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Equivariant stable homotopy and Sullivan's conjecture

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Introduction

Let G be a p-group, and let X be a G-complex. Let EG denote a contractible space on which G acts freely. By the "homotopy fixed point set" of X, we mean the fixed point set $F(EG, X)^G$, where $F(EG, X)$ denotes the function space of all maps $EG \to X$, equipped with a G-action by $gf = gfg^{-1}$. If we let * denote the one point space with trivial G-action, we may also consider the G-space $F(*, X)$; it is canonically G-homeomorphic to X. The G-map $EG \to *$ induces a map $\eta: X^G \cong F(*, X)^G$.

In [19], D. Sullivan proposed the following.

Conjecture A. *For X a finite G-complex, the map η is a weak equivalence after p-adic completion.*

In fact, Sullivan only proposed the conjecture in the case $G = \mathbb{Z}/2\mathbb{Z}$. He was interested in the situation of a complex algebraic variety V with real defining equations, with G-action defined by coordinatewise complex conjugation, and showed that an affirmative answer would imply that the 2-adic homotopy type of the variety of real points $V_{\mathbb{R}}$ is determined by the étale homotopy type of V.

A second consequence is that the based mapping space $F(BG, X)$ is p-adically contractible for any finite complex X. This is obtained by considering G-complexes X with trivial action. Conjecture A in this case was proved in 1983 by H.R. Miller [17], using a powerful array of techniques including the Bousfield-Kan unstable Adams spectral sequence and the André-Quillen homology theory of rings. This form of the conjecture has seen extensive application, including the solution by McGibbon and Neisendorfer of an old conjecture of Serre's, and important work of Zabrodsky on maps between classifying spaces.

A third consequence is the existence of a "destabilization spectral sequence". This was pointed out to the author by M. Mahowald and M.J. Hopkins. If we view S^{n+1} as the join $S^n * S^0$, and let $G = \mathbb{Z}/2$ act on S^0 by interchanging points, we

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obtain an action on S^{n+1} whose fixed point set is S^n. Given the validity of Conjecture A, we can 2-adically identify S^n with the homotopy fixed point set $F(EG, S^{n+1})^G$. Filtering EG by its skeleta produces a spectral sequence with $E_2^{p,q} \cong H^p(\mathbb{Z}/2; \pi_{-q}(S^{n+1}))$, and with E_∞ an associated graded version of $\pi_*(S^n)$. This spectral sequence should have important applications.

We now state our theorem. Let X be a finite dimensional G-complex, and $X^G = \coprod_{\alpha \in A} X_\alpha^G$ be a decomposition of X^G into its connected components. Let $\tilde{X}_\alpha^G$ denote the covering space of X_α^G associated to the subgroup $K_\alpha = \operatorname{Ker}(\pi_1(X_\alpha^G, *) + \pi_1(X, *)) \subseteq \pi_1(X_\alpha^G, *)$, and let $L_\alpha = \pi_1(X_\alpha^G, *)/K_\alpha$. Let $\bar{X}_\alpha^G = EL_\alpha \underset{L_\alpha}{\times} (\tilde{X}_{\alpha p}^{G\wedge})$, where $X_p^\wedge$ denotes the Bousfield-Kan mod p completion of X.

Theorem B. *For any finite-dimensional G-complex, the following hold.*

(*a*) *There is a natural map* θ: $F(EG, X)^G \to \coprod_\alpha \bar{X}_\alpha^G$, *which induces an isomorphism on mod-p homology*;

(*b*) *Let* $F(EG, X)_\alpha^G$ *denote the component of* $F(EG, X)^G$ *corresponding to* $\bar{X}_\alpha^G$. *Then the homotopy fibre* $\mathcal{F}$ *of the map* $F(EG, X)_\alpha^G \to \bar{X}_\alpha^G$ *is connected, has abelian fundamental group, and* $\pi_i(\mathcal{F})$ *is uniquely p-divisible for all p*;

(*c*) *Let* $X_p^\wedge$ *denote the mod-p completion of the G-space X.* (*This construction is discussed in §II*)*-Then the map* $(X^G)_p^\wedge \to F(EG, X_p^\wedge)^G$ *is a weak equivalence*;

(*d*) *If* $\pi_i(X) = 0$ *for* $i \leqq N$, *then the map* $\pi_i(\theta)$ *is an isomorphism for* $i < N$, *and a surjection for* $i = N$.

In the case of spaces for which the subgroups K_α are nilpotent and act nilpotently on higher homology, this completion is well understood, and agrees with Sullivan's notion of p-adic completion, and this theorem gives an affirmative answer to Conjecture A. In situations where K_α is not nilpotent, this theorem shows that Sullivan's conjecture is false as stated. As pointed out by H.R. Miller, it shows that the correct formulation of the conjecture should have been the statement in part (c) of Theorem B. It achieves Sullivans goal of recovering the p-adic completion of $V_\mathbb{R}$ from $X_p^\wedge$ rather than from X.

Our method of proof is to use the affirmative solution of Segal's Burnside ring conjecture [9], together with a cosimplicial approximation scheme. One formulation of Segal's conjecture is that the map η: $\mathbb{Q}^G(X)^G \to F(EG, \mathbb{Q}^G(X))^G$ is an equivalence for finite-dimensional G-complexes after p-adic completion. Here $\mathbb{Q}^G X$ denotes the space $\lim \Omega^V \Sigma^V X$, where V ranges over all representations of the group G. The homotopy groups of the fixed point set are the equivariant stable homotopy groups of X, as defined by Segal [18].

We construct the cosimplicial G-space associated to the triple $\mathbb{Q}^G$ (see [4, 7]) to construct a tower of G-fibrations $\ldots \to X_{i+1} \to X_i \to X_{i-1} \to \ldots \to X_0$, so that the inverse limit of the tower approximates X in a suitable sense made precise in the paper. The fibres F_i of the maps $X_i \to X_{i-1}$ are G-spaces to which a slight elaboration of the Segal Conjecture applies. The analysis of these fibres relies on the equivariant Snaith decomposition proved in [14].

The paper is organized as follows.

§I discusses the results concerning equivariant stable homotopy theory which we will need. §II constructs the cosimplicial approximation scheme, and develops its properties. §III develops all the necessary applications of Segal's conjecture. The above mentioned elaboration is Proposition III.9. §IV states some elementary results concerning the universal covering space of a G-complex, which is necessary for the proof of Theorem B, part (a). V shows how to apply the equivariant Snaith

decomposition (Theorem I.10) to analyze the fibres F_i. In §VI, we prove our main theorem. Theorem B, part (c) is Theorem VI.1, Theorem B, part (b) is Theorem VI.5, and Theorem B, part (a) is Theorem VI.11. Finally, Theorem B, part (d) is Corollary VI.13. We have added two appendices to clarify certain technical points in the text.

Some remarks are in order, first H.R. Miller and J. Lannes have independently of me and of each other arrived at proofs of Theorem B, part (c). This is the central part of the theorem; parts (a), (b), and (d) are minor improvements which are possible due to the fact that our cosimplicial approximation scheme produces $\mathbb{Z}$-completions rather than $\mathbb{F}_p$-completions of the fixed point sets. Secondly, the authors original proof involved the construction of an equivariant version of T. Goodwillie's theory of analytic functors. M. Hopkins suggested the possibility of substituting the cosimplicial space associated to the triple $\mathbb{Q}^G$, which drastically shortened and simplified the proof.

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I. Equivariant stable homotopy theory

We use the notation and definitions of [9, §I]. Thus, G-complexes are all based, unless we explicitly state otherwise, and if X is any G-space, X^+ denotes X with a disjoint fixed base point added. If we want to speak of a $G - CW$ complex in the sense of [18], without a choice (or existence) of a fixed basepoint, we call it an *unbased* G-complex. If X and Y are any G-spaces, by $X \times_G Y$ we mean the orbit space $X \times Y/G$, where G acts diagonally. Similarly, if X and Y are G-complexes, we write $X \wedge_G Y$ for $X \wedge Y/G$. If X is any G-space, and Y is a G-complex, we write $X \ltimes Y$ and $X \ltimes_G Y$ for $X^+ \wedge Y$ and $X^+ \wedge_G Y$ respectively. If $H \subseteq G$ is a subgroup, $N_G(H)$ will denote the normalizer of H in G. $W_G(H)$, the "Weyl group" of H in G, is defined to be $N_G(H)/H$. Given a group G and $\Gamma \lhd G$ a normal subgroup, let $\mathscr{F}_G(\Gamma)$ be the family of all subgroups $K \subseteq G$ such that $K \cap \Gamma = \{e\}$.

Definition I.1. A G-space X is said to be an $E_G\Gamma$-space if
(*a*) $X^K = \Phi$ for all $K \in \mathscr{F}_G(\Gamma)$
(*b*) X^K is contractible for all $K \notin \mathscr{F}_G(\Gamma)$

For all G and $\Gamma \lhd G$, $E_G\Gamma$-spaces exist, may be taken to be unbased G-complexes, and are unique up to weak G-equivalence. See [11] for proofs of these facts. (A G-map $f: X \to Y$ is said to be a weak G-equivalence if each of the maps $f^H: X^H \to Y^H$ is a weak equivalence for all subgroups $H \subseteq G$. A weak G-equivalence between (unbased) G-complexes is a G-homotopy equivalence.) The orbit space $E_G\Gamma/\Gamma$ is called a $B_G\Gamma$ space; it is equipped with a G/Γ-action, and is unique up to weak G/Γ-equivalence (see [11]). Due to this uniqueness, we will abuse notation and simply refer to $E_G\Gamma$ and $B_G\Gamma$ as well-defined G and G/Γ-spaces, respectively.

Proposition I.2. *Let G and Γ be as above, and let H denote a subgroup of G/Γ. Let $\pi: G \to G/\Gamma$ denote the projection. Then $E_G\Gamma$, viewed as an unbased $\pi^{-1}(H)$-complex, is an $E_{\pi^{-1}(H)}\Gamma$-space. Consequently, $B_G\Gamma$, viewed as an H-space, is a $B_{\pi^{-1}(H)}\Gamma$-space.*

Proof. One simply notes that if $K \subseteq \pi^{-1}(H)$, $K \in \mathscr{F}_{\pi^{-1}(H)}\Gamma$ if and only if $K \in \mathscr{F}_G \Gamma$, when viewed as a subgroup of G. By the definition of $E_G \Gamma$, this gives the result. □

Let G and Γ be as above, and let X be an unbased G-complex. We will need to study the fixed point sets of the G/Γ-space $E_G \Gamma \times_\Gamma X$.

Proposition I.3. *Let X be an unbased G-complex. Then $(E_G \Gamma \times_\Gamma X)^{G/\Gamma} = \coprod_K EW_G(K) \times_{W_G(K)} X^K$, where K ranges over all G-conjugacy classes of subgroups K of G in $\mathscr{F}_G(\Gamma)$, which also have $\pi(K) = G/\Gamma$.*

Proof. This is proved in [20], Theorem 12. □

Corollary I.4. *Let X be any G-complex. Then $(E_G \Gamma^+ \wedge_\Gamma X)^{G/\Gamma} \cong \bigvee_K EW_G(K) \wedge_{K_G(K)} X^K$, where K ranges over G-conjugacy classes of subgroups in $\mathscr{F}_G(\Gamma)$, so that $\pi(K) = G/\Gamma$.*

We will now specialize to the case $G = H \times \Gamma$, where H is a finite group and Γ is a subgroup of the symmetric group Σ_n. Let X be a G-space. Then the n-fold product X^n becomes a G-space, by permitting H to act diagonally, and Γ to act by permuting coordinates via its inclusion $\Gamma \subseteq \Sigma_n$. Let K be any subgroup in $\mathscr{F}_G(\Gamma)$, so that $\pi(K) = H$. K determines a homomorphism $\phi_K \colon H \to \Gamma$ by the condition $(h, \phi_K(h)) \in K \;\forall\, h \in H$, and hence determines an action of H on the set $\{1, \ldots, n\}$. Let $i_1, \ldots, i_g$ denote a system of orbit representatives for this action. Let $H_1, \ldots, H_s$ denote the stabilizers of $i_1, \ldots, i_s$, respectively.

Proposition I.5. *$(X^n)^K$ is homeomorphic to $X^{H_1} \times \ldots \times X^{H_s}$*

Proof. This is a routine computation which we omit. □

Corollary I.6. *If $X^{\wedge n}$ denotes the n-fold smash product of X with itself, equipped with $H \times \Gamma$ action as above, and $K \in \mathscr{F}_G(\Gamma)$, $\pi(K) = H$, then $(H^{\wedge n})^K \cong X^{H_1} \wedge \ldots \wedge X^{H_s}$, where $H_1, \ldots, H_s$ are defined as above*

Corollary I.7. *Let X be a G-complex with connected fixed point sets. Let Γ be a subgroup of $\sum_N$. $X^{\wedge N}$ becomes an $H \times \Gamma$-complex as above. Then $(E_{H \times \Gamma}\Gamma^+ \wedge_\Gamma X^{\wedge N})^H$* is *$([N/|H|] - 1)$-connected.*

Proof. By Corollary I.4, we have that $(E_{H \times \Gamma}\Gamma^+ \wedge_\Gamma X^{\wedge N})^H \cong \bigvee_K EW_G(K)^+ \wedge_{W_G(K)} (X^{\wedge N})^K$, where $G = H \times \Gamma$, and K ranges over G-conjugacy classes of subgroups in $\mathscr{F}_G(\Gamma)$ for which $\pi(K) = H$. As before, such a subgroup determines a homomorphism $\phi_K \colon H \to \Gamma$, and hence an action of H on the set $\{1, \ldots, N\}$. By Corollary I.6, $(X^{\wedge N})^K \cong X^{H_1} \wedge \ldots \wedge X^{H_s}$, where s is the number of orbits of this action, and H_i denotes a stabilizer of a point in the i-th orbit. Note that $s \geqq [N/|H|]$, since each orbit has order at most $|H|$. By hypothesis, X^{H_i} is connected, so $X^{H_1} \wedge \ldots \wedge X^{H_s}$ is $(s - 1)$-connected, hence $([N/|H|] - 1)$-connected. It follows that $EW_G(K)^+ \wedge_{W_G(K)} (X^{\wedge N})^K$ is $([N/|H|] - 1)$-connected, which was to be shown. □

We now recall how these constructions relate to equivariant stable homotopy theory. From this point on, let G be finite. Suppose V is an orthogonal representation of G; we let S^V denote its one point compactification, with the point at infinity as its base point. If X is a G-complex, $\sum^V X$ is defined to be the G-complex $S^V \wedge X$. By $\Omega^V X$, we mean the space of continuous based maps from S^V to X in the compact open topology, equipped with a G-action defined by $(g \cdot f)(z) = gf(g^{-1}z)$.

Definition I.8. *Let X be a G-complex. We let ε denote a one dimensional trivial representation of G, and let R denote the regular representation of G. $n\varepsilon$ and nR denote a direct sum of n copies of ε and R, respectively. We define*
(a) $\bar{Q}^G(X) = \lim\limits_{\vec{n}} \Omega^{n\varepsilon}\sum^{n\varepsilon} X$
(b) $Q^G(X) = \lim\limits_{\rightarrow} \Omega^{nR}\sum^{nR} X$

(The maps in the directed system of G-spaces arise from suspension, using the fact that $S^{V\oplus W} \cong S^V \wedge S^W$.)

Both these spaces are G-spaces, and there is a natural inclusion $\bar{Q}^G(X) \to Q^G(X)$, induced by the inclusion of ε in R as the unique trivial summand in R. $\bar{Q}^G$ and Q^G are functors from the category of based G-spaces to itself.

Lemma I.9. *$\bar{Q}^G$ and Q^G convert G-cofibrations into G-quasifibrations; i.e., if Y is a G-subcomplex of X, the sequences of maps*

$$\bar{Q}^G(Y) \to \bar{Q}^G(X) \to \bar{Q}^G(X/Y)$$
$$Q^G(Y) \to Q^G(X) \to Q^G(X/Y)$$

are G-quasifibrations. A map of G-spaces is said to be a G-cofibration (G-quasifibration) if the fixed point maps f^H are cofibrations (quasifibrations) for all subgroups $H \subseteq G$.

Proof. It is immediate that $\bar{Q}^G(X)^H = Q(X^H)$, and according to [10], $Q^G(X)^H \cong \prod_{K \subseteq H} Q(EW_H(K)^+ \wedge_{W_H(K)} X^K)$, where the product runs over all conjugacy classes of subgroups of G. These descriptions together with the well-known fact that $Q(Y) \to Q(X) \to Q(X/Y)$ is a quasifibration for any cofibration $Y \to X \to X/Y$, give the result. □

Theorem I.10 [14]. *Let X be a G-complex, all of whose fixed point sets are connected. Then there is a natural weak G-equivalence.*

$$Q^G Q^G(X) \cong \prod_{n \geq 1} Q^G(E_{G \times \sum_n} \textstyle\sum_n^+ \underset{\sum_n}{\wedge} X^{\wedge n})$$

Proof. This is the equivariant Snaith splitting; its proof may be found in [14]. □

Corollary I.11. *Let X be a G-complex, all of whose fixed point sets are connected. Then there is a weak G-equivalence*

$$(Q^G)^r(X) \cong \prod_{\alpha \in A} Q^G(E_{G \times \Gamma_\alpha} \Gamma_\alpha \underset{\Gamma_\alpha}{\wedge} X^{n_\alpha})$$

where α is an indexing set, and Γ_α denotes a subgroup of $\sum_{n_\alpha}$, acting on $X^{\wedge n_\alpha}$ via its inclusion in $\sum_{n_\alpha}$. (Remark. The equivalence goes naturally to the sum rather than the product. The fact that the fixed point sets are all connected assures that for any k, all but a finite numbers of the summands $Q^G(E_{G \times \Gamma_\alpha} \Gamma_\alpha \wedge_{\Gamma_\alpha} X^{n_\alpha})$ have k connected H-fixed point sets for all $H \subseteq G$.)

Finally, we wish to study the splitting of the fixed point set $(Q^G(X))^H$, where X is a G-complex, and $H \lhd G$ is a normal subgroup. Note that $(Q^G(X))^H$ is a G/H-space. We describe its G/H-equivariant homotopy type. We first recall from [1] the definition of the equivariant transfer. Let Γ be any finite group, and suppose $\Gamma_0 \lhd \Gamma$ is a normal subgroup. Let X be a finite Γ-complex, on which Γ_0 acts freely

off the basepoint. Adams constructs a "stable Γ-map" $\tau_X: X/\Gamma_0 \to X$, enjoying the usual properties of a transfer. Here, a "stable Γ-map" means a Γ-map $S^V \wedge (X/\Gamma_0) \to S^V \wedge X$, for some orthogonal representation of Γ. Now suppose we have a finite group G, $K \subseteq G$ any subgroup, and a finite $N_G(K)$-complex X on which K acts freely off the basepoint. By Adams' construction, we obtain a stable $N_G(K)$-map τ_X: $X/K \to X$. We may apply the functor $G^+ \wedge_{N_G(K)}$ to this map, to obtain a stable G-map τ: $G^+ \wedge_{N_G(K)} (X/K) \to G^+ \wedge_{N_G(K)} X$. After passing to direct limits over finite subcomplexes of an $N_G(K)$-complex X, one obtains a G-map τ: $Q^G(G^+ \wedge_{N_G(K)} X/K) \to Q^G(G^+ \wedge_{N_G(K)} X)$ for any G-complex X. Now, let X be any G-complex, $H \lhd G$ a normal subgroup, and $K \subseteq H$ a subgroup. Note that $W_H(K)$ is a subgroup of the group $W_G(K)$, and hence the $W_G(K)$-space $E_{W_G(K)} W_H(K)$ is defined. Of course, $W_G(K)$ acts on X^K. We now define a G/H-equivariant map ξ_K: $Q^{G/H}(G/H^+ \wedge_{N_G(K)} (E_{W_G(K)} W_H(K)^+ \wedge_{W_H(K)} X^K)) \to Q^G(X)^H$ to be the composite

$$Q^{G/H}\left(G/H^+ \underset{N_G(K)}{\wedge} \left(E_{W_G(K)} W_H(K)^+ \underset{W_H(K)}{\wedge} X^K\right)\right)$$

$$(\underset{\sim}{a})\, Q^G\left(G^+ \underset{N_G(K)\cdot H}{\wedge} \left(E_{W_G(K)} W_H(K)^+ \underset{W_H(K)}{\wedge} X^K\right)\right)^H$$

$$(\underset{\sim}{b})\, Q^G\left(G^+ \underset{N_G(K)\cdot H}{\wedge} \left(E_{W_G(K)} W_H(K)^+ \wedge X^K\right)\right)^H$$

$$(\underset{\sim}{c})\, (Q^G(X))^H$$

Here, (a) is induced by the inclusion of $\mathbb{R}[G/H]$ as the H-fixed point space in $\mathbb{R}[G]$, where H acts by left multiplication, (b) is induced by the above mentioned transfer, and (c) is induced by the maps

$$G^+ \underset{N_G(K)\cdot H}{\wedge} (E_{W_G(K)} W_H(K)^+ \wedge X^K) \to G^+ \underset{N_G(K)\cdot H}{\wedge} (S^0 \wedge X)^K \simeq G^+ \underset{N_G(K)\cdot H}{\wedge} X \to X .$$

Definition I.12. *Let G be a finite group, $H \lhd G$ a normal subgroup, and X a G-complex. We define a G/H-space $\theta(G, H, X)$ to be*

$$\prod_K Q^{G/H}\left(G/H^+ \underset{N_G(K)}{\wedge} \left(E_{W_G(K)} W_H(K)^+ \underset{W_H(K)}{\wedge} X^K\right)\right).$$

The product is over G-conjugacy classes of subgroups of H.

Proposition I.13. *Let G and H be as above, and let $G^1 \subseteq G$ be a subgroup containing H. Then, ξ as a G^1/H-space, $\theta(G, H, X)$ has the weak G^1/H-homotopy type of $\theta(G^1, H, X)$.*

Proof. This is a tedious but elementary verification using Proposition I.2. □

Proposition I.14. The map $\prod_K \xi_K$: $\theta(G, H, X) \to (Q^G(X))^H$ *is a G/H weak equivalence, where the product is over a system of representatives for the G-conjugacy classes of subgroups of H.*

Proof. This is just the zeroth space version of Theorem V. 11.1 of [14]. □

II. Triples and cosimplicial spaces

By a triple in a category C, we mean a functor $T: C \to C$, equipped with natural transformations $\phi: Id \to T$ and $\psi: T^2 \to T$, which satisfy the identities $(T\phi)\phi = (\phi T)\phi$, $\psi(T\psi) = \psi(\psi T)$, and $\psi(T\phi) = Id = \psi(\phi T)$ (see [7] for a discussion of this.) We discuss certain triples on the category of spaces, where by a space, we mean simplicial set. If we mean an actual space, we will write "topological space". Let R be a ring. In [7], a triple $X \to RX$ is defined as follows. We first define a triple $\tilde{R}$ on the category of sets. For a set X, $\tilde{R}X$ is defined to be the subset of the free R-module on X consisting of all sums $\sum_i r_i x_i$, $r_i \in R$, $x_i \in X$, such that $\sum_i r_i = 1$. ϕ is here the natural inclusion of X into the free module on X, which certainly factors through $\tilde{R}X$. ψ is defined by $\psi(\sum_i r_i(\sum_j r_j x_{i_j})) = \sum r_i r_j x_{i_j}$. It is easy to check that ψ preserves the subset $\tilde{R}X$ in the free R-module on X. Due to the naturality of these transformation this triple immediately gives a triple on the category of spaces.

A second triple of interest occurs in the category of topological spaces. First, for any based topological space X, we define QX to be $\lim_{r\to\infty} \Omega^n \Sigma^n X$. Note that we have an inclusion $X \to QX$ and an evaluation map $QQX \to X$, which makes Q as it stands into a triple on the category of based topological spaces. For later purposes, we will need a definition which does not take into account base points. Given any topological space X, let X^+ denote X with a disjoint base point $*$ added. Let $p: X^+ \to S^0$ be the based map sending X to the non-base point in S^0. Then we define $\tilde{Q}(X) \subseteq Q(X^+)$ by $\tilde{Q}(X) = \{f: S^n \to \sum^n(X^+) | Q(p)\circ f \simeq \mathrm{id}_n\}$. (This is the analogue for Q of the condition $\sum r_i = 1$ for the triple $\tilde{R}$.) It is easy to check that $\tilde{Q}$ now becomes a triple on the category of topological spaces.

We now define an associated triple Q on the category of simplicial sets as follows. Let τ be the functor from sets to topological spaces defined by $\tau(X) = \tilde{Q}(X)$, where X is viewed as a discrete topological space. Applying τ levelwise, we obtain a functor from simplicial sets to simplicial topological spaces. Applying Sin levelwise, we obtain a functor from simplicial sets to bisimplicial sets. We finally compose with the "diagonal simplicial set functor" to obtain a functor Q from simplicial sets to simplicial sets. Note that $Q(X)_k = F(\Delta_k, \tilde{Q}(X_k))$, where F denotes the set of continuous functions, and where Δ_k denotes the standard k-simplex viewed as a topological space. We observe that there is a continuous map $\tilde{Q}(F(\Delta_k, X)) \to F(\Delta_k, \tilde{Q}X)$, where $F(\Delta_k, X)$ is given the discrete topology and $F(\Delta_k, \tilde{Q}X)$ is given the compact open topology. Q now becomes a triple as follows. The simplicial map $X \to Q(X)$ is given in level k by the map $N: X_k \to F(\Delta_k, \tilde{Q}(X_k))$, defined by $N(x)(z) = j(X)$, where $j: X_k \to \tilde{Q}(X_k)$ is the inclusion. The map $QQX \to QX$ is given in level k by the composite $F(\Delta_k, \tilde{Q}(F(\Delta_k, \tilde{Q}(X_k)))) \to F(\Delta_k, F(\Delta_k, \tilde{Q}\tilde{Q}(X_k)) \to F(\Delta_k, F(\Delta_k, \tilde{Q}(X_k))) \to F(\Delta_k, \tilde{Q}(X_k))$, where the first map is induced by the map $\tilde{Q}(F(\Delta_k, X)) \to F(\Delta_k, \tilde{Q}X)$ described above ($\tilde{Q}(X_k)$ is substituted for X), the second is induced by the map $\tilde{Q}\tilde{Q}(X_k) \to \tilde{Q}(X_k)$, and where the final map is obtained by pulling back along the diagonal map $\Delta_k \to \Delta_k \times \Delta_k$. We note that for any simplicial set X, there is a weak equivalence $|Q(X.)| \to \tilde{Q}(|X.|)$. Let $\mathbf{Z}$ denote the triple on the category of simplicial sets associated to the ring $\mathbf{Z}$.

Then there is a natural transformation (respecting the triple structure) $Q \xrightarrow{\nu} \mathbf{Z}$, defined in level $\mathbf{k}$, obtained by observing that a typical element of $F(\Delta_k, \tilde{Q}(X_k))$ can by adjointness be identified with a continuous map $f: S^n \wedge (\Delta_{k^+}) \to S^n \wedge (X_{k^+})$ and

defining $v(f)$ to be $H_n(f)(i_n) \in H_n(S^n \wedge (X_{k^+}); \mathbf{Z}) \cong \mathbf{Z}(X_k)$, where i_n denotes the usual generator for $H_n(S^n \wedge (\Delta_{k^+}); \mathbf{Z})$. Since for any ring R, there is a map of rings $\mathbf{Z} \to R$, we get a map of triples from Q to the triple associated with R.

We now consider the functors $\bar{Q}^G$ and Q^G on the category of based topological G-spaces. Note that precisely the same procedure as for Q above turn these into triples on the category of unbased G-simplicial sets; we will call these triples $\bar{Q}^G$ and Q^G as well.

Furthermore, for any ring R and G-simplicial set X, RX becomes a G-simplicial set, and R becomes a triple on the category of G-simplicial sets. As in the non-equivariant case, we obtain maps of triples

$$Q^G \to R \text{ and } Q^G \to R.$$

We now wish to define a cosimplicial space associated to a triple. See [7] for detailed information about cosimplicial spaces. Recall that a cosimplicial object in a category $\mathscr{C}$ is a family of objects $C_n \in ob\ \mathscr{C}$, together with coface maps $\delta^i\colon C_{n-1} \to C_n$, $0 \leqq i \leqq n$, and codegeneracy maps $s^i\colon X^{n+1} \to X^n$, $0 \leqq i \leqq n$, satisfying the cosimplicial identities (dual to the standard simplicial identities. See [7], p. 267). Given a triple on a category $\mathscr{C}$, and $X \in ob\,\mathscr{C}$, we define a cosimplicial object $T.X$ in $\mathscr{C}$, with $T_n X \doteq T^{n+1}(X) = T \circ \ldots \circ T(X)$, $n+1$ factors, and coface and codegeneracy maps defined as in [7], p. 20.

Given a triple T on the category of simplicial sets, the cosimplicial space associated to T is not necessarily fibrant in the sense of [7], p. 275. However, Bousfield and Kan have shown that a cosimplicial space can be made fibrant in a functorial way, preserving the weak homotopy type of the cosimplicial space. Since all of our constructions and proofs depend only on the weak homotopy type of the cosimplicial spaces in question, we adopt the convention that for any non-fibrant cosimplicial space X which appears, X may be replaced by X made fibrant without comment. This convention should not create any confusion. We also note that Bousfield and Kan's functor respects G-actions and fixed point sets, and that if X is a G-cosimplicial space, then the associated fibrant cosimplicial space with G-action has fibrant fixed point sets.

When $\mathscr{C}$ is the category of spaces (= simplicial sets), there is a "total space" functor from the category of cosimplicial spaces to spaces. For its total space, Tot$X$ is actually defined as the inverse limit of a tower of fibrations $\mathrm{Tot}_s X \to \mathrm{Tot}_{s-1} X \to \ldots$. See [7], Ch. X, for a complete description of this construction. When X is an augmented cosimplicial space, we obtain a map $X_{-1} \to \mathrm{Tot}X$. In particular, when T is any triple on the category of spaces, we have the augmented cosimplicial space, $T.X$ and hence a map $X \to \mathrm{Tot}(T.X)$. When $T = R$, defined for a ring R, we call $\mathrm{Tot}(T.X)$ the *R-completion* of X; we will often write, as in [7], $\underset{\sim}{R}X$ for the cosimplicial space $R.X$ and $R_s X$ for $\mathrm{Tot}_s(\underset{\sim}{R}X)$. We admit $s = \infty$, so the R-completion will be denoted $R_\infty X$. More generally, when T is any triple on the category of spaces, we will write $\underset{\sim}{T}X$ for the cosimplicial object $T.X$, $T_s X$ for $\mathrm{Tot}_s \underset{\sim}{T}X$ and $T_\infty X$ for $\mathrm{Tot}(\underset{\sim}{T}X)$. Bousfield and Kan have shown that $\mathbb{Z}_\infty X$ is often well related to X.

Theorem II.1. ([7]) *If X is a space whose components will have nilpotent fundamental groups, and for each component X_0 of X, $\pi_1(X_0)$ acts nilpotently on the homology groups of X_0, then the map $X \to \mathbb{Z}_\infty X$ is a weak homotopy equivalence.*

We now need a preliminary lemma which will be useful in comparing the various completions we will be studying. From this point on, we let $\mathscr{S}$ and $\mathscr{S}_*$

denote the categories of space and pointed spaces respectively. The following lemma is only a restatement of lemma 10.6 of [6], for a general triple T instead of the triple R.

Lemma II.2. *Let B be any functor from $\mathcal{S}_*$ to abelian groups, and let T be a triple defined on $\mathcal{S}_*$. Then $B\underline{T}X$ becomes an augmented cosimplicial abelian group. If the natural inclusion $BX \to BTX$ admits a natural left inverse $BTX \to BX$, then $\pi^s(B\underline{T}X) = 0$ for $s > 0$, and $\pi^0(B\underline{T}X) \cong BX$.*

Proof. We view $B\underline{T}X$ as a cochain complex, and let the operator $H_i\colon BT^iX \to BT^{i-1}X$ denote the above mentioned natural inverse, evaluated at $T^{i-2}X$. Then we have the equations $H_i\delta_0 = Id$, $H_i\delta_j = \delta_{j-1}H_i$, due to the naturality of the inverse. This means that θH_i gives a contracting homotopy for the augmented version of $B\underline{T}X$, which yields the desired result. □

Proposition II.3. *The functor $X \to Q_\infty X$ preserves disjoint unions.*

Proof. The proof follows exactly the proof of Theorem 7.1 of [7, Ch. I]. We indicate briefly what changes are necessary. $\underline{R}(X, a)$ is replaced by $\underline{Q}(X, a) = \{x \in Q^n(X, a)$ *s.t.* $px =$ constant map in $Q^n(\pi_0 X)\}$. The two cochain functors which must be compared are $(\pi_t\underline{Q}(X_a), d)$ and $(\pi_t\underline{Q}(X, a), d)$. The role of T is now played by the triple Q. □

Corollary II.4. *$Q_\infty X$ and $\mathbb{Z}_\infty X$ are naturally equivalent.*

Proof. The preceding proposition permits us to assume that X is connected. We consider the bicosimplicial space $\underline{S}\underline{T}X$, where S is the triple $\mathbb{Z}$, and T is the triple Q. If we consider the "total space" in the S direction, we find that the total space of $\underline{S}\underline{T}X$ is the total space of the cosimplicial space $\mathbb{Z}\underline{T}X$. Since each space T^nX is an infinite loop space, II.1 implies that $T^nX \cong \mathbb{Z}_\infty T^nX$, so that the total space of $\underline{S}\underline{T}X$ is $Q_\infty X$. We now wish to show that $\underline{S}\underline{T}X$ is also equivalent to $\mathbb{Z}_\infty X$. To see this, we will show that $\mathbb{Z}^nX \simeq \operatorname{Tot}\mathbb{Z}^n\underline{Q}X$. From the existence of the homotopy spectral sequence and its associated mapping lemma [7, Ch. IX, §5.2], it will suffice to show that the cosimplicial abelian groups $\pi_i(\mathbb{Z}^n\underline{Q}X)$ have vanishing cohomotopy groups in positive dimensions, and that $\pi^0\pi_i(\mathbb{Z}^n\underline{Q}X) \cong \pi_i(\mathbb{Z}^nX)$. According to Lemma II.2, it will now be enough to produce a natural inverse to the inclusion $\pi_i(\mathbb{Z}^nX) \to \pi_i(\mathbb{Z}^nQX)$. But such a map arises from the natural map $ZQX \to \mathbb{Z}\mathbb{Z}X \to \mathbb{Z}X$, where the map $QX \to \mathbb{Z}X$ is defined above. □

We now move to the categories $\mathcal{S}^G$ and $\mathcal{S}^G_*$ of G-spaces and based G-spaces, and study the completions $\bar{Q}^G_\infty$ and Q^G_∞.

Proposition II.5. *Let X be a G-space.*

(*a*) *$\bar{Q}^G(X)^H \cong Q(X^H)$ for any subgroup $H \subseteq G$*

(*b*) *$\bar{Q}^G(X)$ is the zeroth space of the cosimplicial space $\bar{Q}^G(X)$, and we have the two coface maps δ^0, δ^1: $\bar{Q}^G(X) \to \bar{Q}^G\bar{Q}^G(X)$. The set $\{\alpha \in \pi_0\bar{Q}^GX^H) | \delta^0(\alpha) = \delta^1(\alpha)$. in $\bar{Q}^G\bar{Q}^G(X)^H\}$ is in bijective correspondence via the augmentation $X^H \to (\bar{Q}^GX)^H$ with $\pi_0(X^H)$.*

(*c*) *Viewing Q^GX as the zeroth space of $(\tilde{Q})^G(X)$ we again have the two coface maps δ^0, δ^1: $Q^GX \to Q^GQ^GX$. For any subgroup $H \subseteq G$, the set $\{\alpha \in \pi_0 Q^GX^H) | \delta^0(\alpha) = \delta^1(\alpha)$ in $Q^GQ^G(X)^H\}$ is in bijective correspondence with π_0X^H via the augmentation $X^H \to (Q^GX)^H$.*

(*d*) *If $X^H = \varnothing$, then $Q^G(X)^H = \varnothing$.*

Proof. (a) is immediate from the definitions. For (b), part (a) shows that we must only show that $\{\alpha \in \pi_0 Q(X^H) \mid \delta^0(\alpha) = \delta^1(\alpha)\}$ is in one to one correspondence with $\pi_0(X^H)$ via the augmentation. But $\pi_0(Q(X^H)) \cong Z[\pi_0(X)]$, and $\pi_0 QQ(X^H) \cong Z[Z[\pi_0 X]]$. Thus, the calculation reduces to a standard one for integral homology. For (c), we must describe $\pi_0((Q^G X)^H)$ and $\pi_0((Q^G Q^G X)^H)$ A straightforward application of Tom Dieck's result (1.13) shows that

$$\pi_0(Q^G X)^H \cong \pi_0(Q^H X)^H = \bigoplus_{\mathbf{K} \subseteq \mathbf{H}} \mathbf{Z}[\pi_0(X^K)/N_H(K)]\,.$$

Similarly,

$$\pi_0(Q^G Q^G X)^H \cong \bigoplus_{\mathbf{K} \subseteq \mathbf{H}} \mathbf{Z}\left[\left(\bigoplus_{L \subseteq \mathbf{K}} \mathbf{Z}[\pi_0(X^L)/N_K(L)]\right)\Big/ N_H(K)\right)\right]$$

δ^0 corresponds to the inclusion on the factor $K = H$ of this decomposition. δ^1, on the other hand, corresponds to the inclusion $\oplus_{\mathbf{K} \subseteq \mathbf{H}} i_K$, where i_K: $\mathbf{Z}[\pi_0(X^K)/N_H(K)]) \to \mathbf{Z}[(\oplus_{L \subseteq \mathbf{K}} \mathbf{Z}[\pi_0(X^L)/N_K(L)])/N_H(K)]$ is the inclusion on the factor corresponding to $L = K$. If $\delta^0(\alpha)$ is to equal $\delta^1(\alpha)$, then, we must clearly have that α is in the summand $\mathbf{Z}[\pi_0(X^H)] \subseteq \underline{\pi_0((Q^G X)^H)}$. We are now reduced to case (b), since the natural transformation $\pi_0(\bar{Q}^G \bar{Q}^G (X)^H) \to \pi_0(Q^G Q^G (X)^H)$ is injective. (d) is direct from (c). □

Proposition II.6. *Let X be any G-space. Then $(\bar{Q}^G_\infty X)^H \cong \mathbb{Z}_\infty(X^H)$.*

Proof. Proposition II.5 shows that $((\bar{Q}^G)^K X)^H \cong Q^K(X^H)$, and the equivalence is achieved via a natural inclusion $Q(X^H) \to \bar{Q}^G(X)^H$. Consequently, the cosimplicial space $(\underset{\sim}{\bar{Q}}^G X)^H$ is canonically isomorphic to the cosimplicial space $\underset{\sim}{Q}(X^H)$. Now, Corollary II.4 gives the result. □

We must now relate $Q^G_\infty X$ to $\bar{Q}^G_\infty X$.

Theorem II.7. *The natural inclusion $\bar{Q}^G \to Q^G$ induces a G-equivalence $\bar{Q}^G_\infty X \to Q^G_\infty X$ for any G-space X.*

Proof. Let S be the triple $\bar{Q}^G$ and T be the triple Q^G. We will study the bicosimplicial space $(\underset{\sim}{S}\underset{\sim}{T}X)$. If we "totalize" in the S-direction first, we obtain the cosimplicial space $\mathbb{Z}_\infty \underset{\sim}{T}X^H$. But the functor $TX \to (TX)^H$ is infinite loop space valued, so the natural map $(\underset{\sim}{T}X)^H \to \mathbb{Z}_\infty(\underset{\sim}{T}X)^G$ is a weak equivalence. The conclusion is that $\mathrm{Tot}(\underset{\sim}{S}\underset{\sim}{T}X)^G \cong (T_\infty X)^G$. On the other hand, if we totalize in the T direction first, and can show that the map $(S^k X)^G \to \mathrm{Tot}(S^k \underset{\sim}{T}X)^G$ is an equivalence, we will also have shown that $\mathrm{Tot}(\underset{\sim}{S}\underset{\sim}{T}X)^G \cong (S_\infty X)^G$, which would give the result. We note that for any space X, $S^k X$ is a simplicial monoid whose component set is a group. By II.5(c), it will suffice to prove that for any basepoint in X^H, the map $\pi_n((S^k X)^H) \to \pi_n(\mathrm{Tot}(S^k \underset{\sim}{T}X)^H$ is an isomorphism for all $n \geqq 0$. By the existence of the homotopy spectral sequence, this reduces to verifying that $\pi^s(\pi_n(S^k \underset{\sim}{T}X)^H) = 0$ for $s > 0$, and $\pi^0(\pi_n(S^k \underset{\sim}{T}X)^H) \cong \pi_n(S^k X)^H$. Lemma II.2 then asserts that it is sufficient to find a left inverse to the natural transformation $\pi_n(S^k X^G) \to \pi_n(S^k TX^G)$. The role of B is being played by the abelian group valued functor $X \to \pi_n(S^k X^G)$. To define this, we note that there is a natural map $\theta\colon (TX)^G \to S(X^G) \cong (SX)^G$, defined by $(f\colon S^V \to S^V \wedge X) \to (f^G\colon S^{V^G} \to S^{V^G} \wedge X^G)$. Now, we define a map $\eta\colon (S^k TX)^G \to (S^k X)^G$ to be the composite $(S^k TX)^G \to S^k((TX)^G) \xrightarrow{S^k\theta} S^k(SX^G) \xrightarrow{\sim} S^k(SX)^G \overset{\approx}{\to} S^{k-1}(SX)^G \overset{\approx}{\to} (S^k X)^G$. Our left

inverse to $B \to B \circ T$ is now defined to be $\pi_n((S^kTX)^G) \xrightarrow{\pi_n\eta} \pi_n((S^kX)^G)$. Lemma II.2 and the homotopy spectral sequence now show that the map $(S^kX)^G \to \mathrm{Tot}(S^k\underline{T}X)^G$ is an equivalence, so the map $(S_\infty X)^G \to (T_\infty X)^G$ is an equivalence. For any subgroup $H \subseteq G$, there are natural equivalences of H-spaces $Q^GX \to Q^HX$ and $\bar{Q}^GX \to \bar{Q}^HX$, which show that $Q^G_\infty X$, viewed as an H-space, is H-equivalent to $Q^H_\infty X$. This means that we have shown that the map $(\bar{Q}^G_\infty X)^H \to (Q^G_\infty X)^H$ is a weak equivalence for all $H \subseteq G$, and hence that the G-spaces $\bar{Q}^G_\infty X$ and $Q^G_\infty X$ are weakly G-equivalent. □

We also wish to discuss equivariant R-completion, where $R = \mathbb{Z}/p\mathbb{Z}$, and G is a p-group. The triple R defined above immediately gives a triple on the category of G spaces, where the G-action on RX is obtained from the functoriality of R. We wish to show that the functor $X \to R_\infty X$ from $\mathscr{S}^G_*$ to itself commutes with fixed point sets, i.e. $(R_\infty X)^G = R_\infty(X^G)$. In order to do this, we introduce an auxiliary triple S: $\mathscr{S}^G \to \mathscr{S}^G_*$, defined by $S(X) = X \bigcup_{X^G} R(X^G)$. It is quite clear that $S(X)^G = S(X^G)$, so that $(S_\infty X)^G \cong R_\infty(X^G)$. Of course, $(RX)^G \neq R(X^G)$, and we will need to use a comparison of $S_\infty X$ with $R_\infty X$ to obtain the desired result.

Proposition II.8. *$(R_\infty X)^G$ is weakly equivalent to $R_\infty(X^G)$, if G is a p-group, and $R = \mathbb{Z}/p\mathbb{Z}$.*

Proof. We first study the behaviour on π_0. It is readily verified that $\pi_0((RX)^G)$ can be described as follows. Let $A(X) = \bigoplus_{K \subseteq G} A_K(X)$, where $A_K(X) = R\pi_0(X^K/\bigcup_{K \subseteq L} X^L)$. When $X^K = \varnothing$, we interpret $\pi_0(X^K/\bigcup_{K\subseteq L} X^L)$ as the empty set, and $A_K(X) = \{0\}$. When $K = G$, we interpret $A_G(X)$ as $R\pi_0(X^G)$. We define a homomorphism $A(X) \xrightarrow{\phi_X} R$ by $\phi_X | A_K(X) \equiv 0$ when $K \neq G$, and for $A_G(X)$, $\phi_X(\alpha) = 1$ for each component α of X^G. $\pi_0((RX)^G)$ is now described as the set of all $x \in A(X)$ such that $\phi_X(x) = 1$. This is clearly empty if X^G is, so $(R_\infty X)^G = \varnothing$ if $X^G = \varnothing$. From now on, we assume $X^G \neq \varnothing$. We apply the above analysis to RX, so $\pi_0((RRX)^G)$ is identified with the subset of $x \in A(RX)$ such that $\phi_{RX}(x) = 1$. $\pi_0(\delta^0)$ and $\pi_0(\delta^1)$ provide two functions from $\pi_0((RX)^G)$ into $\pi_0((RRX)^G)$. From the definitions of the cosimplicial space of a triple, it is clear that $\pi_0(\delta^0)$ and $\pi_0(\delta^1)$ may be described as follows. $\pi_0(\delta^0)$ is the restriction to $\{x \in A(X) | \phi_X(x) = 1\}$ of the homomorphism $\bigoplus_{K\subseteq G}\psi_K$: $\bigoplus_{K\subseteq G} A_K(X) \to \bigoplus_{K \subseteq G} A_K(RX)$, where ψ_K: $A_K(X) \to A_K(RX)$ is induced by the inclusion $X \hookrightarrow RX$. $\pi_0(\delta^1)$, on the other hand, is the restriction to $\{x \in A(X) | \phi_X(x) = 1\}$ of the inclusion $\pi_0((RX)^G) \to \pi_0(R((RX)^G)) \cong A_G(RX) \hookrightarrow \bigoplus_{K\subseteq G} A_K(RX) \cong A(X)$. Now, suppose x, $y \in \pi_0((RX)^G)$, and $\pi_0(\delta^0)(x) = \pi_0(\delta^1)(y)$. Then from the description of $\pi_0(\delta^1)$, both x and y must have zero coordinates in all $A_K(X)$ for all K a proper subgroup of G, so we may assume that $x, y \in A_G(X) = R\pi_0(X^G)$. But now, both $\pi_0(\delta^1)$ and $\pi_0(\delta^0)$ applied to such an element belongs to $\pi_0(RR(X^G))$, and it is clear that $x = y$ and $x \in \pi_0(X^G)$, by the component analysis for the R-completion of the space X^G.

It now will suffice to show that for any point $x \in X^G$, the map of cosimplicial groups $\pi_i((\underline{S}X)^G, x) \to \pi_i((\underline{R}X)^G, x)$ induces an isomorphism on cohomology groups. To prove this, we use the standard technique of studying a related bicosimplicial space.

We study the bicosimplicial space $(\underline{S}\underline{R}X)^G$. If we take the total space in the S direction, the result is $R_\infty((\underline{R}X)^G)$. Since each space $(R^nX)^G$ is a simplicial

$\mathbb{Z}/p\mathbb{Z}$-vectorspace, $R_\infty((R^n X)^G) \cong (R^n X)^G$, by the theory of [7]. Consequently, $\mathrm{Tot}(\underset{\sim}{S}\underset{\sim}{R}X)^G \cong (R_\infty X)^G$. We will now consider the total space in the R-direction, and show that it is equivalent to $(S_\infty X)^G \cong R_\infty(X^G)$. To study this second total space, we apply Lemma II.2. Let $B\colon \mathscr{S}^G \to Ab$ be the functor $X \to \pi_n((S^l X)^G)$, and let T be the triple R. We must show that there is a natural left inverse to the canonical map $B \to B \circ T$. In order to construct this inverse, we first show that there is a natural (up to homotopy) left inverse to the map $R(X^G) \to (RX)^G$. Since the category of simplicial $\mathbb{Z}/p\mathbb{Z}$-vectorspaces and linear maps is equivalent to the category of chain complexes over $\mathbb{Z}/p\mathbb{Z}$, it will suffice to produce an inverse to the inclusion $C_*(X^G) \to C_*(X)^G$, where C_* denotes the complex of normalized chains on X. In every dimension n, we have a splitting $C_*(X_n)^G \cong \bigoplus_{K\subseteq G} C_*(X_n^K - \bigcup_{L\supset K} X_n^L)$. Moreover, the boundary map on $C_*(X_n^K - \bigcup_{L\supset K} X_n^L)$ has values in $\bigoplus_{K\subseteq K'} C_*(X_n^K - \bigcup_{L\supset K'} X_n^L)$, and is equivariant. We wish to show that $C_*(X^G)$ is naturally a direct factor in $C_*(X)^G$. This amounts to showing that the $C_*(X^G)$ coordinate in ∂x, where $x \in \bigoplus_{K\nsubseteq G} C_*(X^K - \bigcup_{L\supset K} X^K)^G$, is zero. But every element in $C_*(X^K - \bigcup_{L\subseteq K} X^K)^G$ is of the form $\sum_{x\in\mathcal{O}} X$, where $\mathcal{O}$ is a non-trivial orbit of X. Since ∂ is G equivariant, the $C_*(X^G)$-coordinates of ∂X and $\partial g X$ are identical. Therefore, the $C_*(X^G)$-coordinate of $\partial(\sum_{x\in 0} x)$ is $|\mathcal{O}|\cdot y$, where y is the $C_*(X^G)$-coordinate of X. But, G is a p-group, so $|\mathcal{O}|\cdot y = 0$, which gives the splitting $C_*(X)^G \to C_*(X^G)$. Using the equivalence of the category of simplicial $\mathbb{Z}/p\mathbb{Z}$-vectorspaces with the category of $\mathbb{Z}/p\mathbb{Z}$-chain complexes, we have produced a homotopy class of maps $R(X)^G \xrightarrow{\theta(X)} R(X^G)$, natural in X. The required inverse for $B \to B \circ T$ is the composite

$$\pi_n((S^l(RX))^G) \cong \pi_n(S^l((RX)^G)) \xrightarrow{\pi_n(S^l(\theta(X)))}$$

$$\pi_n(S^l(R(X^G))) \overset{\approx}{\to} \pi_n(S^l S(X^G)) \to \pi_n((S^{l+1}X)^G) \to \pi_n((S^l X)^G)\,.$$

The last map uses the triple structure of S. According to Lemma II.2, we have now shown that the map $S^l X^G \to \mathrm{Tot}(S^l \underset{\sim}{R}X)^G$ is a weak equivalence, and hence $\mathrm{Tot}(S\underset{\sim}{R}X)^G \cong \mathrm{Tot}(\underset{\sim}{S}X)^G \cong R_\infty(X^G)$, which was to be shown.

We need, finally, to make some technical adjustments to cover the case of non-connected fixed point sets. We first work in the non-equivariant setting. Let $\underset{\sim}{\mathbf{N}}$ denote the cosimplicial set $\mathbf{N}_n = \{0, 1, \ldots, n\}$, with coface and codegeneracy maps the various order-preserving maps between these sets. If X is any space, we may form the cosimplicial space $\underset{\sim}{\mathbf{N}}X$, defined by $\mathbf{N}_n X = \{0, 1, \ldots, n\} * X$, where $*$ denotes join. Notice that $\mathbf{N}_n X$ is connected for each n (in fact, it is a suspension). Note also that there are inclusions $\theta_n\colon X \to \{0, 1, \ldots, n\} * X$, whose image consists of those points whose "join coordinate" is 1, and that these are compatible with all coface and codegeneracy maps. Hence we have a map $\theta\colon X \to \mathrm{Tot}\,\underset{\sim}{\mathbf{N}}X$. If T is any triple on the category of spaces, we may form the bicosimplicial space $\underset{\sim}{T}\underset{\sim}{\mathbf{N}}X$, and attempt to analyze its total space.

Proposition II.9. *Let T denote either the triple associated to a ring R or the triple Q. Then the natural map $T_\infty X \to \mathrm{Tot}\,T\underset{\sim}{\mathbf{N}}X$ is a weak equivalence.*

Proof. This is proved in Appendix I. □

Remark. The $\underset{\sim}{\mathbf{N}}X$ construction has been considered by M. Barratt [5] and later M.J. Hopkins [12] in his Oxford thesis. It is also the basis of T. Goodwillie's theory of "analytic functors".

We note that if T is any triple on the category of G-spaces, we may form the cosimplicial G-space $\underset{\sim}{T}\underset{\sim}{N}X$ in the evident way.

Corollary II.10. *Let T be any of the triples Q^G or $\bar{Q}^G$. Then the natural map $T_\infty X \to \operatorname{Tot} \underset{\sim}{T}\underset{\sim}{N}X$ is a weak G-equivalence.*

Proof. The proof is carried out exactly as is the proof of Proposition II.9 in the Appendix. □

Corollary II.11. *Let $R = F_p$, and let $R_\infty \underset{\sim}{Q}^G\underset{\sim}{N}X$ denote the bicosimplicial G-space obtained by applying R_∞ levelwise to $\underset{\sim}{Q}^G\underset{\sim}{N}X$, for any G-space X. Then the natural map*

$$R_\infty X \to \operatorname{Tot} R_\infty \underset{\sim}{Q}^G\underset{\sim}{N}X$$

is a weak G-equivalence, where Tot *denotes "totalization" taken in both directions.*

Proof. $\operatorname{Tot} R_\infty \underset{\sim}{Q}^G\underset{\sim}{N}X$ is in fact the total space of a tricosimplicial space $R\underset{\sim}{Q}^G\underset{\sim}{N}X$, which has the space $R^{k+1}(Q^G)^{l+1}(\{0, 1, \ldots, m\} * X)$ in codimension (k, l, m). By II.9, if we totalize in the m variable, we are left with the bicosimplicial space $\underset{\sim}{R}\underset{\sim}{Q}^G(X)$, and we must prove that the natural map $R_\infty X \to \operatorname{Tot} \underset{\sim}{R}\underset{\sim}{Q}^G(X)$ is a weak G-equivalence. But this is proved in a manner entirely analogous to the proofs of II.4 and II.7. The key ingredient here is the existence of a splitting of the inclusion $RX \to RQ^GX$, via the composite $RQ^GX \to R\,RX \to RX$, where $Q^GX \to RX$ is the map of triples given above. □

III. Consequences of the Segal Conjecture

We first recall a generalization of the Segal Conjecture due to Adams, Haeberly, Jackowski, and May.

Theorem III.1 [2]. *Let G be a finite group, let H be a normal subgroup of G, and let X be a finite G-complex. Let EG/H denote a contractible space on which G/H acts freely, and let $EG/H^{(k)}$ denote the k-skeleton of EG/H. Then the pro-group $\{\pi_n(F(EG/H^{(k)+}, Q^G(X))^G)\}$ is pro-isomorphic to the pro-group $\{\pi_n(Q^G(X)^G)/I_H^k\pi_n(Q^G(X)^G)\}$, where $I_H \subseteq A(G)$ denotes the kernel of the restriction map $A(G) \to A(H)$*

Corollary III.2. *Let G be a p-group, H a normal subgroup of G, and let X be a finite G-complex such that $\pi_n^H(X)$ is a finite p-group for all n and $H \subseteq G$. Then the map of pro-groups $\{\pi_n^G(X)\} \to \{\pi_n(F(EG/H^{(k)+}, Q^G(X))^G)\}$ is a proisomorphism. (The domain is simply a constant pro-group. π_n^G will denote G-equivariant stable homotopy.)*

Proof. It will clearly suffice to prove that for all $l \in \mathbb{Z}$, there exists an integer K so that $I_H^k \subseteq p^l \cdot I_H$. This is a straight forward exercise with the Burnside ring. □

Let G be a p-group, H a normal subgroup and X a G-complex. Then the spaces $(Q^GX)^G$ and $F(EG/H^+, Q^GX)^G$ are infinite loop spaces, and the map ε: $(Q^GX)^G \to F(EG/H^+, Q^GX)^G$ is an infinite loop map. Consequently, if we pick a basepoint $*$ in X^G, then the homotopy fibre of ε at $*$, $\mathscr{F}(\varepsilon, *)$ is an infinite loop space.

Proposition III.3. *Let G, and H be as above; and let X be a finite-dimensional G-complex with connected fixed point sets. Then $F(EG/H^+, Q^GX)^G$ is connected, and $\pi_i(\mathscr{F}(\varepsilon, *))$ is uniquely p-divisible for each i.*

Proof. Filtering by skeleta, and using change of groups results as in [9], we may reduce to proving the result for $Q^G(\bigvee_{\alpha\in A} S^i_\alpha)$, where $i \geqq 1$, A is an indexing set, and S^i_α denotes a copy of the i-sphere with trivial G-action. Of course, $Q^G(\bigvee_{\alpha\in A} S^i_\alpha)$ is G-equivalent to the weak product $\prod_{\alpha\in A} Q^G(S^i_\alpha)$, and thus the pro-group $\{\pi_n(F(EG/H^{(k)+}, Q^G(\bigvee_{\alpha\in A} S^i_\alpha))^G)\}$ is isomorphic to the infinite *sum* $\bigoplus_{\alpha\in A}\{\pi_n(F(EG/H^{(k)+}, Q^G(S^i_\alpha))^G)\}$. By Theorem III.1 this is isomorphic to the pro-group $\bigoplus_{\alpha\in A}\{\pi^G_n(S^i)/I^k_H\pi^G_n(S^i)\}$. Note that since all the maps in this inverse system are surjective, $\lim^1$ vanishes for each n. Since $i \geqq 1$, the system $\bigoplus_{\alpha\in A}\{\pi^G_0(S^i)/I^k_H\pi^G_0(S^i)\}$ is identically zero, and since $\lim^1$ vanishes, we find that $\pi_0(F(EG/H^+, Q^G(\bigvee_{\alpha\in A} S^i))^G) = 0$. We must therefore only check that the map $\pi^G_n(\bigvee_{\alpha\in A} S^i_\alpha) \to \lim_k \pi^G_n(\bigvee_{\alpha\in A} S^i_\alpha)/I^k_H\pi^G_n(\bigvee_{\alpha\in A} S^i_\alpha)$ has uniquely p-divisible kernel and cokernel. Let π denote $\pi^G_n(\bigvee_{\alpha\in A} S^i_\alpha)$, and let $I = I_H$. Then we have a short exact sequence of pro-groups

$$\text{(A)} \qquad 0 \to \{I\pi/I^k\pi\} \to \{\pi/I^k\pi\} \to \{\pi/I\pi\} \to 0$$

and a correspondence short exact sequence

$$\text{(B)} \qquad 0 \to I\pi \to \pi \to \pi/I\pi \to 0\,,$$

which maps to sequence (A). The right hand progroup in (A) is a constant one, so to prove the result, it will suffice to prove that the map $I\pi \to \varprojlim I\pi/I^k\pi$ has uniquely p-divisible kernel and cokernel. But it is easy to show that the progroups $\{I\pi/I^k\pi\}$ and $\{I\pi/p^k\cdot I\pi\}$ are pro-isomorphic, hence it will suffice to show that the map $I\pi \to \varprojlim I\pi/p^k\cdot I\pi$ has uniquely p-divisible kernel and cokernel. This is equivalent to the assertion that $I\pi \to (I\pi)^\wedge_p$ has uniquely p-divisible kernel and cokernel.

Recall from [7, Ch. 6] that we have a six term long exact sequence $0 \to \mathrm{Hom}(\mathbf{Z}/_{p\infty}, I\pi) \to \mathrm{Hom}\left(\mathbf{Z}\frac{1}{p}, I\pi\right) \to \mathrm{Hom}(\mathbf{Z}, I\pi) \to \mathrm{Ext}(\mathbf{Z}/_{p\infty}, I\pi) \to \mathrm{Ext}\left(\mathbf{Z}\frac{1}{p}, I\pi\right) \to \mathrm{Ext}(\mathbf{Z}, I\pi) \to 0$, and that $\mathrm{Ext}(\mathbf{Z}/_{p\infty}, I\pi)$ is canonically identified with $(I\pi)^\wedge_p$. Moreover, $\mathrm{Hom}(\mathbf{Z}, I\pi) \simeq I\pi$, and under these identifications, the map in the diagram is identified with the inclusion $I\pi \to (I\pi)^\wedge_p$. Thus, it suffices to show that Cokernel $\left(\mathrm{Hom}(\mathbf{Z}/_{p\infty}, I\pi) \to \mathrm{Hom}\left(\mathbf{Z}\frac{1}{p}, I\pi\right)\right)$ and kernel $\left(\mathrm{Ext}\left(\mathbf{Z}\frac{1}{p}, I\pi\right) \to \mathrm{Ext}(\mathbf{Z}, I\pi)\right)$ are uniquely p-divisible. Since $I\pi$ is isomorphic to an infinite sum of the same finitely generated abelian group, it is clear that $\bigcap_k p^kI\pi = 0$, so $\mathrm{Hom}\left(\mathbf{Z}\frac{1}{p}, I\pi\right) = 0$, so the first group actually vanishes. For the second group, $\mathrm{Ext}(\mathbf{Z}, I\pi) = 0$, and $\mathrm{Ext}\left(\mathbf{Z}\frac{1}{p}, I\pi\right)$ is evidently uniquely p-divisible, so we have the result. □

Let R be the ring $\mathbb{F}_p$. We wish to find a useful model for $R_\infty QX$, where X is a connected space. Let M_{p^i} denote the $\mathbb{Z}/p^i\mathbb{Z}$-Moore space, with $H_1(M_{p^i}) \cong \mathbb{Z}/p^i\mathbb{Z}$. