y, t)| h(y)\, dy \\ &\leqq C_6(m, K, T) \left(\sup_M |h| \right) r(x, x') t^{-1/2} , \end{aligned}$$

and the result follows.

Proposition 2.4 *Let M be a complete noncompact Riemannian manifold with Ricci curvature bounded below by $-K$. Let h be a bounded C^1 function on M so that $\sup_M |\nabla h| < \infty$. If*

$$u(x, t) = \int_M H(x, y, t) h(y)\, dy ,$$

then

$$\sup_M |\nabla u(\cdot, t)| \leqq \exp(Kt) \sup_M |\nabla h| ,$$

for all $t > 0$.

Proof. Let us first assume that h has compact support. Then

$$d_x u(x, t) = \int_M H_1(x, y, t)\, dh(y)\,, \tag{2.7}$$

where $H_1(x, y, t)$ is the fundamental solution for the heat equation on 1-forms. By [D-L] and the assumption on h, we conclude that

$$\sup_{M \times (0, t)} |\nabla u| < \infty\,, \tag{2.8}$$

for all $t > 0$, and

$$\lim_{t \to 0} \nabla u(x, t) = \nabla h(x)\,. \tag{2.9}$$

Since u satisfies the heat equation on $M \times (0, \infty)$, Bochner's formula implies that

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla u| \geqq - K |\nabla u|\,.$$

Hence by considering $\exp(-Kt)|\nabla u|(x, t)$, using (2.8), (2.9) and the proof of Theorem 1 in [K-L] (see also [Lo-T]), one concludes that

$$\sup_M \exp(-Kt)|\nabla u|(\cdot, t) \leqq \sup_M |\nabla h|\,.$$

This establishes the proposition for compactly supported h.

In general, let h be a bounded C^1 function with $\sup_M |\nabla h| < \infty$. One can find a sequence of compactly supported smooth functions, h_v, which are uniformly bounded, converging uniformly on compact subsets of M to h, and

$$\limsup_{v \to \infty} \sup_M |\nabla h_v| \leqq \sup_M |\nabla h|\,. \tag{2.10}$$

Let $u_v(x, t) = \int_M H(x, y, t) h_v(y)\, dy$. Then the sequence u_v is uniformly bounded and are solutions of the heat equation. The Schauder estimates implies that, by passing to a subsequence, u_v together with their gradients converge uniformly on compact subsets of M. However for each fixed x, $\lim_{v \to \infty} u_v(x, t) = u(x, t)$. Hence the sequence u_v converges to u in C^1. Using the result for compactly supported functions and (2.10), the proposition is proved.

Proposition 2.5 *Let M be a complete noncompact manifold and $f: M \times [0, T) \to \mathbf{R}$ is a bounded function. Suppose $H(x, y, t)$ is the minimal fundamental solution of the heat equation of M obtained by taking compact exhaustion. If*

$$u(x, t) = - \int_0^t d\tau \int_M H(x, y, t - \tau) f(y, \tau)\, dy\,,$$

then

$$\sup_{M \times (0, t)} |u| \leqq t \sup_{M \times (0, t)} |f|\,. \tag{2.11}$$

Proof. This follows from the facts that $H(x, y, t) > 0$ and $\int_M H(x, y, t)\,dy \leqq 1$.

Lemma 2.6 *Let M^m be a complete noncompact Riemannian manifold with Ricci curvature bounded from below by $-K \leqq 0$. Let $H(x, y, t)$ be the heat kernel of M. For any $0 < \alpha \leqq 1$ and $T > 0$, there exists a constant $C_7(\alpha, m, K, T)$ such that for $0 < t < T$, we have*

$$\int_M H(x, y, t) r^\alpha(x, y)\,dy \leqq C_7 t^{\alpha/2}\,, \tag{2.12}$$

where $r(x, y)$ is the distance between x and y. When $K = 0$, the constant C_7 can be taken to depend on α and m alone.

Proof. Let $B_x(r)$ be the geodesic ball of radius r will center at x and let $V_x(r)$ be its volume. The integral

$$\begin{aligned}\int_M H(x, y, t) r^\alpha(x, y)\,dy &= \int_{B_x(2\sqrt{t})} H(x, y, t) r^\alpha(x, y)\,dy \\ &\quad + \int_{M - B_x(2\sqrt{t})} H(x, y, t) r^\alpha(x, y)\,dy \\ &\leqq (2\sqrt{t})^\alpha + \int_{M - B_x(2\sqrt{t})} H(x, y, t) r^\alpha(x, y)\,dy\,,\end{aligned} \tag{2.13}$$

where we have used the fact that $H(x, y, t) > 0$ and $\int_M H(x, y, t)\,dy \equiv 1$.

The upper estimates of the heat kernel of [L-Y] implies that, for $0 < t < T$, there exists a constant $C_8(K, T)$ such that

$$H(x, y, t) \leqq C_8 V_x^{-1/2}(\sqrt{t}) V_y^{-1/2}(\sqrt{t}) \exp\left(-\frac{ar^2(x, y)}{t}\right),$$

where $a = \frac{2}{9}$. The volume comparison theorem asserts that, for $r = r(x, y) \geqq 2\sqrt{t}$,

$$V_x(\sqrt{t}) \leqq V_y(r + \sqrt{t}) \leqq V_y(\sqrt{t}) \frac{V(K, r + \sqrt{t})}{V(K, \sqrt{t})}\,,$$

where $V(K, s)$ is the volume of the geodesic ball of radius s in the simply connected space form of constant $-\dfrac{K}{m-1}$ curvature. Hence

$$\begin{aligned}\int_{M - B_x(2\sqrt{t})} H(x, y, t) r^\alpha(x, y)\,dy &\leqq C_8 \int_{2\sqrt{t}}^{\infty} \left(\frac{V(K, r + \sqrt{t})}{V(K, \sqrt{t})}\right)^{1/2} \exp\left(-\frac{ar^2}{t}\right) r^\alpha A_x(r)\,dr \\ &\leqq C_8 V_x^{-1}(\sqrt{t}) \int_{2\sqrt{t}}^{\infty} \left(\frac{V(K, 2r)}{V(K, \sqrt{t})}\right)^{1/2} \\ &\quad \times \exp\left(-\frac{ar^2}{t}\right) r^\alpha A_x(r)\,dr\,,\end{aligned} \tag{2.14}$$

where $A_x(r)$ is the area of $\partial B_x(r)$.

Integrating by parts gives

$$
\begin{aligned}
I &= \int_{2\sqrt{t}}^{\infty} \left(\frac{V(K,2r)}{V(K,\sqrt{t})}\right)^{1/2} \exp\left(-\frac{ar^2}{t}\right) r^{\alpha} A_x(r)\, dr \\
&= \left(\frac{V(K,2r)}{V(K,\sqrt{t})}\right)^{1/2} \exp\left(-\frac{ar^2}{t}\right) r^{\alpha} V_x(r) \Bigg]_{2\sqrt{t}}^{\infty} \\
&\quad - \left(\int_{2\sqrt{t}}^{\infty} V_x(r) \left\{ \alpha r^{\alpha-1} \left(\frac{V(K,2r)}{V(K,\sqrt{t})}\right)^{1/2} \exp\left(-\frac{ar^2}{t}\right) \right.\right. \\
&\quad + r^{\alpha} \frac{A(K,2r)}{(V(K,2r)V(K,\sqrt{t}))^{1/2}} \exp\left(-\frac{ar^2}{t}\right) - \frac{2ar^{1+\alpha}}{t} \left(\frac{V(K,2r)}{V(K,\sqrt{t})}\right)^{1/2} \\
&\quad \left.\left. \times \exp\left(-\frac{ar^2}{t}\right) \right\} dr \right) \\
&\leqq \int_{2\sqrt{t}}^{\infty} \frac{2ar^{1+\alpha}}{t} \left(\frac{V(K,2r)}{V(K,\sqrt{t})}\right)^{1/2} \exp\left(\frac{-ar^2}{t}\right) V_x(r)\, dr\,,
\end{aligned}
$$

where we have used the volume comparison theorem and the fact that

$$V(K,r) \leqq C_9 \exp((m-1)\sqrt{K}r)\,.$$

Letting $s = \dfrac{r}{\sqrt{t}}$, and using the volume comparison theorem again,

$$
\begin{aligned}
I &\leqq t^{\alpha/2} \int_2^{\infty} 2\alpha s^{1+\alpha} \left(\frac{V(K,2s\sqrt{t})}{V(K,\sqrt{t})}\right)^{1/2} \exp(-as^2) V_x(s\sqrt{t})\, ds \qquad (2.15) \\
&\leqq 2\alpha t^{\alpha/2} V_x(\sqrt{t}) \int_2^{\infty} s^{1+\alpha} \left(\frac{V(K,2s\sqrt{t})}{V(K,\sqrt{t})}\right)^{3/2} \exp(-as^2)\, ds\,.
\end{aligned}
$$

However, for $s \geqq 2$ and $0 < t < T$,

$$
\begin{aligned}
\frac{V(K,2s\sqrt{t})}{V(K,\sqrt{t})} &= \frac{\int_0^{2s\sqrt{t}} (\sinh(\sqrt{K}\tau)^{m-1}\, d\tau}{\int_0^{\sqrt{t}} (\sinh(\sqrt{K}\tau))^{m-1}\, d\tau} \qquad (2.16) \\
&\leqq C_{10} s^m \exp(C_{11} s)\,,
\end{aligned}
$$

where C_{10} and C_{11} are constants depending only on m, K, and T. Inequality (2.12) now follows from (2.13)–2.16).

Theorem 2.7 *Let M be as in Theorem* 2.6. *Suppose h is a bounded continuous function on M, such that*

$$[h]_{\alpha, M} = \sup_{\substack{x, x' \in M \\ x \neq x'}} \frac{|h(x) - h(x')|}{r^{\alpha}(x, x')} < \infty\,, \text{ for some } 0 < \alpha \leqq 1\,.$$

For any $T > 0$, there exists a constant $C_{12}(\alpha, m, K, T)$ so that for $0 < t < T$, if

$$u(x,t) = \int_M H(x,y,t) h(y)\, dy\,,$$

then

$$|u(x, t) - h(x)| \leqq C_{12}[h]_{\alpha, M} t^{\alpha/2} .$$

When $K = 0$, the constant C_{12} can be taken to depend on α and m alone.

Using Theorem 2.3, Theorem 2.7 and the semigroup property of the heat equation, the next theorem can be proved similarly.

Theorem 2.8 *Let M be a complete noncompact Riemannian manifold with Ricci curvature bounded below by $-K \leqq 0$. Let $u(x, t)$ be a solution of the heat equation on $M \times [0, T)$ with initial condition h. Then there exists a constant $C_{13}(m, K, T)$ such that for $x \in M$ and $0 < t < t' < T$,*

$$|u(x, t') - u(x, t)| \leqq C_{13}\left(\sup_M |h|\right)\left(\frac{t' - t}{t}\right)^{1/2} .$$

When $K = 0$, the constant C_{13} can be taken to depend on m alone.

3 Short time solutions of the parabolic harmonic map equation

We will now apply the results in §1 and §2 to study harmonic maps on noncompact manifolds. Let M^m and N^n be complete Riemannian manifolds with metrics $g_{ij}dx^i dx^j$ and $h_{\alpha\beta}du^\alpha du^\beta$, respectively. During the course of this discussion we will always implicitly assume that M is noncompact. Suppose $h: M \to N$ is a C^1 map so that $h(M)$ is bounded in N, then there exists an open neighborhood N' of $h(M)$ with compact closure so that N' can be embedded into $\mathbf{R}^q$ isometrically for some q. If necessary, by choosing a smaller neighborhood, we may assume that there is a bounded tubular neighborhood $\tilde{N}$ of N' in $\mathbf{R}^q$ exists. Let $\pi: \tilde{N} \to N'$ be the nearest point projection denoted by $\pi = (\pi^1, \ldots, \pi^q) = (\pi^A)_{1 \leqq A \leqq q}$. By choosing an even smaller N', we may assume that π can be extended smoothly to the whole $\mathbf{R}^q$ so that each π^A is compactly supported. Hence, in particular, π^A, $\pi^A_B = \dfrac{\partial \pi^A}{\partial z^B}$, $\pi^A_{BC} = \dfrac{\partial^2 \pi^A}{\partial z^B \partial z^C}$, etc. are bounded, where $z = (z^A)$ are the standard coordinates of $\mathbf{R}^q$. Let us consider the following system of differential equations for a map $u: M \times [0, T) \to \mathbf{R}^q$ with $u = (u^A)$, given by

$$\left(\Delta - \frac{\partial}{\partial t}\right) u^A = \pi^A_{BC}(u) \frac{\partial u^B}{\partial x^i} \frac{\partial u^C}{\partial x^j} g^{ij} \tag{3.1}$$

$$= \pi^A_{BC}(u) \langle \nabla u^B, \nabla u^C \rangle$$

on $M \times (0, T)$ for $A = 1, \ldots, q$, satisfying the initial condition

$$u(x, 0) = h(x), \qquad x \in M . \tag{3.2}$$

Here and from here on, the summation convention is taken on repeated indices.

