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Small cancellation theory and automatic groups: Part II

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In this article we continue the work of [GS] (which we shall henceforth refer to as Part I). There we showed that small cancellation groups satisfying additional conditions are automatic (we refer the reader to Part I for definitions and further references). Recall that a group is said to be automatic if there is

(1) a finite set of semigroup generators,
(2) a finite state automaton accepting a language L of words in these generators which contains at least one representative for each element of G,
(3) for each generator a there is a finite state automaton which accepts the set of pairs of words in L which represent elements of G which differ by multiplication on the right by a.

The main results obtained in Part I can be rephrased in the language of root systems and Euclidean planar tessellations as saying that finite $A_1 \times A_1$ and A_2 piecewise Euclidean (abbreviated PE) complexes of non-positive curvature have automatic fundamental groups. (A 2-dimensional complex is a $A_1 \times A_1$, respectively A_2, PE complex if every 2-cell can be identified with a unit square, resp., equilateral triangle in the Euclidean plane, in such a way that the induced metrics agree on intersections. Such a complex has non-positive curvature if the link of each vertex contains no circuits without backtracking of length less than 2π).

Here we prove using similar techniques that finite 2-complexes modelled on the other two root systems, B_2 and G_2, corresponding to tessellations by 45° right angle triangles and by 30°, 60° right angle triangles, also have automatic fundamental groups (Theorems 2.3 and 3.2). To be precise,

Theorem 1 *A finite 2-complex of non-positive curvature modelled on the $A_1 \times A_1$, A_2, B_2 or G_2 root system has automatic fundamental group.*

From this it follows that a torsion-free discrete group acting, with compact quotient, by isometries of an Euclidean building of rank 2 where all vertices

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have finite links, is automatic (Theorem 4.1). This result provides a rich class of examples of automatic groups.

We also use the B_2 case to extend the result of Part I that groups of prime alternating links are automatic to finite sums of alternating links (Theorem 4.2).

In another direction, we show how to lift the extra conditions (that pieces have length 1, and that no relation be a proper power) we imposed on the small cancellation groups in Part I, using the solution given here of the B_2 and the G_2 cases (Sects. 5 and 6). This means that we prove:

Theorem 2 *A group possessing of finite presentation satisfying a* $C(6)$, $C(4)-T(4)$ *or* $C(3)-T(6)$ *small cancellation condition is automatic.*

In Sect. 7 we conclude with a definition of biautomatic structures (where, for each semigroup generator a there is an automaton whose language consists of accepted words which differ by multiplication on the left by a, as well as an automaton for multiplication on the right). All known examples of automatic groups have such biautomatic structures, and all the structures constructed here, in Part I and in [CEHPT] have this additional property. We prove:

Theorem 3 *A group with a biautomatic structure has a solvable conjugacy problem.*

The corresponding problem for automatic groups is open. The solution of the conjugacy problem for the $C(p)-T(q)$ small cancellation groups (for $(p, q)=(3, 6), (4, 4)$, or $(6, 3)$) was established by Schupp [LS] while for negatively curved (or word hyperbolic) groups the solution was given by Gromov [G, 7.4.B]. The proof offered here gives a unified treatment ot the problem.

Since this work was completed, we have further developed the notion of biautomatic group, and gone some way to understanding the subgroup structure of these groups. In particular we present in [GS 2] an alternative proof of Theorem 3.

1 Definitions and brief review of Part I

Corresponding to the two root systems (covering the plane by 45° right angle triangles and by 30°, 60° right angle triangles), we make the following two definitions of B_2 and G_2 complexes. Note that the G_2 case has a further restriction concerning orientations.

Definition. A B_2 *complex* is a PE complex, where all 2-cells are isometric to an isosceles right angle triangle $\mathbb{E}^2$ with short side of length one.

Let K be a finite PE complex where all 2-cells are oriented isometric to a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle with short side of length $\frac{1}{2}$, subject to further restrictions. The 1-cells in the boundaries of 2-cells thus fall into three equivalence classes, according to their lengths, which are $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$ or 1 – call these type x, y and z edges. We assume in addition the vertices fall into equivalence classes X, Y, Z according to whether they lie opposite an x, y or z edge in some 2-cell. We orient the edges of a standard such triangle in $\mathbb{E}^2$, such that an edge is

oriented from the shorter to the longer of the edges that it meets. We say that K is a G_2 *complex* if it satisfies the following additional conditions:

(1) each 1-cell is oriented
(2) each two cell is oriented isometric to the standard triangle (by an isometry respecting the orientation on the 1-skeleton):

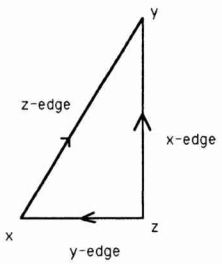

Fig. 1. Standard 2-cell for a G_2 complex

(3) if an x-edge e_1 and a y-edge e_2 have a common initial vertex, then there is a 2-cell containing these two edges.

In the non-positive curvature context in which we are principally interested, the 2-cell whose existence is asserted in condition (3) is necessarily unique.

The tessellations of the euclidean place $\mathbb{E}^2$ by isosceles right angle triangles and by (oriented) 30°, 60° right angle triangles we shall refer to as B_2 and G_2 tessellations.

A PE complex K has *non-positive curvature* (resp. *negative curvature*) if every nontrivial circuit without backtracking in the link of a vertex v subtends an angle of at least 2π (resp. greater than 2π) at v. To understand the subtended angle, we look at the *star graph* for the vertex v. This graph has a vertex for each oriented edge of $K^{(1)}$. Thus the vertices of the star graph occur in pairs $e, \bar{e}$, where $\bar{e}$ denotes the edge oppositely oriented to e. The attaching map of a 2-cell in K gives a circuit in $K^{(1)}$, defined up to inversion and starting point. If the syllable ab occurs in an attaching map of a 2-cell α of K (and thus $\partial_1(a)=\partial_0(b)$) then join the vertex labelled a to the vertex labelled b^{-1} in the star graph by an edge, and assign to this edge a length equal to the angle at the corner corresponding to ab in α measured in radians.

A PE complex has non-positive curvature if the sum of the lengths in a nontrivial circuit without backtracking in the star graph of any vertex is at least 2π. For instance, the 2-complex corresponding to the presentation $\langle a, b \mid abab^{-1}b^{-1}\rangle$ has non-positive curvature.

Some definitions recalled from Part I

We shall use the techniques of Part I. Let $\langle X; R\rangle$ be a presentation of the group G; as usual we take the set $\mathscr{A}=X\cup X^{-1}$ as a set of semigroup generators for G, and let $\mathscr{A}^*$ denote the free semigroup on $\mathscr{A}$.

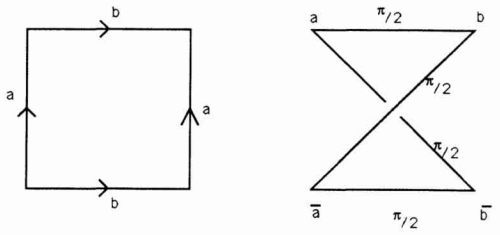

Fig. 2. Star graph for $\mathbb{Z}$

We denote the element of G represented by the word $w \in \mathscr{A}^*$ by $\bar{w}$. The main theorem used to assure the existence of an automatic structure is:

Theorem 1.1 ([CEHPT], see also [BGSS, 2.10] and Part I) *Suppose that there is a regular language of words in $\mathscr{A}^*$ which maps naturally on to the group G as geodesic (i.e. shortest) representatives. Suppose in addition that there is a constant k such that words in this language which begin and end at most distance 1 apart in the Cayley graph lie in a k-Hausdorff neighborhood of each other. Then G is automatic, and L is the language of an automatic structure for G.*

We describe the language in terms of turning angles, as in Part I. Let K be a PE complex, and let $v = a_1 \dots a_n$ be a word in the labels on the edges of a closed path in $K^{(1)}$. If the word v represents the trivial word in $\pi_1(K)$, then there is a singular PE disc diagram whose boundary label is v (see Part I) and whose regions are isometric copies of the 2-cells of K (all diagrams are assumed reduced). If a segment of the boundary of a singular disc diagram is labelled by the word w, we say that the diagram *includes* w. Let D be a singular PE disc diagram which includes the word $w = a_1 \dots a_n$. Then to each a_i, $1 < i < n$, D associates an ordered pair of turning angles (θ, θ') ocurring at the initial and final vertex of the edge of ∂D labelled a_i. If the edges corresponding to adjacent letters a_i and a_{i+1} do not lie on the topological boundary of an open 2-ball in D, then we assign the value $-\infty$ to the turning angle at the vertex between these two edges. In fact, for each $i = 1, \dots, n-1$ the diagram D associates an angle θ_i to the vertex common to the boundary edges of D corresponding to a_i and a_{i+1}, and thus associates a sequence $\theta_1, \dots, \theta_{n-1}$ of turning angles to the word w.

2 B_2 complexes of non-positive curvature

Let K be a finite B_2 complex of non-positive curvature.

We follow the steps of Part I, first giving explicitly a regular language for the fundamental group $\pi_1(K) = G$, and then establishing the various propositions required to show that this language satisfies the conditions of Theorem 1.1, and thus forms part of an automatic structure for G. To do this we shall consider reduced (singular) disc diagrams which inherit the PE structure from the B_2 complex, by pulling back the structure from K. We shall restrict our interest to the case when the boundary contains no hypotenuse edge: we call these

(singular) B_2 disc diagrams. Notice that a B_2 diagram becomes a singular disc diagram where all cells are rectangles by deleting all hypotenuse edges.

Let K be a B_2 complex, and let X be the set of oriented edges, and R the set of cyclic words in X corresponding to the attaching maps of the 2-cells. The fundamental group of K is a free factor of the group $\langle X; R\rangle$, which is the fundamental group $\pi_1(K)$ if K has just one vertex (we shall not specify the base point here). Let $\mathscr{A}$ denote the set of non hypotenuse edges; clearly these elements generate $\pi_1(K)$. Let $\Gamma_{\mathscr{A}}$ be the Cayley graph of $\pi_1(K)$ with respect to the generating set $\mathscr{A}$. Distance in the Cayley graph is measured by the metric induced by regarding each edge of $\Gamma_{\mathscr{A}}$ as an isometric copy of the unit interval.

Definition. A word $w=a_1 \ldots a_n$ in $\mathscr{A}^*$ is called a *B_2-geodesic* if in all B_2 diagrams which include w, the sequence $\theta_1, \ldots, \theta_{n-1}$ of associated turning angles never contains a non-negative turning angle followed by a positive turning angle.

We now follow the sequence of Part I to show that Theorem 1.1 applies.

Proposition 2.1 *Every element of $\pi_1(K)$ has a B_2-geodesic representative.*

Proof. Given an element $g \in G$, let w be a geodesic word w in $\mathscr{A}^*$ representing g.

Consider the various cases where a positive turning angle follows a non-negative turning angle in the boundary of a singular B_2 disc diagram which includes the word w. Notice that all turning angles associated are multiples of $\pi/2$ because of the restriction to words which contain no hypotenuse edges.

If a $\pi/2$ turning angle is followed a $\pi/2$ turning angle, then w could be shortened.

If a turning angle of $\pi/2$ follows a turning angle of zero, say at the letter a_i, then the following change is made, to give the word w'.

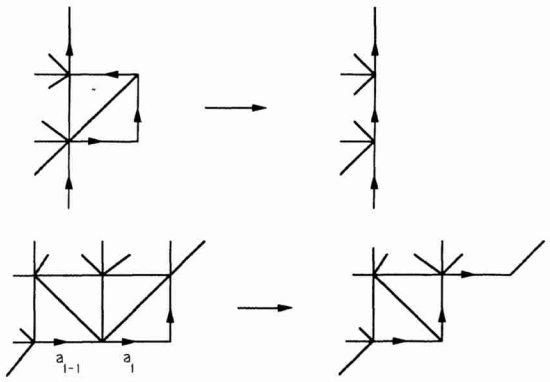

Fig. 3. Making a word into a B_2-geodesic

Notice that if the subword $a_{i-1}\, a_i$ occurs in another reduced disc diagram, then the turning angle between a_{i-1} and a_i cannot be $\pi/2$. If this were the case, then the non-positive curvature condition would be violated.

Let $H(i, w)$ be the number of values of j, $1<j\leqq i$, such that in some B_2 diagram which contains w, the turning angle θ_j is $\pi/2$; here $i\leqq n-1$ where n is the length of w.

Then, as in Proposition 5.4 of Part I,

$$\sum_2^n H(i, w)<\sum_2^n H(i, w')<n(n+1)/2,$$

where w' is the word obtained from w by the change illustrated in Fig. 3.

It follows that this process of 'pushing right angles to the front' terminates and gives a B_2-geodesic. □

To determine whether a word w is a B_2-geodesic or not, it suffices to check whether w contains one of a finite list of subwords. This remark proves:

Proposition 2.2 *The language of B_2-geodesics is regular.*

As all turning angles are multiples of $\pi/2$, deleting hypotenuse edges from a B_2 singular disc diagram gives a PE disc diagram where all cells are rectangles (i.e. an $A_1\times A_1$ diagram). The methods of Sect. 5 of Part I can now be used, and the proofs of Propositions 5.6 and 5.7 show that

Proposition 2.3 (i) *A B_2-geodesic is geodesic.*
(ii) *If two B_2-geodesics begin and end distance at most 1 apart in the Cayley graph $\Gamma_{\mathscr{A}}$, then they lie in a 2-Hausdorff neighborhood of each other.*

It thus follows that:

Theorem 2.4 *The fundamental group of a finite B_2 complex of non-positive curvature is automatic.*

3 G_2 Complexes of non-positive curvature

Recall that in a G_2 complex K, each 2-cell is oriented isometric to a standard triangle (Fig. 1).

Choose as base point for K some X-vertex. Let $\mathscr{A}$ be the set of y- and z-edges; clearly these elements generate $\pi_1(K)$. A G_2 disc diagram is defined analogous to a B_2 disc diagram, where here we do not have x edges on the boundary. As for B_2 complexes, we define the set of G_2 geodesics by conditions on associated turning angles, but now, instead of just considering adjacent turning angles, we consider triples of turning angles.

Definition. In a G_2 diagram which includes the word $w=a_1\ldots a_n\in\mathscr{A}^*$ let $\theta_1, \ldots, \theta_{n-1}$ be the associated turning angles. We say that w is a G_2 *geodesic* if for all G_2 diagrams which include w,

(1) if a_i is a z-label then we do not have $\theta_{i-1}\geqq 0$ and $\theta_i>0$;
(2) if a_j, a_{j+1} are y-labels, then we do not have $\theta_{j-1}\geqq 0$ and $\theta_j=0$ and $\theta_{j+1}>0$.

Proposition 3.1 *Every element of $\pi_1(K)$ has a G_2 geodesic representative.*

Proof. Let w be a word representing $g\in\pi_1(K)$, where w is chosen to minimise the G_2-length $\ell_G(w)=2\,\#\{z\text{-labels in } w\}+\#\{y\text{-labels in } w\}$. We show how to alter w to make it satisfy (1) and (2) above, without altering the G_2 length.

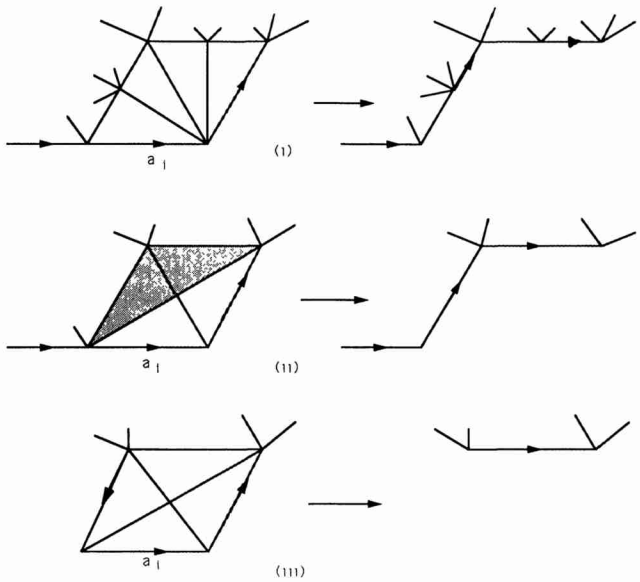

Fig. 4. Configurations prohibited by (1)

Case (1). *Suppose* (1) *is not satisfied*

The indicated changes are made. Notice that in case (ii) the condition (3) in the definition of G_2 complex is required to guarantee the existence of the shaded 2-cells. Case (iii) does not occur as it would lead to a reduction of G_2-length.

Case (2). *Suppose* (2) *is not satisfied*

The path continues past the edge labelled a_{j+1}. The path continues past the

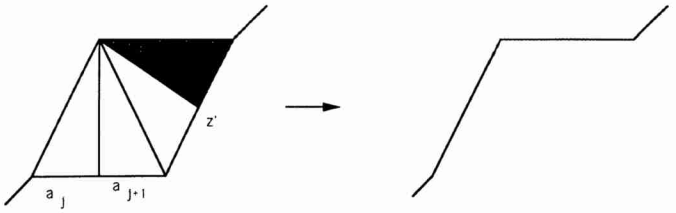

Fig. 5. Case (2)

Z-vertex z', as it must end at an X-vertex. The shaded triangle exists by condition (3) of the definition, so the indicated change can be made, without increasing the G_2 length.

Now set $H(i, w)$ to be the number of turning angles θ_j associated to w with $i \leqq j$ which can be given a value $< \pi$. As before, the above changes increase this value, so the process of replacement cannot continue indefinitely. □

Now everything reduces to the A_2 case, as a G_2 disc diagram which has no X-edges on the boundary, becomes an A_2 diagram when the X-edges are deleted from the interior. Thus § 5 of Part I shows that

Theorem 3.2 *The fundamental group of a G_2 complex of non-positive curvature is automatic.*

4 Applications

If we widen the concept of Euclidean building to allow the $A_1 \times A_1$ tesselation of the plane[1] we obtain:

Theorem 4.1 *Let X be a Euclidean building of rank 2, such that the link $\mathrm{Link}_X(v)$ of each vertex v is a finite complex. Let G be a discrete torsion-free group of isometries of X with compact quotient.*

Then G is automatic.

Proof. Let Γ denote the group of isometries of X; this is a locally compact group. The stabilizer Γ_v of the vertex v is profinite, i.e. the inverse limit of a sequence of finite groups, and thus compact: one sees this as follows. Let $B_v(n)$ be the ball of radius n about v, and let $\Gamma_v(n)$ be the subgroup of Γ which restricts to the identity on $B_v(n)$. Then we have projections

$$\Gamma_v \twoheadrightarrow \Gamma_v/\Gamma_v(n)$$

and

$$\bigcap_n \Gamma_v(n) = 1 \quad \text{so} \quad \Gamma_v \xrightarrow{1-1} \varprojlim \Gamma_v/\Gamma_v(n).$$

Now G is a discrete group, and so the stabilizer G_v of v is a discrete subgroup of a compact group, and so is finite. As G is assumed to be torsion-free, G_v is trivial, and the action is thus free. It follows that the quotient map $X \to X/G$ is a regular covering map, and so the quotient is a $A_1 \times A_1$, A_2, B_2 or G_2 complex. It remains to show that the non-positive curvature condition is satisfied in the quotient complex.

As the action is free, it is sufficient to show that the curvature condition is satisfied in X. Let C be a non-trivial reduced circuit in $\mathrm{Link}_X(v)$. If we cone the circuit C to the vertex v, we get a Euclidean cone; let θ denote the cone angle. If $\theta < 2\pi$, there are pairs of points in the cone which have two geodesic ares joining them, as can be seen as follows. If we cut the cone along a straight line from the cone point, and flatten this into a planar sector of internal angle θ, we see that there are two straight lines from a point on the angle bisector to points which were previously identified. In this way we see that geodesic arcs between points are not unique; this cannot happen in a Euclidean building (see [B, part (4) of the Theorem on p. 152]). □

[1] A building as defined in [B] is a simplicial complex. We understand buildings in the original sense of [BT] to be polysimplicial complexes subject to their axioms. In this latter sense the $A_1 \times A_1$ tesselation of $\mathbb{E}^2$ is an apartment.

Remark. We need G to be torsion-free in 4.1 since it is not known otherwise that G is linear, and hence whether (by Selberg's lemma) it contains a torsion-free subgroup of finite index.

In Part I, we used Weinbaum's description [W] of an alternating link group as a free factor of a $C(4)-T(4)$ (in fact the fundamental group of a finite $A_1 \times A_1$ complex) group, to show that such groups are automatic. We can now generalize this to:

Theorem 4.2 *The group of a connected sum of alternating links in automatic.*

Proof. Given a planar projection of a knot or link k, we can obtain a presentation (known as the Dehn presentation) of $\pi_1(S^3-k)*\mathbb{Z}$ where the generators correspond to the regions into which the projected curve divides plane, as follows. Enclose the projection in a large circle, and take a generator m_i for each compact region. We think of m_i as the homotopy class of a loop based at $+\infty$, the point from which projection takes place (or the viewer's eye), down through the region of the plane of projection corresponding to m_i, and on down to $-\infty$. As we are in S^3, this is in fact a loop. This provides a set of generators for the link group.

The relations for the presentation are of the form $m_1 m_2^{-1} m_3 m_4^{-1}$, when the regions labelled m_1, m_2, m_3, m_4 meet at a crossover.

When the link projection under consideration is alternating, the link is prime, and in addition there are no trivial crossings, there are four distinct regions at every crossing (see [W]). Moreover, the presentation obtained by the above method satisfies the $C(4)-T(4)$ small cancellation conditions. A minimal alternating projection of a prime knot or link satisfies all the above conditions.

For instance, the projection of Fig. 6 for a figure of eight knot gives a presentation of $\pi_1(S^3-k)*\mathbb{Z}$ with generators $m_1, m_2, m_3, m_4, m_5, m_6$ and relators

$$m_1 m_2^{-1} m_3 m_4^{-1}, m_1 m_6^{-1} m_3 m_2^{-1}, m_3 m_6^{-1} m_5 m_4^{-1}, m_5 m_6^{-1} m_1 m_4^{-1}.$$

Killing the generator corresponding to the region which has the extra added curve on its boundary gives the link group, but it is convenient to work with the free product as it is the above presentation which satisfies the small cancellation condition.

We deal first with the case of a sum of two prime links. In the above presentation, if m, m' are generators corresponding to neighboring regions, then mm'^{-1} is a meridian of the knot. The two presentations obtained give $A_1 \times A_1$ complexes K_1 and K_2.

Cutting each square into two triangles along a hypotenuse (in each square choose one of the two ways of doing this) turns K_i into a B_2 complex K_i'. Let m_i, m_i' be neighboring regions in the alternating projection of k_i, such that m_i, m_i' are neighboring non-hypotenuse edges in some triangle in K_i'. Then

$$\pi_1(S^3-(k_1 \# k_2)) = \pi_1(S^3-k_1) *_{m_1 m_1'^{-1} = m_2 m_2'^{-1}} \pi_1(S^3-k_2).$$

So we add to the disjoint union of the complexes K_i a cylinder triangulated as shown in Fig. 7, with the ends identified to the edges labelled m_i, m_i', and then identify all vertices.

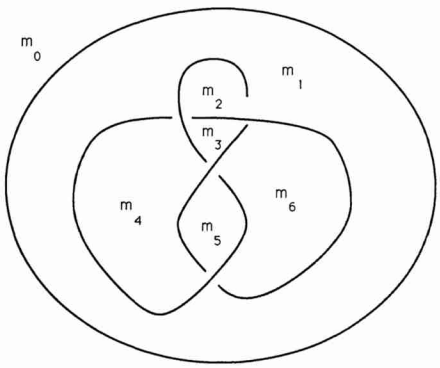

Fig. 6. A figure of eight knot

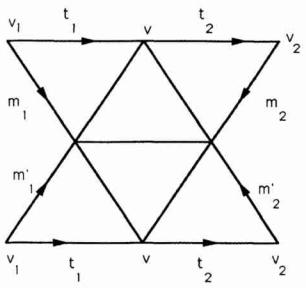

Fig. 7.

The fundamental group of the resulting B_2 complex is

$$\pi_1(S^3-(k_1\#k_2))*F_4$$

and the result follows for the sum of two prime alternating links.

Now to add on another alternating link to the sum already obtained, we follow the same procedure, again obtaining a B_2 complex. □

5 $C(4)-T(4)$ groups are automatic

We refer the reader to Part I for the relevant definitions of small cancellation conditions. For the rest of this section $\mathscr{P}=\langle X\,|\,R\rangle$ is a finite $C(4)-T(4)$ presentation of the group G, and K is the corresponding 2-complex. Recall that a non-trivial word $p\in F(X)$ is called a *piece* if there are two different words r_1, r_2 which are cyclic conjugates of a word (or the inverse of a word) in R such that $r_1=pr_1'$ and $r_2=pr_2'$. Each (cyclic) word $r\in R$ can be decomposed into no fewer than four pieces, possibly in several different ways. In Part I we avoided

the problem of different decompositions of a given relator by insisting that the pieces all be single letters.

The proof proceeds by first obtaining a new B_2-like 2-complex K' related to K where we can define turning angles, though lengths are not well-defined. In this new complex we choose a new set of generators for the fundamental group which we use to define representative words which have the same form as B_2-geodesics. The automatic structure is then constructed as in the B_2 case.

We build K' as follows: take a stellar decomposition of the 2-cell corresponding to $r=a_1 \ldots a_n$. This introduces new 1-cells e_i from the barycenter to the initial vertex of the edge a_i, which we orient outwards from the barycenter. The 2-cell for r is thus replaced by n 2-cells, with attaching maps $e_i a_i e_{i+1}^{-1}$. For each piece $p=a_i \ldots a_j$ of length greater than 1 occurring in the cyclic word r, add a new 2-cell to K with attaching map $e_i p e_{j+1}^{-1}$. Finally identify all the new barycenter vertices with the original vertex.

Let $E(r)=\{e_i | i=1, \ldots, n\}$, and $E=\bigcup_{r \in R} E(r)$.

For instance, take the presentation $\langle x, y; x^3=y^2 \rangle$; the relator has three decompositions into pieces:

$$x \cdot x \cdot x = y \cdot y, \quad (x^2) \cdot x = y \cdot y, \quad x(x^2) = y \cdot y$$

which give the 2-cells shown in Fig. 8.

It is easy to see via Tietze transformations that:

Lemma 5.1 *$\pi_1(K')$ is the free product of G and a finitely generated free group.*

In K', we model each 2-cell on a 45° triangle, where the hypotenuse edge corresponds to the piece on the boundary. Notice that this does not impose a piecewise Euclidean structure on K', as different subdivisions of a relator would in general have to assign different lengths to the 1-cells common to different pieces.

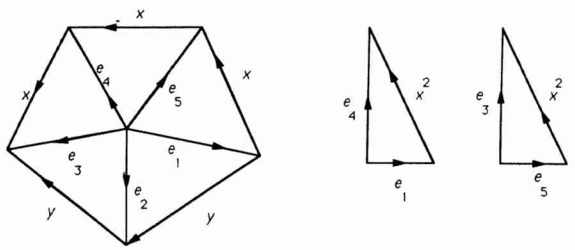

Fig. 8. Decomposition of a 2-cell

We now study maps of pairs $(D, \partial D) \to (K', K'^{(1)})$. After a homotopy, the cell structure of K' pulls back to a cell decomposition of a simply connected 2-complex which we call a singular disc diagram D. The orientations of the edges labelled in E divide the vertices in the interior of D into two classes. We refer to the vertices of D which correspond to barycenters, where the edges labelled

in E are all oriented outwards, as B-vertices. The others, where the edges labelled in E are all labelled inwards, we call V-vertices. If the boundary ∂D spells out the word w in $\pi_1(K'^{(1)})$, then we say that D is a disc diagram for $w=1$ in $\pi_1(K')$. Notice that the set E generates $\pi_1(K')$.

If two faces have an edge in common, and have the same label on their boundaries, then they can be cancelled, giving a diagram with fewer faces. A diagram is *reduced* if no faces can be cancelled. From now on, unless otherwise stated, all diagrams are assumed to be reduced.

Definition. We define an *admissible* disc diagram to be a singular disc diagram such that whenever two hypotenuse edges in D have a non-empty subsegment in common, then these entire edges are identified in D.

This means that in an admissible disc diagram D, the preimages of the vertex of the complex K' which lie in the interior of a hypotenuse edge meet no such other edge. It follows from the definition of admissible diagram that a hypotenuse edge in the interior is labelled by a piece, and that two hypothenuse edges either coincide or have no subsegment in common.

Definition. The regular language for $C(4)-T(4)$ presentations.

An *allowed* word is a cyclically reduced word in $F(E)$ corresponding to an edge path forming a circuit without backtracking in $K'^{(1)}$, satisfying the B_2 turning angle condition: in any admissible diagram which includes the word w, a non-negative turning angle is never followed by a positive turning angle (see Figs. 3, 4).

To obtain a diagram of non-positive curvature from a given diagram, we show how to remove interior vertices of positive curvature (i.e. where the sum of the corner angles is less than 2π) in a cell decomposition of a singular disc diagram.

Lemma 5.2 *If w is an allowed word representing the identity element of $\pi_1(K')$, then there is an admissible singular disc diagram of non-positive curvature at all interior vertices.*

Proof. Let D be a diagram for $w=1$ in $\pi_1(K')$. We first show that we can choose D to be admissible, and then we show how to remove interior vertices of positive curvature.

The important thing to note is that a non-trivial subword of a piece is also a piece, and so there is a 2-cell in K' with the hypotenuse edge labelled by that subword. Suppose that the hypotenuse edge h of a face f meets the hypotenuse edge h' of the face f' in a proper subsegment $h'' \subsetneq h$. The faces f, f' can be subdivided into faces corresponding to the subpieces labelling the segment h'' and the complement of h'' in h. In this way the diagram D can be made an admissible.

So from now on we assume that D is admissible.

Case 1: B-vertices of positive curvature

As all corner angles at a B-vertex are $\pi/2$, the only cases to consider are B-vertices meeting 2 or 3 edges.

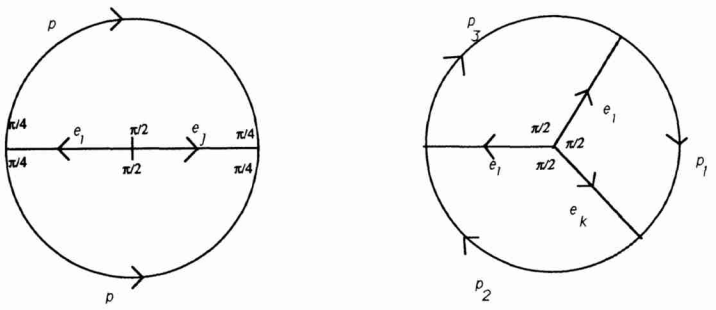

Fig. 9. *B*-vertices of positive curvature

In case (a), the two faces which have the vertex v on their boundary can be cancelled, as the diagram is not reduced (e_i and e_j define a unique pice p).

In case (b), there are neighboring faces in D, as the edges opposite the B-vertex cannot lie on the boundary of the diagram. The three pieces p_1, p_2, p_3 come from the same relator, but the $C(4)$ condition implies that three pieces do not cover the relator. Thus, for some choice of indices, $p_1 = p_3^{-1} p_2^{-1}$ and the diagram can be altered locally by subdivision and cancellation as shown in Fig. 10.

Although this change increases the curvature at two of the vertices, the number of faces and the number of vertices in the diagram is decreased.

Case 2: V-vertices of positive curvature

Interior V-vertices have even order, as E-edges alternate with hypotenuse edges.

In case (a), the two faces cancel, as before.

In case (b), the four faces can be replaced by two faces, as the two edges corresponding to pieces in $\pi_1(K)$ can be combined into a single piece, thus reducing the number of faces.

In case (c), the pieces p_j and p_{j+1} ($j = 1, 2, 3$, subindices mod 3) are subwords of a (cyclic) relator $r_j \in R$, as they lie on the boundary of faces f_j, f_j' which have a common edge labelled in E.

Either (i) we can write r_j as $p_j r_j' p_{j+1}^{-1}$, or (ii) the shorter, say p_j, of the words p_j, p_{j+1}, of the two words is an initial segment of the longer.

In subcase (ii), subdivide the faces f_2 and f_2' adjacent to the hypotenuse edge p_2 so that this common edge is divided in two, the first segment being labelled p_1. This creates two new faces, one of which can be cancelled with face f_1', producing a diagram with the same number of faces as before.

So suppose that subcase (ii) does not hold for any j. Then we can write the relators r_1, r_2, r_3 as:

$$r_1 = p_1 r_1' p_2^{-1}, \quad r_2 = p_2 r_2' p_3^{-1}, \quad r_3 = p_3 r_3 p_1^{-1}.$$

Then $r_j r_{j+1}$ is not cyclically reduced, and so by the $T(4)$ condition on G, two of these relators are identical, and two of the faces f_j', f_{j+1}' of D can be cancelled to give a diagram with fewer faces.

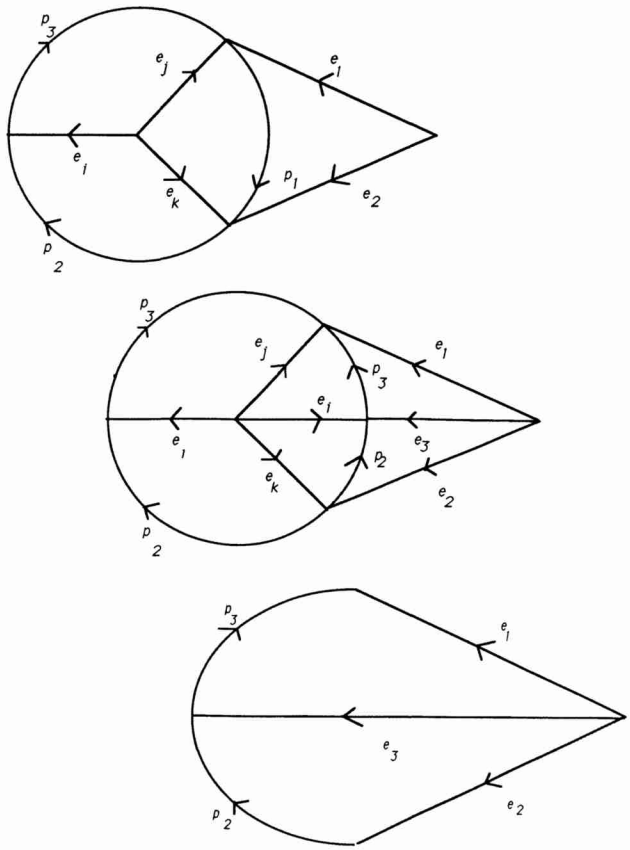

Fig. 10. Removal of a positive curvature B_2-vertex

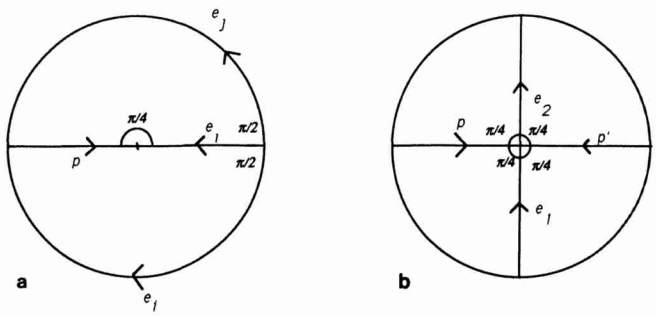

Fig. 11 a, b. V-vertices of order 2 and 4

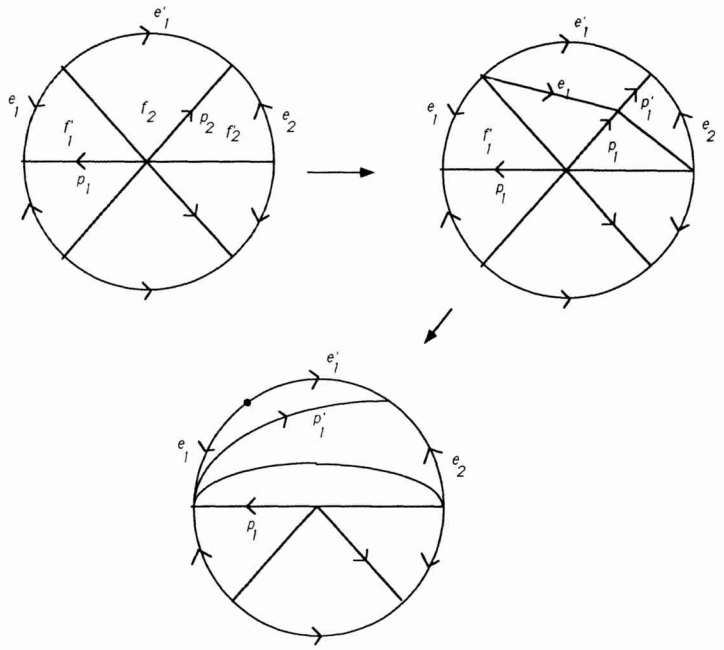

Fig. 12. Removing a V-vertex of order 6

It follows that a minimal admissible diagram for $w=1$ in $\pi_1(K')$ has non-positive curvature at all interior vertices, and the proof of Lemma 5.2 is complete. □

To establish the fact that the $C(4)-T(4)$ small cancellation groups are automatic, we now follow the argument for the B_2 case. We say that a word $w \in F(E)$ is an allowed representative of $g \in \pi_1(K')$ if it is shortest in $F(E)$ representing g, and satisfies the B_2 turning angle condition (i.e. this is just like an allowed word, but without the cyclically reduced condition, as we are not now considering cyclic words).

As in the B_2 case, each element of $\pi_1(K')$ has an allowed representative (see Proposition 2.1). For instance, given a geodesic representative w for the element $g \in G$, if a turning angle of $\frac{\pi}{2}$ follows a turning angle of 0 in a diagram D, then we can change w as in Fig. 3.

Only a finite number of configurations of subwords are used to define allowed representatives, so the language of allowed representatives is regular.

By considering admissible diagrams of non-positive curvature, the proof of Proposition 2.3 shows that two allowed representatives which begin and end distance at most 1 apart in the Cayley graph $\Gamma_E(\pi_1(K'))$ lie in a 2-Hausdorff neighborhood of each other.

Thus:

Proposition 5.3 *A group admitting a finite presentation satisfying a* $C(4)-T(4)$ *small cancellation condition is automatic.*

$C(6)-T(3)$ groups are automatic

The proof of this follows the $C(4)-T(4)$ case, though here we shall make use of the A_2 complexes.

Take the stellar subdivision of each 2-cell from the barycenter of each as before, adding a set E of 1-cells. For each relation r_i, we add a 2-cell for each piece, and identify the new barycenter vertices with the original vertex of K. Call the resulting complex K', where we give each 2-cell the structure of an equilateral triangle. As in the $C(4)-T(4)$ case, this does not define a piecewise Euclidean structure on K'. We now consider diagrams corresponding to maps of discs into the 2-complex K'.

Definition. An *admissible* diagram is a singular disc diagram such that if two edges in the interior of the diagram are labelled by pieces (i.e. when they are not E-edges) and intersect in a segment with non-empty interior, then the edges are identified. (An edge labelled by a piece of length greater than one consists of several segments, each of which we can think of as being labelled by a generator. If two pieces have a common subword, then in a diagram two cells could meet in the subsegment corresponding to the common subword.)

As before, the orientation of the E-edges divides the vertices of an admissible diagram into two classes, the B vertices corresponding to the barycentre vertices, where all edges are E-edges and oriented outwards, and V-vertices, where the edges are alternately E-edges (oriented inwards) and edges labelled by pieces.

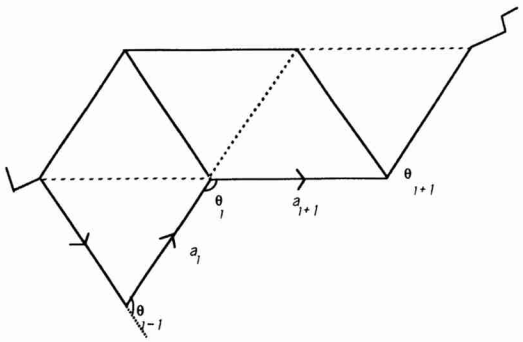

Fig. 13. Disallowed configuration of type (2) (dotted lines are non E edges)

An allowed word is a cyclically reduced word w in $F(E)$ corresponding to a circuit without backtracking, which satisfies the A_2 turning angle condition (1), and another more complicated condition (2). In any admissible A_2 diagram which includes $w=a_1 \dots a_n$, if $\theta_1 \dots \theta_{n-1}$ are the associated turning angles, then

(1) a positive turning angle does not follow a non-negative turning angle,

(2) we do not have $\theta_{i-1}=\frac{2\pi}{3}$, $\theta_i=-\frac{\pi}{3}$, $\theta_{i+1}>0$.

The reason for this more complicated second condition is to prohibit situations as in Fig. 14 below, where the paths grow arbitrarily far apart, though a positive turning angle never follows a non-negative one.

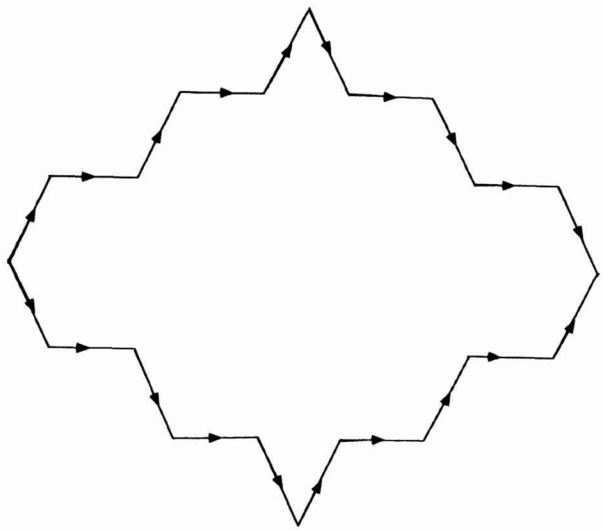

Fig. 14.

Lemma 6.1 *If w is an allowed word representing the identity element of $\pi_1(K')$, then there is an admissible disc diagram for w with non-positive curvature at all interior vertices.*

Proof. As in the proof of Lemma 5.2, a diagram can be made admissible by subdivision of edges labelled by pieces, so we can restrict our attention to admissible diagrams.

In the complex K', all corner angles are $\frac{\pi}{3}$, so that a vertex in the interior of a diagram has positive curvature if it meets fewer than 6 edges.

Case 1: B-vertices of positive curvature

These cases are treated in the same way as the case of B-vertices in Lemma 5.2.

A B-vertex of order 2 can be removed by cancellation of faces.

The $C(6)$ condition says that no relation is a product of fewer than 6 pieces. If $p_1, \ldots, p_n$, where $2<n<6$ are the pieces opposite a B-vertex, then as in subcase (b) of 5.2, for some index j, p_j^{-1} is an initial segment of the piece p_{j+1} or p_{j+1}^{-1}

is an initial segment of p_j (considering the indices cyclically). Subdivision of the corresponding face leads to cancellation and reduction of the number of vertices in the diagram.

Case 2: V-vertices of positive curvature

The fact that edges meeting a V-vertex are alternately E-edges and edges corresponding to pieces means that there an even number of edges, and so the only cases of interest are when there are 2 or 4 edges at a V-vertex. These two cases are dealt with as in the analogous subcases (a) and (b) of 5.2. □

We now continue as in Sect. 5. Define an allowed representative to be a reduced word in $F(E)$ satisfying the conditions for allowed words (without the restriction of being cyclically reduced), and shortest among all words representing the same element of $\pi_1(K')$.

A given word can be converted into an allowed word by operations, as before (specifically see Sect. 5 of Part I).

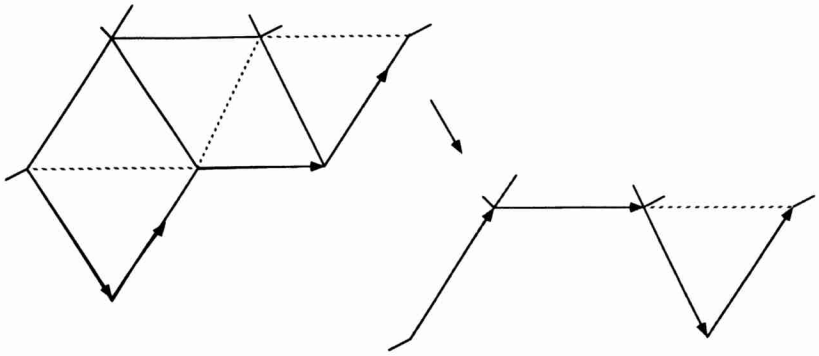

Fig. 15. Changing a disallowed configuration of type ('2)

To show that each element of $\pi_1(K')$ has an allowed representative, we must show that the process of changes cannot continue indefinitely. Here for the function $H(i, w)$ we use a count of the number of turning angles of $-\frac{\pi}{3}$ up to the i-th letter). When a change of type (2) occurs, as in Fig. 15, such an angle is interchanged with an earlier angle of $\frac{\pi}{3}$. Thus the sum of the $H(i, w)$ is definitely increased with each change.

It is clear that the language of allowed representatives is regular, as the conditions prohibit a finite number of ocnfigurations, which can be expressed in terms of finite state automata.

The results on A_2 diagrams of positive curvature contained in Part I, Sect. 6, show that allowed words which begin and end distance at most one apart lie in a 2-Hausdorff neighborhood of each other. Thus we have

Proposition 6.2 *A group which has a finite presentation satisfying the small cancellation conditions $C(6)-T(3)$ is automatic.*

7 $C(3)-T(6)$ groups are automatic

This case follows exactly the proof of Sect. 6. After subdivision of 2-cells etc., each triangle is given the structure of an isoceles $\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}$ triangle with short sides of length one. We can think of this triangle as two copies of the standard triangle of the G_2 type, identified along the shortest side.

We define admissible diagram and allowed words as in § 6. In this case, the following configurations are disallowed:

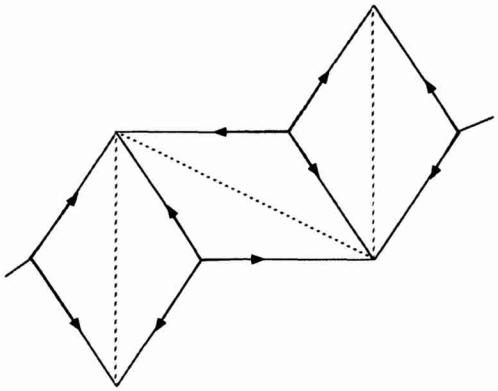

Fig. 16. Disallowed configuration

Lemma 7.1 *If an allowed word w represents the identity element of* $\pi_1(K')$ *then there is an admissible diagram for w with non-positive curvature at each interior vertex.*

Proof. As in 5.2, we can restrict our attention to admissible diagrams. Now all angles at a B-vertex are $\frac{2\pi}{3}$, and all angles at a V-vertex are $\pi/6$.

Case 1: B-vertices of non-positive curvature

As the angles involved are $\frac{2\pi}{3}$, the only case to consider is a B-vertex of order 2, which can be removed by cancellation of the neighboring faces as in 5.2.

Case 2: V-vertices of non-positive curvature

Here we must consider V-vertices of order less than 12. At a V-vertex, the E-edges and the edges labelled by pieces alternate. The proof is concluded by

treating such a vertex as in case 2(c) of 5.2, where the $T(6)$ condition is used instead of the $T(4)$ condition. □

The proof now concludes exactly as in § 5. We define an admissible representative as in § 6, and then we see that every element of $\pi_1(K')$ has such a representative, and that admissible words beginning and ending distance one apart lie in a 2-Hausdorff neighborhood of each other. Thus we have:

Proposition 7.2 *A group which has a finite presentation satisfying* $C(3)-T(6)$ *small cancellation condition is automatic.*

8 Remarks on the conjugacy problem

One of the interesting facts about automatic groups is that they have solvable word problem (see [CEHPT] or [BGSS, § 2.4]). At present it is unknown whether automatic groups have solvable conjugacy problem. The automatic structures constructed here and in Part I, have an additional property which gives a solution to the conjugacy problem using finite state automata. A way to think of this is that up until now, an automatic structure provides finite state automata which tell when two words differ by multiplication on the right. We can however equally well consider multiplication on the left, and multiplication on both sides. Let $\mathscr{A}$ be a finite set of semigroup generators for the automatic group G, and let L be the language of accepted words in $\mathscr{A}^*$ for an automatic structure on G. We say that L is the language of a *biautomatic structure* for G if in addition for each $a \in \mathscr{A}$, the language

$$\{(u, v)^{\$} \mid u, v \in L, \quad \text{and} \quad \bar{u} = \overline{a}\,\overline{v}\}$$

is a regular sublanguage in the padded language $v(\mathscr{A}^* \times \mathscr{A}^*)$. Recall that in the product language we pad the shorter of the two words u, v by adding a string of padding symbols \$ at the end of the shorter so that the two words have the same length, giving the set $v(\mathscr{A}^* \times \mathscr{A}^*) = \{(u, v)^{\$} \mid u, v \in L\}$ – see Part I, § 1.

Lemma 8.1 *If* L *is the language of a biautomatic structure on* G, *then for any words* $x, y \in \mathscr{A}^*$, *the language*

$$L(x, y) = \{(u, v)^{\$} \mid u, v \in L, \quad \textit{and} \quad \bar{u} = \overline{x}\,\overline{v}\,\overline{y}\}$$

is regular.

Proof. We prove by induction on the length of the words x, y, using the fact that the class of regular languages is closed under the operations of intersection, union, direct product and projection onto factors (see for instance [BGSS, Sect. 2][HU, Chapt. 3]).

We first show that the language

$$L(x) = \{(u, v)^{\$} \mid u, v \in L \quad \text{and} \quad \bar{u} = \overline{x}\,\overline{v}\}$$

is regular (this is analogous to [BGSS, Prop. 2.7]). By definition this is true for words x of length 1.

Now suppose that the language $L(x)$ is regular for words of length $<n$. Let $x=ax'$ be a word of length n, where $a\in\mathscr{A}$, and $\ell(x')=n-1$. Then $L(x')$ and $L(a)$ are both regular by the induction hypothesis, and so the padded language

$$L(a)\times L(x')=\{(u, v, w, z)\,|\,\bar{u}=\overline{av} \quad \text{and} \quad \bar{w}=\overline{x'z}\}$$

is also regular. Now

$$L(M_0)=\{(v, w)\,|\,v, w\in L \quad \text{and} \quad \bar{v}=\bar{w}\}$$

is the regular language accepted by the equality checker M_0 of the automatic structure on G, so that

$$(\mathscr{A}^*\times L(M_0)\times\mathscr{A}^*)\cap(L(a)\times L(x'))$$

is the intersection of two regular padded languages and so is itself regular. Notice that this language consists of padded quadruples $(u, v, w, z)^\$$ where $\bar{u}=\overline{av}=\bar{a}\bar{w}=\overline{ax'z}$. We now project this language onto the product of the first and last factors, to give the regular language $L(x)$.

Similarly the language

$$L'(y)=\{(u, v)^\$\,|\,u, v\in L \quad \text{and} \quad \bar{u}=\overline{vy}\}$$

is regular, and the language $L(x, y)$ can be obtained by projecting the regular language

$$(L(x)\times L'(y))\cap(L\times L(M_0)\times L)$$

onto the product of the first and last factors. □

We have shown that all the languages that we have constructed in Part I and here have the essential property that geodesic words which begin and end distance at most one apart lie in a k-Hausdorff neighborhood of each other for some k. It follows that as in Part I, § 1, it is possible to build a standard automaton which accepts the languages $L(a)$, by simply altering the automaton given in the proof of 1.1 of Part I by making the start state $(s_0, s_0, \bar{a}^{-1})$, and the accept states to be $Y\times Y\times\{1\}$. Thus the proof of Theorem 1.1 of Part I shows that:

Lemma 8.2 *If G has an automatic structure in which the language L of accepted words consists of geodesics with the property that words which begin and end distance one apart lie in a k-Hausdorff neighborhood, then L is part of a biautomatic structure for G.*

Our constructions thus show that negatively curved groups have biautomatic structures, and that small cancellation groups are free factors of groups with biautomatic structures (and so themselves are biautomatic, following the argument of [BGSS, § 4]).

Theorem 8.3 *A group with a biautomatic structure has solvable conjugacy problem.*

Proof. Given two words $x, y\in\mathscr{A}^*$, then from the above result the language $L(x, y)$ is regular. But if $\bar{x}$ and $\bar{y}$ are conjugate in G, then there is some element

$g \in G$ such that $\bar{x} = g\bar{y}g^{-1}$. This means that there is some word $w \in L$ such that $\overline{xw} = \overline{wy} \Leftrightarrow \bar{w} = \overline{x^{-1}wy}$, or alternatively, that the regular language

$$L(x^{-1}, y) \cap \{(w, w) | w \in L\}$$

is non-empty. But the problem of deciding whether a given regular language is empty is algorithmically solvable (see [HU, Chapt. 3]). □

Notice that given a small cancellation group G, we obtained languages of geodesics with the required properties which put biautomatic structures on a group $H = G * F$, where F is a finitely generated free group. But as the conjugacy problem is solvable in H, it is also solvable in G.

The conjugacy problem was originally solved for small cancellation groups by Schupp (see [LS, Chapt. V, Sect. 7)] and for negatively curved groups by Gromov [G].

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