Lemma 3.1 *Suppose the image of u lies in N', then u satisfies the parabolic equation for harmonic maps from $M \times (0, T) \to N$, if and only if u satisfies* (3.1).

Proof. See [E-S, p. 140].

Lemma 3.2 *Suppose there exists a point $p \in M$ and a constant $C > 0$ so that $V_p(r) \leqq \exp(C(1 + r^2))$ for all r. Let u be a solution of* (3.1) *which is continuous on $M \times [0, T)$ with $u(x, 0) \in N$ for all $x \in M$. Suppose $u(x, t) \in \tilde{N}$ on $M \times [0, T)$, then $u(x, t) \in N$ for all $(x, t) \in M \times [0, T)$.*

Proof. For $z \in \mathbf{R}^q$, let us define $\rho : \mathbf{R}^q \to \mathbf{R}^q$ by $\rho(z) = z - \pi(z)$. The function $\rho(u(x, t))$ is bounded on $M \times [0, T)$ because $\tilde{N}$ is bounded. Consider

$$\varphi(x, t) = \sum_{A=1}^{q} (\rho^A(u(x, t))^2$$

$$= \text{the square of the distance from } u(x, t) \text{ to } N'.$$

By [E-S, p. 141], we have

$$\left(\Delta - \frac{\partial}{\partial t}\right)\varphi(x, t) = 2\sum_{A} |\nabla \rho^A(u(x, t))|^2$$

$$\geqq 0 .$$

Since $\varphi \geqq 0$ is bounded and $\varphi(x, 0) = 0$, the proof of Theorem 1 in [K-L] (also see [Lo-T]) implies that $\varphi \equiv 0$ on $M \times (0, T)$. This establishes the lemma.

In what follows, we will prove short time existence for the initial value problem of (3.1) and (3.2).

Theorem 3.3 *Let M be a complete noncompact Riemannian manifold with Ricci curvature bounded from below by $-K \leqq 0$. Given $h \in C^1(M, N)$ so that $h(M)$ is bounded and the energy density of h is also bounded. Then there exists $T_0 > 0$, such that* (3.1) *and* (3.2) *has a solution on $M \times [0, T_0)$.*

Proof. We will follow the argument of [E-S] on successive approximation. Let us set

$$\Lambda = \sup_{\mathbf{R}^q,\, A,\, B,\, C,\, D} (|\pi^A_{BC}|, |\pi^A_{BCD}|) .$$

The hypotheses on h imply that $\sup_M |h^A| < \infty$, and $\sup_M |\nabla h^A| < \infty$ for $1 \leqq A \leqq q$. For $\nu = -1, 0, 1, 2, \ldots$, let us define inductively a sequence of maps $u^\nu : M \times [0, 1) \to \mathbf{R}^q$ as follows:

$$u^{-1, A} = 0 ; \tag{3.3}$$

$$u^{0, A}(x, t) = \int_M H(x, y, t) h^A(y)\, dy ; \tag{3.4}$$

and

$$u^{\nu, A}(x, t) = -\int_0^t d\tau \int_M H(x, y, t - \tau) F^{\nu-1, A}(y, \tau)\, dy + u^{0, A}(x, t) \tag{3.5}$$

for $\nu \geqq 1$. Here $H(x, y, t)$ is the heat kernel of M, and

$$F^{\nu, A} = \pi^A_{BC}(u^\nu)\langle \nabla u^{\nu, B}, \nabla u^{\nu, C} \rangle$$

for $v \geqq -1$, $A = 1, \ldots, q$. The function u^{-1} is introduced for the sake of convenience. Note that $F^{-1} = 0$. Obviously, u^{-1} and u^0 are well-defined. Also it is easy to see that u is smooth on $M \times (0, 1)$.

For $0 < t < 1$, let us consider the function

$$p_v(t) = \sup_{M \times (0,t)} \left(\sum_A |\nabla u^{v,A}|^2 \right)^{1/2} . \tag{3.6}$$

Clearly p_v is nondecreasing in t, and there is a constant $C_1(m, q)$ so that

$$\sup_{M \times (0,t)} |F^{v,A}| \leqq C_1 p_v^2(t) . \tag{3.7}$$

In order to prove that the functions u^v are well-defined, it suffices to show that $p_v(t) < \infty$ for all v and for all $0 < t < 1$. Let us first consider the case when $v = 0$. By Proposition 2.4 and the definition of u^0, we have

$$p_0(t) \leqq C_2(q) \exp(2Kt) \sup_{M,A} |\nabla h^A| , \tag{3.8}$$

where $C_2(q)$ is a constant depending only on q. Hence u^1 is well-defined and is smooth on $M \times (0, 1)$. Suppose u^v is defined, $p_{v-1}(t) < \infty$ for $0 < t < 1$, and u^v is smooth on $M \times (0, 1)$. Theorem 2.2 implies that for $0 < t < 1$,

$$\begin{aligned} \sup_{M \times (0,t)} |\nabla u^{v,A}| &\leqq C_3(m, K) t^{1/2} \sup_{M \times (0,t)} |F^{v-1,A}| + p_0(t) \\ &\leqq C_4(m, q, K, \Lambda) t^{1/2} p_{v-1}^2(t) + p_0(t) . \end{aligned}$$

Hence,

$$p_v(t) \leqq C_5(m, q, K, \Lambda) t^{1/2} p_{v-1}^2(t) + p_0(t) . \tag{3.9}$$

By induction hypothesis, we conclude that u^{v+1} is well-defined and is smooth on $M \times (0, t)$.

Now choose $0 < T_1 < 1$ such that

$$C_2 C_5 T_1^{1/2} \exp(2K) \sup_{A,M} |\nabla h^A| \leqq \frac{1}{4} . \tag{3.10}$$

By (3.8), we have

$$C_5 T_1^{1/2} p_0(t) \leqq \frac{1}{4} . \tag{3.11}$$

If $C_5 T_1^{1/2} p_{v-1}(t) \leqq \frac{1}{2}$ on $(0, T_1)$, then (3.9) implies that for $0 < t < T_1$,

$$\begin{aligned} C_5 T_1^{1/2} p_v(t) &\leqq (C_5 T_1^{1/2} p_{v-1}(t))^2 + C_5 T_1^{1/2} p_0(t) \\ &\leqq \frac{1}{4} + \frac{1}{4} \\ &\leqq \frac{1}{2} . \end{aligned}$$

Hence one can prove inductively that for all $\nu \geqq 0$,

$$C_5 T_1^{1/2} p_\nu(t) \leqq \frac{1}{2} \tag{3.12}$$

on $(0, T_1)$. In particular, $p_\nu(t)$ are uniformly bounded on $(0, T_1)$. Also by the definition of u^ν, inequality (3.7), and the fact that h is bounded, we conclude that u^ν are uniformly bounded on $M \times (0, T_1)$. Therefore the following functions X_ν and $\bar{X}_\nu$ are well-defined

$$\begin{aligned} X_\nu(t) = &\sup_M \sum_A |u^{\nu, A}(\cdot, t) - u^{\nu-1, A}(\cdot, t)| \\ &+ \sup_M \left(\sum_A |\nabla u^{\nu, A}(\cdot, t) - \nabla u^{\nu-1, A}(\cdot, t)|^2 \right)^{1/2} \end{aligned} \tag{3.13}$$

and

$$\bar{X}_\nu(t) = \sup_{0<\tau<t} X_\nu(\tau)\,, \tag{3.14}$$

for $0 < t < T_1$. Now

$$\begin{aligned} F^{\nu, A} - F^{\nu-1, A} &= \pi^A_{BC}(u^\nu) \langle \nabla u^{\nu, B}, \nabla u^{\nu, C} \rangle - \pi^A_{BC}(u^{\nu-1}) \langle \nabla u^{\nu-1, B}, \nabla u^{\nu-1, C} \rangle \\ &= (\pi^A_{BC}(u^\nu) - \pi^A_{BC}(u^{\nu-1})) \langle \nabla u^{\nu, B}, \nabla u^{\nu, C} \rangle \\ &\quad + \pi^A_{BC}(u^{\nu-1}) \langle \nabla u^{\nu, B} - \nabla u^{\nu-1, B}, \nabla u^{\nu, C} \rangle \\ &\quad + \pi^A_{BC}(u^{\nu-1}) \langle \nabla u^{\nu-1, B}, \nabla u^{\nu, C} - \nabla u^{\nu-1, C} \rangle\,. \end{aligned}$$

Applying the mean-value theorem to

$$\pi^A_{BC}(u^\nu) - \pi^A_{BC}(u^{\nu-1})\,,$$

we have

$$\sup_{M \times (0, t)} |F^{\nu, A} - F^{\nu-1, A}| \leqq C_6 \bar{X}_\nu(t)(p_\nu^2(t) + p_\nu(t) + p_{\nu-1}(t))\,,$$

where $C_6 = C_6(q, \Lambda)$. Inequality (3.12) asserts that for $0 < t < T_1$,

$$\sup_{M \times (0, t)} |F^{\nu, A} - F^{\nu-1, A}| \leqq C_7 \bar{X}_\nu(t)\,, \tag{3.15}$$

where $C_7 = C_7(m, K, q, \Lambda)$. On the other hand, Proposition 2.4 implies that for $0 < t < T_1$ and for $x \in M$,

$$\begin{aligned} |u^{\nu+1, A}(x, t) - u^{\nu, A}(x, t)| &\leqq t \sup_{M \times (0, t)} |F^{\nu, A} - F^{\nu-1, A}| \\ &\leqq C_7 t \bar{X}_\nu(t)\,. \end{aligned} \tag{3.16}$$

Using (3.15) and Theorem 2.2,

$$\begin{aligned} |\nabla u^{\nu+1, A}(x, t) - \nabla u^{\nu, A}(x, t)| &\leqq C_8(m, K) t^{1/2} \sup_{M \times (0, t)} |F^{\nu, A} - F^{\nu-1, A}| \\ &\leqq C_9(m, q, K, \Lambda) t^{1/2} \bar{X}_\nu(t)\,. \end{aligned} \tag{3.17}$$

Combining (3.16), (3.17) and the definition of $\bar{X}_\nu$, we have

$$\bar{X}_{\nu+1}(t) \leqq C_{10}(m, q, K, \Lambda) t^{1/2} \bar{X}_\nu(t) , \tag{3.18}$$

for $0 < t < T_1$.

Let us now choose $0 < T_0 < 1$ to be sufficiently small so that $C_{10} T_0 < 1$. If $0 < t < T_0$, then by induction we conclude that

$$\begin{aligned} \bar{X}_{\nu+1}(t) &\leqq (C_{10} t^{1/2})^\nu \bar{X}_0(t) \\ &\leqq (C_{10} T_0^{1/2})^\nu \bar{X}_0(t) \\ &\leqq C_{11}(C_{10} T_0^{1/2})^\nu \left(\sup_{M, A} |h^A| + \sup_{M, A} |\nabla h^A| \right) , \end{aligned} \tag{3.19}$$

where C_{11} is a constant depending only on m, K and q. Here we have used Proposition 2.4, and the fact that $|u^{0, A}| \leqq \sup_M |h^A|$ for all A.

Since $C_{10} T_0^{1/2} < 1$, the sequence of functions $\sum_{\nu=1}^{\infty} X_\nu(t)$ converges uniformly on $(0, T_0)$. Hence $u^{\nu, A}$ and $\nabla u^{\nu, A}$ converges uniformly on $M \times (0, T_0)$. Let $u = \lim_{\nu \to \infty} u^\nu$. Then ∇u^A exists and is the uniform limit of $\nabla u^{\nu, A}$. By (3.15) it is easy to see that $F^{\nu, A}$ also converges uniformly to $F^A = \pi^A_{BC}(u) \langle \nabla u^B, \nabla u^C \rangle$ on $M \times (0, T_0)$ for all A. Hence on $M \times (0, T_0)$, we have

$$u^A(x, t) = - \int_0^t d\tau \int_M H(x, y, t - \tau) F^A(u, \tau) dy + \int_M H(x, y, t) h^A(y) dy . \tag{3.20}$$

Note that each u^ν is smooth on $M \times (0, T_0)$, and satisfies $\left(\Delta - \frac{\partial}{\partial t} \right) u^{\nu, A} = F^{\nu-1, A}$ for $A = 1, \ldots, q$. By (3.5), (3.7) and (3.12) it is easy to see that F^ν and u^ν are uniformly bounded on $M \times (0, T_0)$. By [L-S-U, p. 211 Theorem 11.1], for any compact subset $\Omega \subset M$ and $0 < t_1 < t_2 < T_0$, there exists $C_{12} > 0$ and $0 < \beta < 1$ independent of ν and A, so that

$$|\nabla u^{\nu, A}(x, t) - \nabla u^{\nu, A}(x', t')| \leqq C_{12}((r(x, x'))^\beta + |t - t'|^{\beta/2})$$

for all $x, x' \in \Omega$ and for all $t_1 < t, t' < t_2$. Letting $\nu \to \infty$, we have

$$|\nabla u^A(x, t) - \nabla u^A(x', t')| \leqq C_{12}((r(x, x'))^\beta + |t - t'|^{\beta/2}) .$$

Hence by (3.20) one can conclude that u satisfies (3.1) in $M \times (0, T_0)$. It is easy to see that $\lim_{t \to 0} u^A(x, t) = h^A(x)$ and the proof of the theorem is completed.

Theorem 3.4 *Let M and N be complete Riemannian manifolds where M is noncompact with Ricci curvature bounded below by $-K \leqq 0$. Let $h \in C^1(M, N)$ be a map with bounded energy density such that $h(M)$ is also bounded. Then there exists $T_0 > 0$ and a map u which satisfies the parabolic equation for harmonic maps on $M \times (0, T_0)$ with initial data h.*

Proof. We embed a bounded nieghborhood N' of $h(M)$ as described in the beginning of this section. By Theorem 3.3, one can solve (3.1) and (3.2) for short

time. Theorem 3.4 then follows from Proposition 2.5, Theorem 2.7, Lemma 3.1 and Lemma 3.2.

Remark 3.1 From the proof of the theorem, it is easy to see that T_0 depends only on M, N, a neighborhood of $h(M)$ and the bound on the energy density of h. Furthermore, the energy density and the image of the solution u is bounded on $M \times [0, T_0)$ by a constant depending only on the known quantities mentioned above. The limit $\lim_{t\to 0} u(x, t) = h(x)$ is uniform on M, and also by (3.20), (2.6) and (2.9) we have $\lim_{t\to 0} e(u)(x, t) = e(h)(x)$ uniformly on compact subsets.

We will now prove a uniqueness theorem for the parabolic equation of harmonic maps.

Theorem 3.5 *Let M and N be complete Riemannian manifolds where M is noncompact. Suppose there exists a point $p \in M$ and a constant $a > 0$ such that $V_p(r) \leqq \exp(a(1 + r^2))$ for all $r > 0$. Let u_1 and u_2 be two solutions of the parabolic equation for harmonic maps from $M \times (0, T)$ to N with the same initial condition. Suppose the images and the energy densities of u_1 and u_2 are bounded on $M \times [0, T)$, then $u_1 = u_2$.*

Proof. By assumptions, there exists a bounded open set N' which contains the images of u_1 and u_2. Using the method of embedding N' in $\mathbf{R}^q$ described in the beginning of this section, Lemma 3.1 implies that u_1 and u_2 satisfy (3.1) in $M \times (0, T)$. Let

$$\varphi = \sum_A (u_1^A - u_2^A)^2 .$$

Then φ satisfies

$$\left(\Delta - \frac{\partial}{\partial t}\right)\varphi = 2\sum_A (u_1^A - u_2^A)\left(\Delta - \frac{\partial}{\partial t}\right)(u_1^A - u_2^A) + 2\sum_A |\nabla(u_1^A - u_2^A)|^2$$

$$= 2\sum_{A,B,C} (u_1^A - u_2^A)(\pi_{BC}^A(u_1)\langle \nabla u_1^B, \nabla u_1^C\rangle - \pi_{BC}^A(u_2)\langle \nabla u_2^B, \nabla u_2^C\rangle) \tag{3.21}$$

$$+ 2\sum_A |\nabla u_1^A - u_2^A)|^2 .$$

As in the proof of Theorem 3.3, one can compute and show that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\varphi \geqq -C_{13}\left(\sum_A |u_1^A - u_2^A|\right)\left[\sum_B (|u_1^B - u_2^B| + |\nabla(u_1^B - u_2^B)|)\right]$$

$$+ 2\sum_A |\nabla(u_1^A - u_2^A)|^2$$

$$\geqq -C_{14}\varphi ,$$

where C_{13} and C_{14} are constants depending only on q, the upper bounds of $|\pi_{BC}^A|$ and $|\pi_{BCD}^A|$ for $1 \leqq A, B, C, D \leqq q$, and the upper bounds of the energy densities of u_1 and u_2. Since φ is bounded and $\varphi(x, 0) \equiv 0$, by [K-L, Theorem 1], we conclude that $\varphi \equiv 0$ on $M \times [0, T)$. Hence u_1 and u_2 are equal.

4 Harmonic maps: $K = 0$ and $\kappa = 0$

In this section we will show that if the upper bound of the sectional curvature κ of N is 0, then equation (0.1) possesses a smooth solution for all time. If in addition, the Ricci curvature of M is nonnegative, then one can deform the initial map to a harmonic map through the solution of the parabolic equation.

Theorem 4.1 *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by $-K \leqq 0$, and let N be a complete Riemannian manifold with nonpositive sectional curvature. Let $h \in C^1(M, N)$ be such that $h(M)$ is bounded in N and $\sup_M e(h) < \infty$. Then (0.1) has a unique smooth solution u defined on $M \times (0, \infty)$ with $u(x, 0) = h(x)$, which satisfies the properties that:*

(a) $u(M \times [0, T])$ is bounded for all $T > 0$; and

(b) $\sup_{M \times [0, T]} e(u) < \infty$, for all ∞.

Proof. Let T^* be the supremum of those T_1 such that (0.1) has a solution u on $M \times [0, T_1)$ satisfying (a) and (b) for all $T < T_1$. By Theorem 3.4 and Remark 3.5, we have $T^* > 0$. Let us assume that $T^* < \infty$. The assumption that N has nonpositive curvature implies that (see [E-S]) the energy density satisfies

$$\left(\Delta - \frac{\partial}{\partial t}\right) e(u) \geqq -2Ke(u) \tag{4.1}$$

on $M \times (0, T^*)$. By Remark 3.5, we have

$$\lim_{t \to 0} e(u(x, t)) = e(h)(x) \tag{4.2}$$

uniformly on compact sets of M. Inequality (4.1) asserts that the function

$$\exp(-2Kt) e(u(x, t))$$

is a subsolution of the heat equation for functions. By (4.2) and the maximum principle in [L-K] (see also [Lo-T]) we have

$$\exp(-2Kt) e(u(x, t)) \leqq \sup_M e(h) .$$

Hence

$$\sup_{M \times [0, T^*)} e(u) \leqq \exp(2KT^*) \sup_M e(h) . \tag{4.3}$$

On the other hand, by the definition of T^*, $u\left(M \times \left[0, \frac{T^*}{2}\right]\right)$ is in a bounded open set N' of N. Let $\delta > 0$ be the injectivity radius of N'. By Theorem 3.6, Remark 3.5, and the proof of Theorem 3.3, we conclude that as $t \to 0$, $u(x, t)$ converges uniformly on M to $h(x)$. Therefore, there exists $t_0 > 0$ such that $d(u(x, t), h(x)) < \delta$ for all $x \in M$ and $0 < t < t_0$, where d is the distance function of N.

Consider the map $v(x, t) = u(x, t + t_0)$ for $0 \leqq t < T^* - t_0$. Clearly, v also satisfies (0.1) on $M \times [0, T^* - t_0)$ with initial condition $v(x, 0) = u(x, t_0)$. For $x \in M$, let $\sigma(x, s)$ for $0 \leqq s \leqq 1$, be the unique minimal geodesic joining $v(x, 0)$ to

$u(x, 0)$. For $0 < t < T^* - t_0$, let $\sigma(x, t, \cdot)$ be the geodesic parametrized by arclength joining $v(x, t)$ to $u(x, t)$ which is homotopic to the curve

$$f(x, s) = \begin{cases} v(x, t, t - s) , & 0 \leqq s \leqq t ; \\ \sigma(x, s - t) , & t \leqq s \leqq t + 1 ; \\ u(x, s - t - 1) , & t + 1 \leqq s \leqq 2t + 1 . \end{cases}$$

Since N has nonpositive curvature, $\sigma(x, t, \cdot)$ is unique. Let $\rho(x, t)$ be the length of $\sigma(x, t, \cdot)$. Then ρ^2 is smooth on $M \times (0, T^* - t_0)$ (see [S-Y3]) and the function $\varphi = (\rho^2 + 1)^{1/2}$ must satisfy

$$\left(\Delta - \frac{\partial}{\partial t}\right)\varphi \geqq 0$$

and

$$|\nabla\varphi|^2 \leqq 2(e(u) + e(v))$$

on $M \times (0, T^* - t_0)$. Hence by (4.3), the fact that M has at most exponential volume growth and Lemma 3.1 in [Lo-T], we have

$$\sup_M \varphi(\cdot, t) \leqq \sup_M \varphi(\cdot, 0) \leqq (\delta^2 + 1)^{1/2}$$

for all $0 < t < T^* - t_0$. In particular

$$\sup_{M \times [0, T^* - t_0)} d(v, u) \leqq \sup_{M \times [0, T^* - t_0)} \rho \leqq \delta .$$

Since $u\left(M \times \left[0, T^* - \frac{t_0}{2}\right]\right)$ is bounded, by the definition of v and the above inequality, we conclude that $u(M \times [0, T^*))$ is bounded. Using this fact together with (4.3), we can extend u as a solution of (0.1) beyond T^* because of Theorem 3.3 and Theorem 3.6. This contradicts the assumption that $T^* < \infty$. Hence T^* must be ∞.

In fact, we can remove the restriction that the image of the initial data is bounded. First we need the following lemma.

Lemma 4.2 *Let N^n be a complete simply connected manifold with nonpositive curvature. Let p be a fixed point. For any $R > 0$ there exists a smooth map ϕ_R from N to N such that $\phi_R|_{B_p(R)} = \text{identity}$, $\phi_R(M) \subset B_p(2R)$, and $\sup_M e(\phi_R) \leqq C_1$ for some constant C_1 which is independent of R.*

Proof. Note that N is diffeomorphic to $\mathbf{R}^n$ under the exponential map at p. In geodesic polar coordinates at p, the metric on N can be expressed as

$$ds^2 = dr^2 + g_{\alpha\beta}(r, \theta) d\theta^\alpha d\theta^\beta ,$$

where $(\theta^2, \ldots, \theta^n)$ is the local coordinates on $\mathbf{S}^{n-1}$. Here and below, the Greek indices ranges from 2 to n.

For any $R > 0$, let $\tilde{\phi}$ be a real valued function on $[0, \infty)$, such that $0 \leqq \tilde{\phi} \leqq 1$, $\tilde{\phi} = 1$ on $[0, R]$, $\tilde{\phi} = 0$ on $[2R, \infty)$, and $|\tilde{\phi}'| \leqq \frac{C_2}{R}$ for some C_2 which is independent of R. Now if x has coordinates (r, θ), we define $\phi_R(x)$ to be the point with coordinates $(r\phi_R(r), \theta)$. Then $\phi_R = \textit{identity}$ on $B(R)$ and $\phi_R(M) \subset B(2R)$. It remains to estimate the energy density of ϕ_R. Direct computation shows that

$$\frac{\partial}{\partial r} g_{\alpha\beta} = 2h_{\alpha\beta} ,$$

where $(h_{\alpha\beta})$ is the second fundamental form of $\partial B(r)$ with respect to $\frac{\partial}{\partial r}$. Since N is simply connected and has nonpositive curvature, therefore $\partial B(r)$ is strictly convex and $(h_{\alpha\beta})$ is positive definite. Hence for all $\xi \in \mathbf{R}^n$ and for all $r \leqq r'$,

$$g_{\alpha\beta}(r, \theta)\xi^\alpha \xi^\beta \leqq g_{\alpha\beta}(r', \theta)\xi^\alpha \xi^\beta , \tag{4.4}$$

It is clear that $e(\phi_R) = 1$ on $B(R)$. For x with $r = r(x) \geqq R$, let us choose θ such that $g_{\alpha\beta}(r, \theta) = \delta_{\alpha\beta}$. Then the energy density of ϕ_R is

$$\begin{aligned} e(\phi_R)(x) &= \left(\frac{\partial}{\partial r}(r\tilde{\phi}(r)) \right)^2 + g^{\alpha\beta}(r, \theta) g_{\alpha\beta}(\phi_R(r), \theta) \\ &\leqq (1 + C_2)^2 + \sum_\alpha g_{\alpha\alpha}(\phi_R(r), \theta) \\ &\leqq (1 + C_2)^2 + \sum_\alpha g_{\alpha\alpha}(r, \theta) \\ &\leqq (1 + C_2)^2 + (n - 1) , \end{aligned}$$

where we have used (4.4) and the fact that $\phi_R(r) \leqq r$ for all r. This completes the proof of the lemma.

Theorem 4.1 *Let M and N be as in Theorem* 4.1. *For any $h \in C^1(M, N)$ with bounded energy density, there exists a unique solution u of* (0.1) *on $M \times (0, \infty)$ such that $u(x, 0) = h(x)$ and $\sup_{M \times [0, T]} e(u) < \infty$ for all $T > 0$.*

Proof. Let us first assume that N is simply connected. Let y_0 be a fixed point in N and let $R_\nu \to \infty$. Denote $\Omega_\nu = B_{y_0}(R_\nu)$. By Lemma 4.2, there exists smooth maps g_ν from N to itself such that $g_\nu|_{\Omega_\nu} = \textit{identity}$, $g_\nu(N)$ is compact, and $\sup_{\nu, N} e(g_\nu) < \infty$. Consider $h_\nu = g_\nu \circ h \in C^1(M, N)$. Then there exists a constant C_3, independent of ν such that

$$\sup_M e(h_\nu) \leqq C_3 \tag{4.5}$$

for all ν. Since $h_\nu(M)$ is bounded, by Theorem 4.1 there exists a solution u_ν of (0.1) on $M \times (0, \infty)$. From the proof of Theorem 4.1 and (4.5), it is easy to see that for all $T > 0$ and for all ν, there exists a constant C_4 not depending on ν such that

$$\sup_{M \times [0, T]} e(u_\nu) \leqq C_4 . \tag{4.6}$$

The proof of the main lemma in [H] and the fact that N has nonpositive curvature implies that

$$\left(\Delta - \frac{\partial}{\partial t}\right)|(u_v)_t|^2 \geqq 0 .$$

Let $p \in M$ be a point. As in [Lo-T], there exists a constant C_5 depending only on m such that for t, $R > 0$,

$$\int_0^t \int_{B_p(R)} |(u^v)_t|^2(y, \tau)\, dy\, d\tau \leqq E_v(0, 2R) + \frac{C_5}{R^2}\int_0^t E_v(\tau, 2R)\, d\tau , \tag{4.7}$$

where $E_v(t, R) = \int_{B_p(R)} e(u_v)(y, t)\, dy$.

On the other hand because $|(u^v)_t|^2$ is a nonnegative subsolution for the heat equation, the mean-value inequality Theorem 1.1 gives

$$|(u^v)_t|^2(p, T) \leqq C(K, 1, R, T) \int_{B_p(R)\times[0, T]} |(u^v)_t|^2(y, s)\, dy\, ds$$

for $0 < T \leqq \frac{1}{2}R^2$, where

$$C(K, 1, R, T) = C_6 \frac{V(K, 2R)}{V_p(R)}(R\sqrt{K}\coth(R\sqrt{K}) + 1) \times \exp(C_7 K T)(R^{-m-2} + T^{-(m/2)-1})$$

for some constants C_6 and C_7 depending only on m. Hence by setting $R = 2\sqrt{T}$, this implies

$$|(u^v)_t|^2(p, T) \leqq C_8 T^{-(m/2)-1} \frac{V(K, 4\sqrt{T})}{V_p(K, 2\sqrt{T})} \int_{B_p(2\sqrt{T})\times[0,T]} |u_t|^2(y, s)\, dy\, ds ,$$

where C_8 is a constant depending only on m and the upper bound of T. Together with (4.6), we have

$$|(u^v)_t|^2(p, T) \leqq C_8 T^{-(m/2)-1} \frac{V(K, 4\sqrt{T})}{V_p(K, 2\sqrt{T})}\left(E_v(0, 4\sqrt{T}) + \frac{C_5}{2T}\int_0^T E_v(\tau, 4\sqrt{T})\, d\tau\right).$$

Therefore, given a bounded domain Ω in M and $T > 0$, there exists a constant depending only on m, T, Ω and $\sup_{M\times[0, T]} e(u_v)$, such that for $0 < t \leqq T$ and $x \in \Omega$, we obtain the estimates

$$|(u_v)_t|(x, t) \leqq C_9 t^{-1/2} ,$$

and

$$\operatorname{dist}(u_v(x, t), h_v(x)) \leqq 2C_9 t^{1/2} . \tag{4.8}$$

By (4.6) and (4.7) and $g_v = \textit{identity}$ on Ω_v, we conclude that for any $T > 0$ and $R > 0$ there exists a bounded set $N(T, R) \subset N$ such that

$$u_v(B_p(R)\times[0, T]) \subset N(T, R) \tag{4.9}$$

for sufficiently large v.

From (4.6) and (4.9), it is easy to deduce that if we pass to a subsequence, u_ν will converge to a map u which is a solution of (0.1) on $M \times (0, \infty)$ and by (4.6) and (4.8) we can see that $u(x, 0) = h(x)$. The convergence is uniform on $B_p(R) \times [0, T]$ for all $0 < T < \frac{1}{4}R^2$ together with their first and second derivatives in the space variable and their first derivative in the time variable. Hence u is a solution of (0.1) which satisfies the condition of the theorem. Uniqueness of u follows from Theorem 2.3 in [Lo-T].

In general, let $\bar{M}$ and $\bar{N}$ be the universal coverings of M and N and let π_M and π_N be the projections from $\bar{M}$ to M and from $\bar{N}$ to N respectively. Then one can find h^* from $\bar{M}$ to $\bar{N}$ such that $h \circ \phi_M = \phi_N \circ h^*$ and that h^* is invariant under deck transformations of $\bar{M}$. Note that h^* has bounded energy density. Therefore by the previous argument, there is a solution u^* of the parabolic equation for harmonic maps from $\bar{M} \times (0, \infty)$ to $\bar{N}$ with initial data h^*. Define $u(x, t) = \pi_N(u^*(\bar{x}, t))$, where $\pi_M(\bar{x}) = x$. In order to show that $u(x, t)$ is well-defined and is a solution of (0.1) on $M \times (0, \infty)$ with initial data h, it is sufficient to show that $u^*(\bar{x}, t) = u^*(\sigma(\bar{x}), t)$ for all deck transformations σ on $\bar{M}$. However, this follows from Theorem 2.3 in [Lo-T] and the fact that h^* is invariant under deck transformations.

Theorem 4.3 *Let M^m be a complete noncompact Riemannian manifold with nonnegative Ricci curvature, and let N^n be a complete Riemannian manifold with nonpositive sectional curvature. Let $h \in C^1(M, N)$ with bounded energy density. Then* (0.1) *has a unique solution u on $M \times (0, \infty)$ with initial condition $u(\cdot, 0) = h$, which satisfies $\sup_{M \times [0, T]} e(u) < \infty$ for all $T > 0$. Furthermore, if there exists a point $p \in M$ and a sequence $t_\nu \to \infty$ such that $u(p, t_\nu)$ converges in N (for example when N is compact), then by passing to a subsequence, $u(\cdot, t_\nu)$ converges uniformly on compact subsets together with their first and second derivatives to a harmonic map u_∞.*

Proof. The existence of u is guaranteed by Theorem 4.3. Since the Ricci curvature of M is nonnegative and the sectional curvature of N is nonpositive, Bochner's formula in [E-S] implies that $e(u)$ is a subsolution of the heat equation for functions on M. Theorem 3.1 asserts that $e(u)$ is bounded on $M \times [0, T]$ for all $T < \infty$. Hence by the maximum principle of [K-L], we have

$$\sup_{M \times [0, \infty)} e(u) \leqq \sup_M e(h) < \infty \,. \tag{4.10}$$

Suppose there exist $p \in M$ and $t_\nu \to \infty$ such that $u(p, t_\nu)$ converges. By (4.7), (4.10), the fact that M has nonnegative Ricci curvature, and the volume comparison theorem, we have

$$\lim_{\substack{R \to \infty \\ t \to \infty}} \frac{1}{tV(B)} \int_0^t d\tau \int_{B_p(R)} |u_t|^2(y, \tau)\, d\tau = 0 \,, \tag{4.11}$$

where $V(R)$ is the volume of $B_p(R)$. On the other hand, as in the proof of the main lemma in [H], since N has nonpositive sectional curvature, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right)|u_t|^2 \geqq 0 \,.$$

Hence by Theorem 1.1 and (4.11), a subsequence of $|u_t|^2(\cdot, t_\nu)$ converges to 0, uniformly on compact sets of M.

The Hölder estimates for the gradients, the Schauder estimates for the parabolic equations (see [L-S-U]), (4.10), the fact that u satisfies (0.1), and that $u(p, t_\nu)$ converges, imply that the sequence of maps $v_\nu(x, t) = u(x, t_\nu + t)$ converges together with their first and second derivatives in the space variable and the first derivative of the time variable uniformly on $B_p(R) \times [0, 1]$ for all $R > 0$. Since each v_ν satisfies (0.1), and for all x, $|u_t(x, t_\nu)| \to 0$ as $\nu \to \infty$, the theorem follows.

5 Harmonic maps: $\lambda(M) > 0$ and $\kappa = 0$

In this section we would like to discuss the question of existence of harmonic maps from a manifold M whose spectrum has a positive lower bound. We also assume that the sectional curvature of N is nonpositive. In particular we will study the boundary valued problem at infinity for harmonic maps between two hyperbolic spaces.

Lemma 5.1 *Let M be a complete noncompact manifold with Ricci curvature bounded from below by $-K \leqq 0$ and $\inf_{x \in M} V_x(1) = b > 0$. Assume that the lower bound of the spectrum $\lambda(M)$ of M is positive. Let N be a complete manifold with sectional curvature bounded above by $\kappa \geqq 0$. Let u be a solution of the parabolic equation for harmonic map from $M \times [0, T)$ to N where $1 < T \leqq \infty$, and suppose $|u_t|$ is a bounded function on $M \times [0, T)$. If $\kappa > 0$, we also assume that $\sup_{M \times [0, T)} e(u) = \beta < \infty$. For any $p > 1$, there exists a constant C_1 depending only on K, λ, p, and b such that for all $1 < t < T$ and for all $x \in M$,*

$$(5.1) \qquad |u_t|^2(x, t) \leqq C_1 \exp\left(-\frac{4\lambda(q-1)}{q^2} t + 2\kappa\beta t\right)\left(\int_M |u_t|^{2p}(y, 0)\, dy\right)^{1/p},$$

where q is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. We may assume that the L^p norm of $|u_t|^2(\cdot, 0)$ is finite, otherwise the result is obviously true. As before, by [H] the function $f(x, t) = \exp(-2\kappa\beta t)|u_t|^2(x, t)$ is a subsolution of the heat equation for functions on M. By the maximum principle in [K-L], we have

$$(5.2) \qquad f(x, t) \leqq \int_M H(x, y, t) f(y, 0)\, dy$$

for all $x \in M$ and for all $0 < t < T$.

Let us first estimate the L^q norm of the heat kernel H on M with $q > 1$. For any smooth compactly supported function ϕ, for any $0 < \varepsilon < 1$, and for a real number δ, we have

$$\begin{aligned} \frac{\partial}{\partial t} \int_M \phi^2(y) H^q(x, y, t)\, dy &= q \int_M \phi^2(y) H^{q-1}(x, y, t) \frac{\partial H}{\partial t}(x, y, t)\, dy \\ &= q \int_M \phi^2(y) H^{q-1}(x, y, t) \Delta H(x, y, t)\, dy \end{aligned}$$

$$
\begin{aligned}
&= -q(q-1)\int_M \phi^2(y)H^{q-2}(x,y,t)|\nabla H(x,y,t)|^2\,dy \\
&\quad -2q\int_M \phi(y)H^{q-1}(x,y,t)\langle \nabla\phi(y),\nabla H(x,y,t)\rangle\,dy \\
&= -\frac{4(q-1)}{q}\int_M \phi^2(y)|\nabla H^{q/2}(x,y,t)|^2\,dy \\
&\quad -4\int_M \phi(y)H^{q/2}(x,y,t)\langle \nabla\phi(y),\nabla H^{q/2}(x,y,t)\rangle\,dy \\
&= -\frac{4(q-1)}{q}\Bigg(\int_M \phi^2(y)|\nabla H^{q/2}(x,y,t)|^2\,dy \\
&\quad +\left(\frac{q}{q-1}-2\delta\right)\int_M \phi(y)H^{q/2}(x,y,t) \\
&\quad \langle \nabla\phi(y),\nabla H^{q/2}(x,y,t)\rangle\,dy \\
&\quad +2\delta\int_M \phi(y)H^{q/2}(x,y,t)\langle \nabla\phi(y),\nabla H^{q/2}(x,y,t)\rangle\,dy\Bigg).
\end{aligned}
$$

Using the inequality

$$
\begin{aligned}
&\left|2\delta\int_M \phi(y)H^{q/2}(x,y,t)\langle\nabla\phi(y),\nabla H^{q/2}(x,y,t)\rangle\,dy\right| \\
&\qquad \leqq \varepsilon\int_M \phi^2(y)|\nabla H^{q/2}(x,y,t)|^2\,dy+\frac{\delta^2}{\varepsilon}\int_M |\nabla\phi(y)|^2H^q(x,y,t)\,dy\,,
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_M \phi^2(y)H^q(x,y,t)\,dy \leqq\ & -\frac{4(q-1)}{q}\Bigg((1-\varepsilon)\int_M \phi^2(y)|\nabla H^{q/2}(x,y,t)|^2\,dy \\
&+\left(\frac{q}{q-1}-2\delta\right)\int_M \phi(y)H^{q/2}(x,y,t)\langle\nabla\phi(y),\nabla H^{q/2}(x,y,t)\rangle\,dy \\
&-\frac{\delta^2}{\varepsilon}\int_M |\nabla\phi(y)|^2H^q(x,y,t)\,dy\Bigg).
\end{aligned}
$$

Let us now choose δ such that $2(1-\varepsilon)=\dfrac{q}{q-1}-2\delta$, the identity

$$
\begin{aligned}
&\int_M \phi^2(y)|\nabla H^{q/2}(x,y,t)|^2\,dy+2\int_M \phi(y)H^{q/2}(x,y,t)\langle\nabla\phi(y),\nabla H^{q/2}(x,y,t)\rangle\,dy \\
&\qquad =\int_M |\nabla(\phi(y)H^{q/2}(x,y,t))|^2\,dy-\int_M |\nabla\phi(y)|^2H^q(x,y,t)\,dy
\end{aligned}
$$

and the fact that the lower bound of the spectrum λ is positive imply that

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \phi^2(y)H^q(x,y,t)\,dy &\leqq -\frac{4(q-1)}{q}\Bigg((1-\varepsilon)\int_M |\nabla(\phi(y)H^{q/2}(x,y,t))|^2\,dy \\ &\quad -\left(1-\varepsilon-\frac{\delta^2}{\varepsilon}\right)\int_M |\nabla\phi(y)|^2 H^q(x,y,t)\,dy\Bigg) \\ &\leqq -\frac{4(q-1)}{q}\Bigg(\lambda(1-\varepsilon)\int_M \phi^2(y)H^q(x,y,t)\,dy \\ &\quad +\left(1-\varepsilon-\frac{\delta^2}{\varepsilon}\right)\int_M |\nabla\phi(y)|^2 H^q(x,y,t)\,dy\Bigg).\end{aligned}$$

However, by the estimate of $H(x, y, t)$ in [L-Y], since M has Ricci curvature bounded from below by $-K$ and $\inf_{x\in M} V_x(1) = b > 0$, $H(x, y, t)$ is uniformly bounded from above as a function of y. Together with $\int_M H(x, y, t)\,dy = 1$, this implies that the integral $\int_M H^q(x, y, t)\,dy$ must be finite for $q > 1$. Hence if we choose the cut-off function to be identically 1 on $B_x(R)$, identically 0 on $M\backslash B_x(2R)$, and satisfies $|\nabla\phi|^2 \leqq \frac{3}{R^2}$, the above inequality will approach

$$\frac{\partial}{\partial t}\int_M H^q(x,y,t)\,dy \leqq \lambda(1-\varepsilon)\frac{4(q-1)}{q}\int_M H^q(x,y,t)\,dy$$

as $R \to \infty$. Taking $\varepsilon \to 0$, this yields the estimate

$$\frac{\partial}{\partial t}\int_M H^q(x,y,t)\,dy \leqq -\frac{4\lambda(q-1)}{q}\int_M H^q(x,y,t)\,dy\,.$$

Integrating from 1 to t, and using the estimate of the heat kernel in [L-Y], we have

$$\begin{aligned}\int_M H^q(x,y,t)\,dy &\leqq \exp\left(-\frac{4\lambda(q-1)}{q}(t-1)\right)\int_M H^q(x,y,1)\,dy \\ &\leqq \exp\left(-\frac{4\lambda(q-1)}{q}(t-1)\right)\left(\sup_{y\in M} H^{q-1}(x,y,1)\right)\int_M H(x,y,1)\,dy \\ &\leqq C_2\exp\left(-\frac{4\lambda(q-1)}{q}(t-1)\right),\end{aligned}$$

where C_2 is a constant depending only on K, q, and b.

By (5.1), (5.2) and the Hölder inequality we have for $1 < t < T$, and for $x \in M$,

$$\begin{aligned}f(x,t) &\leqq \left(\int_M H^q(x,y,t)\,dy\right)^{1/q}\left(\int_M f^p(y,0)\,dy\right)^{1/p} \\ &\leqq C_3\exp\left(-\frac{4\lambda(q-1)}{q^2}t\right)\left(\int_M f^p(y,0)\,dy\right)^{1/p},\end{aligned}$$

where $q > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, and C_3 is a constant depending only on K, λ, p, and b. The result now follows from the definition of f.

Recall that the tension field $\tau(u)$ of a map u from M to N is a vector field along u. In local coordinates of M and N, using previous notations,

$$\tau(u)^\alpha = \Delta u^\alpha + \Gamma^\alpha_{\beta\gamma} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} g^{ij} .$$

Theorem 5.2 *Let M^m be a complete manifold with Ricci curvature bounded below by $-K \leqq 0$, and $\inf_{x \in M} V_x(1) = b > 0$. Assume that the lower bound of the spectrum $\lambda(M)$ of M is positive. Let N^n be a complete manifold with nonpositive curvature. Let u be a solution of the parabolic equation for harmonic maps from $M \times [0, \infty)$ to N^n. Suppose that the square of the norm of the tension field of the initial data of u is bounded and in $L^p(M)$ for some $p > 1$ and suppose that the energy density of u is also bounded on $M \times [0, T]$ for all $T < \infty$. Then $u(\cdot, t)$ converges to a harmonic map u_∞ as $t \to \infty$ with the following properties: (i) $\sup_M e(u) < \infty$; (ii) $\sup_{x \in M} \operatorname{dist}(u_\infty(x), u(x, 0)) < \infty$; and (iii) if in addition that the norm of the tension field of the initial data tends to 0 as $x \to \infty$ then $\lim_{x \to \infty} \operatorname{dist}(u_\infty(x), u(x, 0)) = 0$.*

Proof. Since the square of the norm of the tension field of the initial data is bounded and in $L^p(M)$ for some $p > 1$, the equation implies that $|u_t|^2(\cdot, 0)$ is bounded and in $L^p(M)$. By Theorem 1.2, the fact that $|u_t|^2$ is a subsolution of the heat equation, the assumption on the volume of geodesic balls of radius 1, that $e(u)$ is bounded on $M \times [0, T]$ for finite T, and (4.8), we conclude that $|u_t|^2$ is bounded on $M \times [0, T]$ for all finite T. By Lemma 5.1 implies that $|u_t|^2(x, t)$ decays exponentially in t. Therefore as $t \to \infty$, $u(x, t)$ converges uniformly on M to a map u_∞. In order to prove that u_∞ is a harmonic map and satisfies (i), it suffices to prove that there exists a constant $C_4 > 0$ such that

$$e(u)(x, t) \leqq C_4$$

for all x and for all $t \geqq 1$. As in [Lo-T], if ϕ is a smooth function with compact support in M then we have

$$\frac{d}{dt} \int_M \phi^2 e(u) \leqq 4 \left(\int_M |\nabla \phi|^2 |u_t|^2 \right)^{1/2} \left(\int_M \phi^2 e(u) \right)^{1/2} .$$

Using the fact that

$$|u_t|^2(x, t) \leqq C_5 \exp(-at) \tag{5.3}$$

for some constants $C_5 > 0$, $a > 0$ and for all x and t, we have

$$\frac{d}{dt} \int_M \phi^2 e(u) \leqq 4\sqrt{C_5} \exp\left(-\frac{1}{2} at\right) \left(\int_M |\nabla \phi|^2 \right)^{1/2} \left(\int_M \phi^2 e(u) \right)^{1/2} .$$

From this it is easy to see that

$$\left(\int_M \phi^2 e(\cdot, t) \right)^{1/2} \leqq \frac{4\sqrt{C_5}}{a} \left(\int_M |\nabla \phi|^2 \right)^{1/2} + \left(\int_M \phi^2 e(\cdot, 0) \right)^{1/2} .$$

Using a suitable ϕ we see that $\int_{B_x(1)} e(u)(\cdot, t)$ is uniformly bounded by a constant independent of x and t. By the mean-value inequality, we conclude that $e(u)(x, t)$ is uniformly bounded by a constant C_4 independent of x and t for $t \geqq 1$.

It remains to prove that u_∞ satisfies (ii) and (iii). By (5.3) it is easy to see that

$$\sup_{x \in M} \operatorname{dist}(u_\infty(x), (x, 0)) < \infty .$$

Suppose the norm of the tension field of the initial data tends to 0 as $x \to \infty$, then

$$\lim_{x \to \infty} |u_t(x, 0)| = 0 .$$

Since the L^p norm of $|u_t(\cdot, t)|^2$ is less than or equal to the L^p norm of $|u_t(\cdot, 0)|^2$, hence for $\infty > T > 0$ the L^p norm of $|u_t|^2$ on $M \times [0, T]$ is finite. Since $|u_t|^2$ is a subsolution of the heat equation, therefore by Theorem 1.2 we conclude that

$$\lim_{x \to \infty} \sup_{t \in [0, T]} |u_t(x, t)| = 0 .$$

Now

$$d_N(u_\infty(x), u(x, 0)) \leqq \int_0^\infty |u_t(x, t)| = \int_0^T |u_t| + \int_T^\infty |u_t(x, t)| .$$

Given $\varepsilon > 0$, there exists $T > 0$, such that

$$\int_T^\infty |u_t(x, t)| \leqq \varepsilon .$$

For a fixed T, there exists $R > 0$ such that if $d_M(x) \geqq R$, then

$$\int_0^T |u_t(x, t) \leqq \varepsilon ,$$

where $d_M(x)$ is the distance from x to some fixed point. Hence if $d_M(x) \geqq R$, then

$$d_N(u_\infty(x), u(x, 0)) \leqq 2\varepsilon .$$

This complete the proof of the theorem.

As an application of the above theorem, let us consider the boundary value problem at infinity from $M = \mathbf{H}^m$ to $N = \mathbf{H}^n$, the hyperbolic spaces of dimension m and n with constant curvature -1. Let us first introduce some notations. Both M and N can be taken to be the unit balls in $\mathbf{R}^m$ and $\mathbf{R}^n$ with the Poincaré metric. The ideal boundaries of M and N are identified with $\mathbf{S}^{m-1}$ and $\mathbf{S}^{n-1}$, respectively.

Lemma 5.3 *Let f be a C^3 map from $\mathbf{S}^{m-1}$ into $\mathbf{S}^{n-1}$ so that the energy density of f with respect to the standard metrics on $\mathbf{S}^{m-1}$ and $\mathbf{S}^{n-1}$ is nowhere zero. Then f can be extended to a map from M to N so that the energy density is bounded, and the square of the norm of the tension field tends to 0 as $x \to \infty$ and is in $L^k(M)$ for all $k > \dfrac{m-1}{2}$.*

Proof. On M, the metric is

$$ds_M^2 = 4(1 - \rho^2)^{-2} g_{ij} dx^i dx^j ,$$

where $g_{ij} = \delta_{ij}$ and $\rho^2 = \sum_{i=1}^{m} (x^i)^2 < 1$. On N, we denote the metric in terms of polar coordinates by

$$ds_N^2 = 4(1 - r^2)^{-2}(dr^2 + r^2 h_{pq} d\theta^p d\theta^q),$$

where $r^2 = \sum_{\alpha=1}^{n} (u^\alpha)^2 < 1$. The indices p and q ranges from 2 to n, and $h_{pq} d\theta^p \theta^q$ is the standard metric on $\mathbf{S}^{n-1}$. The Christoffel symbols on N are given by

$$\Gamma_{11}^1 = \frac{2r}{1 - r^2},$$

$$\Gamma_{1q}^1 = 0,$$

$$\Gamma_{11}^s = 0$$

$$\Gamma_{pq}^1 = -\frac{r(1 + r^2) h_{pq}}{1 - r^2},$$

$$\Gamma_{1q}^s = \frac{(1 + r^2)\delta_{sq}}{r(1 - r^2)},$$

and

$$\Gamma_{pq}^s = \tilde{\Gamma}_{pq}^s$$

for $2 \leqq s, p, q \leqq n$. The terms $\tilde{\Gamma}_{pq}^s$ are the Christoffel symbols on $\mathbf{S}^{n-1}$ with respect to the standard metric.

For a map u which is given by $r = r(x^1, \ldots, x^m)$, and $\theta^p = \theta^p(x^1, \ldots, x^m)$, let us compute the component of the tension field $\tau(u)$. Direct computation shows that

$$\begin{aligned}
\tau(u)^1 &= \Delta r + \frac{2r|\nabla r|^2}{1 - r^2} - \frac{r(1 + r^2)}{1 - r^2} \sum_{p,q=2}^{n} h_{pq} \langle \nabla\theta^p, \nabla\theta^q \rangle \\
&= \frac{(1 - \rho^2)^2}{4} \Delta_0 r + \frac{1}{4}\Bigg(2(m - 2)(1 - \rho^2)\rho \frac{\partial r}{\partial \rho} \\
&\quad + \frac{r(1 - \rho^2)^2 (2|\nabla_0 r|^2 - (1 + r^2) \sum_{p,q=2}^{n} h_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle)}{1 - r^2} \Bigg).
\end{aligned} \tag{5.4}$$

For $s \geqq 2$, we have

$$\begin{aligned}
\tau(u)^s &= \Delta\theta^s + \frac{2(1 + r^2)}{r(1 - r^2)} \langle \nabla r, \nabla\theta^s \rangle + \sum_{p,q=2}^{n} \tilde{\Gamma}_{pq}^s \langle \nabla\theta^p, \nabla\theta^q \rangle \\
&= \frac{(1 - \rho^2)^2}{4}\Bigg(\Delta_0 \theta^s + \sum_{p,q=2}^{n} \tilde{\Gamma}_{pq}^s \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle \Bigg) \\
&\quad + \frac{1 - \rho^2}{2}\Bigg((m - 2)\rho \frac{\partial \theta^s}{\partial \rho} + \frac{(1 + r^2)(1 - \rho^2) \langle \nabla_0 r, \nabla_0 \theta^s \rangle}{r(1 - r^2)} \Bigg).
\end{aligned} \tag{5.5}$$

Here, ∇ denotes the gradient on M, ∇_0 denotes the Euclidean gradient, and Δ_0 denotes the Euclidean Laplacian operator. We have used the fact that the hyperbolic Laplacian is given by

$$\Delta = \frac{1}{4}\Bigg((1 - \rho^2)^2 \Delta_0 + 2(m - 2)(1 - \rho^2)\rho \frac{\partial}{\partial \rho} \Bigg).$$

Let us extend f near infinity in the following way. Denote

$$f(\eta^2, \ldots, \eta^m) = (f^2(\eta^2, \ldots, \eta^m), \ldots, f^n(\eta^2, \ldots, \eta^m)),$$

where $(\eta^2, \ldots, \eta^m)$ and $(\theta^2, \ldots, \theta^n)$ are coordinates on $\mathbf{S}^{m-1}$ of the domain manifold and $\mathbf{S}^{n-1}$ on the target manifold, respectively. Let $e(f)$ be the energy density of f with respect to the standard metrics on $\mathbf{S}^{m-1}$ and $\mathbf{S}^{n-1}$ which is nowhere zero. Define an extension u of f to M by

$$\theta^p(\rho, \eta^2, \ldots, \eta^m) = f^p(\eta^2, \ldots, \eta^m)$$

and

$$r(\rho, \eta^2, \ldots, \eta^m) = 1 - (1 - \rho)\phi(\eta^2, \ldots, \eta^m),$$

where

$$\phi = \sqrt{\frac{e(f)}{m-1}}.$$

This extension is well defined providing ρ is close to 1.

It is easy to see that $\Delta_0 r$ and $\sum_{p,q=2}^{n} \tilde{\Gamma}^s_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle$ are bounded if ρ is close to 1. We also have,

$$\frac{1-\rho}{1-r} = \frac{1}{\phi}, \tag{5.6}$$

$$\frac{\partial r}{\partial \rho} = \phi, \tag{5.7}$$

$$|\nabla_0 r|^2 = \phi^2 + \frac{(1-\rho)^2}{\rho^2} |\tilde{\nabla}\phi|^2, \tag{5.8}$$

$$\langle \nabla_0 r, \nabla_0 \theta^s \rangle = \frac{1-\rho}{\rho^2} \langle \tilde{\nabla}\phi, \tilde{\nabla} f^s \rangle, \tag{5.9}$$

and

$$\sum_{p,q=2}^{n} h_{pq} \langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle = \frac{e(f)}{\rho^2}, \tag{5.10}$$

where $\tilde{\nabla}$ denotes the gradient on $\mathbf{S}^{m-1}$.

By (5.6) and the fact that ϕ is nowhere zero, there exists a constant $C_7 > 0$ such that

$$C_7^{-1}(1-r) \leqq (1-\rho) \leqq C_7(1-r). \tag{5.11}$$

Hence

$$\begin{aligned}\tilde{\tau}(u)^1 &= 2(m-2)\frac{\partial r}{\partial \rho} + \frac{1-\rho^2}{1-r^2}\left(2|\nabla_0 r|^2 - (1+r^2)\sum_{p,q=2}^{n} h_{pq}\langle \nabla_0 \theta^p, \nabla_0 \theta^q \rangle\right) \\ &= 2(m-2)\phi + \frac{1-\rho^2}{1-r^2}\left(2\phi^2 + \frac{2(1-\rho)^2}{\rho^2}|\tilde{\nabla}\phi|^2 - \frac{(1+r^2)e(f)}{\rho^2}\right)\end{aligned}$$

$$= 2(m-2)\phi + \frac{2(1+\rho)}{1+r}\phi - \frac{(n-1)(1+\rho)(1+r^2)}{\rho^2(1+r)}\phi$$

$$+ \frac{2(1-\rho^2)(1-\rho)^2}{\rho^2(1-r^2)}|\tilde{\nabla}\phi|^2$$

$$= 2\left(\frac{1+\rho}{1+r} - 1\right)\phi + \left(2 - \frac{(1+\rho)(1+r^2)}{\rho^2(1+r)}\right)(n-1)\phi$$

$$+ \frac{2(1-\rho^2)(1-\rho)^2}{\rho^2(1-r^2)}|\tilde{\nabla}\phi|^2$$

$$= 2\frac{\rho - r}{1+r}\phi + \frac{(2\rho+1)(\rho-1) + (\rho-r)(\rho+r+\rho r) - \rho^2(1-r)}{\rho^2(1+r)}(n-1)\phi$$

$$+ \frac{2(1-\rho^2)(1-\rho)^2}{\rho^2(1-r^2)}|\tilde{\nabla}\phi|^2 \tag{5.12}$$

$$= O(1-\rho)$$

as $\rho \to 1$, where we have used the definition of r, (5.11), and the fact that ϕ is nowhere zero.

Similarly, for $n \geqq s \geqq 2$,

$$\tilde{\tau}(u)^s = (m-2)\frac{\partial \theta^s}{\partial \rho} + \frac{(1+r^2)(1-\rho^2)}{1-r^2}\langle \nabla_0 r, \nabla_0 \theta^s \rangle$$

$$= -\frac{(1+r^2)(1-\rho^2)(1-\rho)}{\rho^2(1-r^2)}\langle \tilde{\nabla}\phi, \tilde{\nabla} f^s \rangle$$

$$= O(1-\rho) \tag{5.13}$$

as $\rho \to 1$, where we have used (5.11) and the fact that f is in C^3.

It is easy to see that there is a constant C_8, such that

$$|\tau(u)^\alpha - \frac{1-\rho^2}{4}\tilde{\tau}(u)^\alpha| \leqq C_8(1-\rho)^2 ,$$

for $1 \leqq \alpha \leqq n$. Combining this with (5.12) and (5.13), we conclude that

$$|\tau(u)^\alpha| \leqq C_9(1-\rho)^2$$

for some constant C_9 and for ρ close to 1. Hence by (5.11), the norm of the tension field

$$\|\tau(u)\| = 2(1-r^2)^{-1}|\tau(u)|$$

of the extension is in $L^k \cap L^\infty$ for $k > m-1$.

The energy density is given by

$$e(u) = \left(\frac{1-\rho^2}{1-r^2}\right)^2 \left(\phi^2 + \frac{(1-\rho)^2}{\rho^2}|\tilde{\nabla}\phi|^2 + \frac{r^2}{\rho^2}e(f)\right).$$

which is bounded because of (5.11). Therefore we have extended f near infinity, and it is easy to see that one can extend f further to a map defined on M which satisfies the required conditions.

Theorem 5.4 *With the same notations as in Lemma* 5.3, *let f be a C^3 map from $\mathbf{S}^{n-1}$ to $\mathbf{S}^{n-1}$ with nowhere vanishing energy density. Then there exists a harmonic map from $\mathbf{H}^m$ into $\mathbf{H}^n$ which has the boundary map given by f at infinity.*

Proof. By Lemma 5.3, one can extend f to be a map from M to N which is also denoted by f, such that it has bounded energy density and the square of the norm of its tension field is bounded and in L^k for $k > \dfrac{m-1}{2}$. Let us first assume that f is C^∞. By Theorem 4.1, the parabolic equation for harmonic map from $M \times [0, \infty)$ to N with initial data f has a solution u. Since hyperbolic space satisfies the conditions of Theorem 5.2, by the properties of f, $u(\cdot, t)$ converges uniformly to a harmonic map u_∞. The fact that

$$\sup_{x \in M} \operatorname{dist}(u_\infty(x), f(x)) < \infty$$

implies that u_∞ and f have the same boundary value. Since the computations in Lemma 5.3 involve at most the third derivative of f, hence we may relax the smoothness of f to C^3 by an approximation argument.

Theorem 5.5 *With the same assumptions and notations as in Theorem* 5.4, *if $m = n = 2$ and if the boundary map is a diffeomorphism, then the harmonic map obtained in the theorem is a diffeomorphism.*

Proof. Using the same notations as in Lemma 5.3 with $m = n = 2$. We identify M and N with the unit disk D in $\mathbf{R}^2$ with the Poincaré metric. Let f be a C^3 map from S^1 to S^1 so that $e(f) > 0$. Note that if we extend f as a map from D to D in two different ways to get h and h^*, and if both h and h^* satisfy the conditions in Lemma 5.3, then h and h^* can be deformed to harmonic maps u and u^* through the parabolic equation. Suppose

$$\lim_{x \to \infty} \operatorname{dist}(h(x), h^*(x)) = 0 ,$$

then the two harmonic maps will be the same by property (i) in Theorem 5.2 and the fact that $\operatorname{dist}^2(u(x), u^*(x))$ is subharmonic. Therefore let us be more precise on the extension h of f. Let ψ_1 be a smooth function from $[0, 1]$ to itself, so that $\psi_1 = 0$ on $\left[0, \dfrac{b}{2}\right]$, $\psi_1(\rho) \leqq \rho$ and $\psi_1(\rho) = \rho$ on $[b, 1]$, where $0 < b < 1$ is chosen to satisfy $\left(1 - \dfrac{b}{2}\right) e(f) < 1$. Let ψ_2 be another smooth function from $[0, 1]$ to $[0, 1]$, such that $\psi_2 = 0$ on $\left[0, \dfrac{b}{2}\right]$ and $\psi_2 = 1$ on $[b, 1]$. Let $\frac{1}{2} < a_j < 1$, and $a_j \uparrow 1$. For each j, we extend f to a map h_j from D to D by setting

$$r = r_j(\rho, \eta) = a_j \psi_2(\rho)(1 - (1 - \psi_1(\rho))\phi_j(\eta))$$

and

$$\theta = f(\eta) ,$$

where

$$\phi_j = \sqrt{\frac{1 + a_j^2}{2a_j^2} e(f)} .$$

We take h to be the map defined as above by replacing a_j by 1. It is easy to see that h_j are well-defined smooth maps. The image of $\rho \leqq \frac{b}{2}$ is the origin, the image of h_j is in the set given by $r \leqq a_j$. The boundary of D is mapped bijectively onto $r = a_j$. Moreover, for $\rho \geqq b$,

$$r_j = a_j(1 - (1 - \rho)\phi_j) .$$

In what follows, the constants C_i are independent of j. Consider $b \leqq \rho < 1$, using the same notation as in the proof of Lemma 5.3,

$$\begin{aligned}
\tilde{\tau}(h_j)^1 &= \frac{1 - \rho^2}{1 - r_j^2}(2|\nabla_0 r_j|^2 - (1 + r_j^2)e(f)^2) \\
&= \frac{1 - \rho^2}{1 - r_j^2}\left(2a_j^2\phi_j^2 + \frac{2a_j^2(1 - \rho)^2}{\rho^2}|\tilde{\nabla}\phi_j|^2 - \frac{(1 + r_j^2)e(f)}{\rho^2}\right) \\
&= 2\frac{1 - \rho_2}{1 - r_j^2}\left(\frac{a_j^2(1 - \rho)^2|\tilde{\nabla}\phi_j|^2}{\rho^2} + \frac{e(f)(a_j^2(1 - \rho)^2\phi_j^2 - 2a_j^2(1 - \rho)\phi_j)}{\rho^2}\right) \\
&\quad + \frac{1 - \rho^2}{1 - r_j^2}\left(2a_j^2\phi_j^2 - \frac{e(f)(1 + a_j^2)}{\rho^2}\right) \\
&= 2\frac{1 - \rho^2}{1 - r_j^2}\left(\frac{a_j^2(1 - \rho)^2|\tilde{\nabla}\phi_j|^2}{\rho^2} + \frac{f'^2(a_j^2(1 - \rho)^2\phi_j^2 - 2a_j^2(1 - \rho)\phi_j)}{\rho^2}\right) \\
&\quad + \frac{1 - \rho^2}{1 - r_j^2} \times \frac{2a_j^2\phi_j^2(\rho^2 - 1)}{\rho^2} ,
\end{aligned} \tag{5.14}$$

where we have used the definition of ϕ_j. Since for $\rho \geqq b$,

$$\tau(h_j)^1 = \frac{(1 - \rho^2)^2}{4}\Delta_0 r_j + \frac{r_j(1 - \rho^2)}{4}\tilde{\tau}(h_j)^1 ,$$

there exists a constant C_6 not depending on j, such that for $\rho \geqq b$

$$|\tau(h_j)^1| \leqq C_1(1 - \rho)^2 .$$

Similarly, one can show that there exists a constant C_7 not depending on j such that

$$|\tau(h_j)^2| \leqq C_7(1 - \rho)^2$$

for $\rho \geqq b$. Hence there exists a constant C_8 not depending on j, such that for $\rho \geqq b$,

$$\|\tau(h_j)\|^2 \leqq C_8(1 - \rho)^2 ,$$

where $\|\tau(h_j)\|$ is the norm of the tension field $\tau(h_j)$. Also one can check that, there is C_9, such that,

$$\|\tau(h_j)\| \leqq C_9$$

for $\rho \leqq b$. Let $p > 1$ be fixed, there exists C_{10} not depending on j, such that

$$\int_M \|\tau(h_j)\|^{2p} \leqq C_{10} . \tag{5.14}$$

Also there is a constant C_{11} such that

$$\sup_M e(h_j) \leqq C_{11} . \tag{5.15}$$

Therefore by Theorem 5.2, we can solve the parabolic equation for harmonic maps with initial data h_j. If we denote the solution to be u_j, then letting $t \to \infty$ we obtain a harmonic map $u_{j,\infty}$. As in the proof of Theorem 5.2, noting that the constants in (5.14) and (5.15) do not depend on j, we have

$$|(u_j)_t|^2(x, t) \leqq C_{12} \exp(-at) , \tag{5.16}$$

where C_{12} and $a > 0$ are independent of j. As before, one argues that

$$\sup_{M \times [0, \infty)} e(u_j) \leqq C_{13}$$

and

$$\operatorname{dist}(u_{j,\infty}(x), h_j(x)) \leqq C_{14}$$

for all $x \in M$. Therefore we have two kinds of convergence. Namely, by passing to a subsequence if necessary, noting that $h_j \to h$ as $j \to \infty$ uniformly on compact set, and by uniqueness of the parabolic equation,

$$\lim_{j \to \infty} u_j = u ,$$

where u is the solution of the parabolic equation with initial data h. Also

$$\lim_{j \to \infty} u_{j,\infty} = v ,$$

where v is a harmonic map. The convergence are uniform on compact sets, together with the first and second derivatives in the space variable, and the first derivative in the time variable for the functions u_j. We want to show that v is exactly the harmonic map u_∞ we get in Theorem 5.4. It suffices to show that $\operatorname{dist}(v(x), u_\infty(x)) \to 0$ as $x \to \infty$. Note that

$$\operatorname{dist}(u_{j,\infty}(x), h_j(x)) \leqq \int_0^\infty |(u_j)_t| \, dt . \tag{5.18}$$

By (5.16), given $\varepsilon > 0$ we can find $T > 0$ which is independent of j so that

$$\int_T^\infty |(u_j)_t| \, dt \leqq \varepsilon .$$

For a fixed x, $u_j(x, t)$ tends to $u(x, t)$ uniformly on $[0, T]$. Hence by letting $j \to \infty$ in (5.18), we obtain

$$\operatorname{dist}(v(x), h(x)) \leqq \varepsilon + \int_0^T |u_t| \, dt \, .$$

Therefore $\operatorname{dist}(v(x), h(x))$ tends to 0 as $x \to \infty$, and $v = u_\infty$.

Each $u_{j,\infty}$ can be considered as a harmonic map with boundary data h_j which is a homeomorphism onto the set $\{r = a_j\}$. By [S-Y2], u_j is a diffeomorphism in D and the Jacobian of u_j is nonzero everywhere on D, hence the Jacobian of u_∞ is either nonnegative everywhere, or nonpositive everywhere. We claim that u_∞ is a diffeomorphism. For $0 < \rho_0 < 1$, denote

$$D_{\rho_0} = \{x | \rho(x) < \rho_0\} \, ,$$

and for $0 < r_0 < 1$, denote

$$D_{r_0} = \{u | r(u) < r_0\} \, .$$

By (5.17) and the definitions of h_j, there exists $0 < \rho_1 < 1$ such that

$$u_{j,\infty}(D_{\rho_1}) \supset D_{1/2} \tag{5.19}$$

for all j. Also for any $0 < \rho_2 < 1$, there exists $0 < r_1 < 1$ such that

$$u_{j,\infty}(D_{\rho_2}) \subset D_{r_1}$$

for all j. Hence by Theorem 7.1 in [J2, p. 80], for any $\rho_1 \leqq \rho < 1$, there is a constant $C_{14} > 0$ which is independent of j such that for all $x \in D_\rho$,

$$|J_j(x)| \geqq C_{14} \, ,$$

where J_j is the Jacobian of $u_{j,\infty}$. Letting $j \to \infty$ and then $\rho \to 1$ we conclude that $|J(x)| > 0$ for all $x \in D$, where J is the functional determinant of u_∞. As in the proof of (5.19), one can show that u_∞ is onto. By (5.14) and the fact that u_∞ is a local diffeomorphism, for any point q in D the set $u_\infty^{-1}(q)$ is finite and u_∞ is a covering map. Hence u_∞ must be a diffeomorphism because D is simply connected.

6 Harmonic maps: Small total energy

In this section we would like to generalize a theorem of Mitteau [M], which states that if M and N are compact manifolds, then every $h \in C^1(M, N)$ with sufficiently small energy density can be deformed through the parabolic equation to a harmonic map. Note that in the compact case, a map with small energy density implies that the total energy is also small.

Theorem 6.1 *Let M^m be a complete noncompact Riemannian manifold with Ricci curvature bounded from below by $-K$ for $K \geqq 0$. Suppose there exists a constant $b > 0$ such that $\inf_{x \in M} V_x(1) = b > 0$. Let N^n be a complete Riemannian manifold with sectional curvature bounded from above by $\kappa > 0$. For any $\alpha > 0$, there exists $\varepsilon > 0$, depending only on m, K, b, κ, and α, such that if $h \in C^1(M, N)$ satisfying $\sup_M e(h) \leqq \alpha$, $E(h) \leqq \varepsilon$, and that $h(M)$ is bounded, then* (0.1) *has a solution u on*

$M \times [0, \infty)$ with initial condition $u(x, 0) = h(x)$. If in addition there exists a point $p \in M$ and a sequence $t_\nu \to \infty$ such that $u(p, t_\nu)$ converges in N (for example when N is compact), then by passing to a subsequence, $u(\cdot, t_\nu)$ converges uniformly on compact subsets of M to a harmonic map u_∞.

Proof. Let $h \in C^1(M, N)$ be such that $\sup_M e(h) \leqq \alpha$. In view of Theorem 3.3 and Remark 3.5, to prove long time existence of solution of the parabolic equation it suffices to show that if u is a solution of (0.1) on $M \times [0, T_0)$ satisfying the properties that $\sup_{M \times [0, t)} e(u) < \infty$, and $u(M \times [0, t))$ is bounded for all $0 < t < T_0$, then there exists $\varepsilon > 0$, depending only on m, K, b, κ, and α, such that if $E(h) < \varepsilon$ then $\sup_{M \times [0, T_0]} e(u) \leqq \beta$ for some $\beta > 0$ independent of T_0 and $u(M \times [0, T_0))$ is bounded.

The Bochner formula for the energy density $e(u(x, t))$ of u asserts that

$$\left(\Delta - \frac{\partial}{\partial t}\right) e \geqq -2Ke - 2\kappa e^2 . \tag{6.1}$$

For $\delta > 0$ sufficiently small, let us consider the function

$$g(t) = \frac{K}{\kappa}\left(\frac{1}{1 - 2\kappa e^{A + 2Kt}} - 1\right),$$

with $A = \log\left(\dfrac{\alpha + \delta}{2K + 2\kappa(\alpha + \delta)}\right)$. Obviously, g satisfies the differential equation

$$g' = 2Kg + 2\kappa g^2 ,$$

with initial condition $g(0) = \alpha + \delta$. The function $f(x, t) = g(t) - e(u(x, t))$ satisfies the differential inequality

$$\left(\Delta - \frac{\partial}{\partial t}\right) f \leqq -2Kf - 2\kappa f(g + e) ,$$

with $f(x, 0) \geqq \delta$. We claim that f must be positive on $M \times [0, t]$, for all $t < T_1 = \dfrac{-(A + \log 2\kappa)}{2K}$. If not, let $T < T_1$ be the first t such that $\inf_M f(x, T) = 0$. Using the facts that $f \geqq 0$ on $M \times [0, T]$ and g is an increasing function on $[0, T_1)$, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right) f \leqq -2Kf - 2\kappa g(0) f .$$

This implies that the function $F = e^{-2(K + \kappa g(0))t} f$ satisfies

$$\left(\Delta - \frac{\partial}{\partial t}\right) F \leqq 0 ,$$

with initial condition $F(x, 0) \geqq \delta$. Hence $F \geqq \delta$ on $M \times [0, T]$, which is a contradiction. Therefore we may conclude that there exists a $t_0 > 0$ such that $e(u(x, t)) \leqq 2\alpha$, for all $x \in M$ and $t \in [0, t_0]$ whenever $u(x, t)$ is defined.

To estimate $e(u(x, t_1))$ for $t_1 \geqq t_0$, let $s = \sup_{M \times [0, t_1]} e(u(x, t))$. By the above discussion, we may assume that the value s is achieved at some $t_2 \in (t_0, t_1]$. Inequality (6.1) can be rewritten as

$$\left(\Delta - \frac{\partial}{\partial t}\right) e \geqq -2Ke - 2s\kappa e ,$$

on $M \times (0, t_1]$. Hence the function $h(x, t) = e^{-2(K+s\kappa)t} e(u(x, t))$ must satisfy

$$\left(\Delta - \frac{\partial}{\partial t}\right) h \geqq 0 .$$

For $t_3 \in (t_2 - t_0, t_2)$, let us solve the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) \phi = 0 ,$$

with

$$\phi(x, 0) = h(x, t_3) .$$

By the maximum principle of [K-L], $\phi(x, t_2 - t_3) \geqq h(x, t_2)$. Now

$$\begin{aligned} \phi(x, t_2 - t_3) &= \int_M H(x, y, t_2 - t_3)\phi(y, 0)\, dy \\ &\leqq \sup_{y \in M} H(x, y, t_2 - t_3) \int_M h(y, t_3)\, dy \\ &\leqq \sup_{y \in M} H(x, y, t_2 - t_3)\varepsilon e^{-2(K+s\kappa)t_3} , \end{aligned} \tag{6.2}$$

where we have used the fact (see [Lo-T]) that the total energy is a non-increasing function of t.

On the other hand, the upper estimate of the heat kernel in [L-Y] asserts that

$$\begin{aligned} H(x, y, t) &\leqq H^{-1/2}(x, x, t) H^{-1/2}(y, y, t) \\ &\leqq C_1 V_x(\sqrt{t})^{-1/2} V_y(\sqrt{t})^{-1/2} \exp(C_2 K t) , \end{aligned}$$

for some constants $C_1, C_2 > 0$ depending only on m. Substituting this into (6.2), the volume comparison theorem and the assumption on the volume of unit balls implies that there exists a constant $\gamma > 0$ which depends only on $t_2 - t_3$, b, and m, such that

$$\phi(x, t_2 - t_3) \leqq \gamma\varepsilon e^{-2(K+s\kappa)t_3} .$$

Hence

$$s e^{-2(K+s\kappa)(t_2 - t_3)} \leqq \gamma\varepsilon .$$

One observes that the function $\psi(s) = s e^{-2(K+s\kappa)(t_2 - t_3)}$ has the properties that $\psi(0) = 0$, $\psi(s) \to 0$ as $s \to \infty$, and ψ has exactly one maximum at $s_0 = (2\kappa(t_2 - t_3))^{-1}$ with maximum value at $\psi_0 = (2\kappa(t_2 - t_3))^{-1} \exp\{-2K(t_2 - t_3) - 1\}$. If we first choose t_3 sufficiently closed to t_2 such that $2\alpha \leqq s_0$, then for $\varepsilon < \dfrac{\psi_0}{\gamma}$ we may conclude that $s < s_0$. This proves the energy estimate with $\beta = s_0$.

In order to prove that $u(M \times [0, T_0))$ is bounded, let us first point out that Theorem 2.4 of [Lo-T] asserts that for $0 \leqq t < T < T_0$, we have

$$E(T) + 2\int_t^T d\tau \int_M |u_t|^2(x, \tau)\, dx \leqq E(t)\,, \tag{6.3}$$

where $E(t) = \int_M e(u)(\cdot, t)$. As in the proof of the main lemma in [H], we also have

$$\left(\Delta - \frac{\partial}{\partial t}\right)|u_t|^2 \geqq -2\kappa s_0 |u_t|^2 \tag{6.4}$$

on $M \times (0, T_0)$. Hence the function $g(x, t) = \exp(-2\kappa s_0 t)|u_t|^2(x, t)$ is a subsolution of the heat equation for functions on M. Let $R^2 \geqq T_0 + 1$ be fixed. By Theorem 1.1, for $\dfrac{T_0}{2} \leqq t < T_0$ and for $x \in M$, there exists a constant C_2 which is independent of x and t such that

$$g(x, t) \leqq \frac{C_3}{t V_x(R)} \int_0^t d\tau \int_{V_x(R)} |u_t|^2(y, \tau)\, dy\,. \tag{6.5}$$

Combining this with (6.4) and the assumption that $\inf_{x \in M} V_x(1) = b$, we conclude that

$$\sup_{M \times \left[\frac{T_0}{2}, T_0\right)} g < \infty\,.$$

Hence

$$\sup_{M \times \left[\frac{T_0}{2}, T_0\right)} |u_t|^2 < \infty\,.$$

From the fact that $u\left(M \times \left[\dfrac{T_0}{2}, T_0\right)\right)$ is bounded, the above estimate imply that $u(M \times [0, T_0))$ is also bounded in N. Hence we have long time existence.

Since the energy density of u is uniformly bounded, in order to prove that if $u(p, t_\nu)$ converges for some p and some sequence t_ν, it remains to prove that $|u_t| \to 0$ as $t \to \infty$. Using (6.3) and (6.5), as in the proof of (6.5), one can show that there exists a constant C_4 depending only on κ and β which is the upper bound of $e(u)$ on $M \times [0, \infty)$ such that for all $t > 1$ and $x \in M$

$$|u_t|^2(x, t) \leqq \frac{C_4}{V_x(4)}(E(t-1) - E(t))\,.$$

Since $E(t)$ is non-increasing and is bounded, $\lim_{t \to \infty} E(t)$ has a limit. So $E(t-1) - E(t)$ tends to zero as $t \to \infty$. This implies that $|u_t|^2(x, t)$ tends to zero as $t \to \infty$, and the theorem follows.

In some cases we can remove the restriction that the image of the initial data is bounded.

Definition 6.2. A complete Riemannian manifold N is said to *satisfy condition (A)*, if there exists a compact exhaustion $\{\Omega_\nu\}$ of N and C^∞ maps g_ν from N into itself such that $g_\nu|_{\Omega_\nu} = \text{identity}$, $g_\nu(M)$ is compact, and $\sup_{\nu, N} e(g_\nu) < \infty$.

Obviously if N is compact then it satisfies condition (A). By Lemma 4.3, if N has nonpositive curvature then it satisfies condition (A). Also, if $N = N_1 \times N_2$ with N_1 and N_2 satisfying condition (A), then so does N. In particular, by [C-G], if N is the universal covering of a compact manifold with nonnegative Ricci curvature then N satisfies condition (A).

Theorem 6.3 *Theorem* 6.1 *is still true without assuming that* $h(M)$ *is bounded, provided that* N *satisfies condition* (A). *The constant* ε *also depends on the compact exhaustion* $\{\Omega_\nu\}$ *and the maps* g_ν.

Proof. Since N satisfies condition (A), there exists a compact exhaustion $\{\Omega_\nu\}$ of N and C^∞ maps g_ν from N into itself such that $g_\nu|_{\Omega_\nu} = identity$, $g_\nu(M)$ is compact, and

$$\sup_{\nu, N} e(g_\nu) = C_5 < \infty \ . \tag{6.6}$$

As in the proof of Theorem 4.1′, let $h_\nu = g_\nu \circ h \in C^1$. Note that $h_\nu(M)$ is bounded. Suppose $\sup_M e(h) \leqq \alpha$, noting that $e(h_\nu)(x) \leqq C_5 e(h)(x)$ by (6.6), so

$$\sup_{\nu, M} e(h_\nu) \leqq C_5 \alpha \ .$$

Hence by Theorem 6.1, there exists a constant $\varepsilon > 0$ such that if $E(h_\nu) \leqq C_5 \varepsilon$ then (0.1) has a solution u_ν on $M \times (0, \infty)$ and $u_\nu(x, 0) = h_\nu(x)$. Suppose $E(h) \leqq \varepsilon$ then $E(h_\nu) \leqq C_5 \varepsilon$, and the solutions u_ν exist. From the proof of Theorem 6.1 we see that

$$\sup_{M \times [0, \infty)} e(u_\nu) \leqq \beta < \infty \tag{6.7}$$

for some $\beta > 0$ for all ν. By [H], we see that

$$\left(\Delta - \frac{\partial}{\partial t} \right) |(u_\nu)_t|^2 \geqq -2\kappa\beta |(u_\nu)_t|^2 \ .$$

Hence the function $\exp(-2\kappa\beta)|(u_\nu)_t|^2(x, t)$ is a subsolution of the heat equation for functions on M. As in the proof of Theorem 4.1′, one can show that given a bounded domain Ω in M and $T > 0$ there exists a constant C_6 which is independent of ν such that

$$|(u_\nu)_t|(x, t) \leqq C_6 t^{-1/2} \tag{6.8}$$

for $x \in \Omega$ and $0 < t \leqq T$. From (6.7) and (6.8), we conclude that passing to a subsequence, u_ν converges to a solution u of (0.1) in $M \times (0, \infty)$. Since $g_\nu|_{\Omega_\nu} = identity$, by the definition of h_ν, the fact that $u_\nu(x, 0) = h_\nu(x)$, and (6.8), we conclude that $u(x, 0) = h(x)$. Since the convergence of u_ν together with their derivatives is uniform on compact subsets of $M \times (0, \infty)$, it is easy to see that $\sup_{M \times [0, \infty)} e(u) \leqq \beta$ and $E(u(\cdot, t)) \leqq C_5 \varepsilon$, where we have used the fact that the total energy is nonincreasing in t. Hence, as in the proof of Theorem 6.1, if there exists a point $p \in M$ and a sequence $t_j \to \infty$ such that $u(p, t_j)$ converges in N, then by passing to a subsequence, $u(\cdot, t_j)$ converges uniformly on compact subsets of M to a harmonic map u_∞. This completes the proof of the theorem.

Remark 6.4 From the proof of the Theorem 6.1 and Theorem 6.3, it is easy to see that if $\kappa = 0$, then ε can be any finite number. This recovers one of the result in

[Lo-T], that any map from a complete noncompact manifold to a compact manifold with nonpositive curvature can be deformed to a harmonic map through the parabolic equation, for the case that the domain manifold has Ricci curvature is bounded below.

In [S-Y1], it was shown that any harmonic map with finite total energy from a complete manifold with nonnegative Ricci curvature to a complete manifold with nonpositive sectional curvature must be constant. When M is $\mathbf{R}^m$, Sealey [Se] proved that any harmonic map with finite total energy into an arbitrary manifold N must be constant provided that $m \geqq 3$. Struwe [St] showed that any harmonic map with bounded energy density from $\mathbf{R}^m$ to a compact manifold is constant if the total energy is small enough. In view of Sealey's theorem, Struwe's result is only interesting when $m = 2$. We would like to point out that Sealey's result does not hold for general complete manifolds with nonnegative Ricci curvature. The following theorem is a generalization of these results.

Theorem 6.5 *Let M be a complete noncompact manifold with nonnegative Ricci curvature. Suppose there exists a constant $C_7 > 0$ such that $V_x(r) \geqq C_7 r^2$ for all $x \in M$ and for all $r \geqq 1$. Let N be a complete manifold with sectional curvature bounded above by $\kappa \geqq 0$. Given $\alpha > 0$ there exists $\varepsilon > 0$ depending only on κ, C_7 and α such that if u is a harmonic map from M to N with $\sup_M e(u) \leqq \alpha$ and $E(u) < \varepsilon$ then u is constant.*

Proof. Let $\sup_M e(u) = \beta$. As in [H], we have

$$\Delta e(u) \geqq -2\kappa(e(u))^2 \geqq -2\kappa\beta e(u) .$$

Let us consider $e(x) = e(u)(x)$ as a function of x and t, which is independent of time t, then the function $\exp(-2\kappa\beta t)e(x)$ is a subsolution of the heat equation for functions on M. By the maximum principle, the estimate of the heat kernel in [L-Y], and the assumption on the volume growth of M, we have for all $t \geqq 1$ and for all $x \in M$

$$\begin{aligned} \exp(-2\kappa\beta t)e(x) &= \int_M H(x, y, t)e(y)\,dy \\ &\leqq C_8 V_x^{-1}(\sqrt{t})E(u) \\ &\leqq \frac{C_8}{C_7} t^{-1} E(u) , \end{aligned}$$

where C_8 is an absolute constant. Hence

$$\beta t \exp(-2\kappa\beta t) \leqq \frac{C_8}{C_7} E(u) ,$$

for $t \geqq 1$. Suppose $\alpha \geqq \beta > 0$. If we let $t = \dfrac{\alpha}{\beta} \geqq 1$ in the above inequality then we will have a contradiction if $E(u)$ is small.

Example. We would like to give an example to show that in general the harmonic map obtained in Theorem 6.1 is not constant. More precisely, in the notations in Theorem 6.1, we can find K, κ, b and α, such that given $\varepsilon > 0$, there exists complete surfaces M and N and a harmonic map from M to N, with the following properties:

(1) the curvature of M is bounded below by $-K$, $\inf_{x\in M} V_x(1) > b > 0$, and the curvature of N is bounded above by κ; (2) $\sup_M e(u) \leqq \alpha$, $E(u) < \varepsilon$ and u is not constant.

Let M be a complete surface which is diffeomorphic to a torus minus a point so that its curvature near infinity is zero. Note that $\inf_{x\in M} V_x(1) > 0$. Let $\mathbf{S}^1$ be a circle, with radius not necessarily equal to 1. The one point compactification $\bar{M}$ of M is diffeomorphic to a torus. For $h \in C(\bar{M}, N)$, h can be deformed to a smooth map which is constant near infinity of M. Hence, we may assume h has this property. By Theorem 6.1 and Remark 6.4, h can be deformed through the parabolic equation to a harmonic map u_∞. Note that the energy density of u_∞ is bounded, and $E(u_\infty)$ is finite. We claim that for some h, u_∞ is not constant. Otherwise, we would have $H_1(\bar{M}, \mathbf{Z}) = 0$ which is a contradiction. This proves the claim. If we let the radius of the circle $\mathbf{S}^1$ goes to zero, then $E(u_\infty)$ can be made arbitrarily small. However, we can embed $\mathbf{S}^1$ into a dumb bell shape surface N so $\mathbf{S}^1$ is a geodesic and the curvature of N is bounded above by a constant independent of the radius of the circle. The composition of u_∞ with the embedding clearly is a harmonic map of M into N satisfying properties (1) and (2).

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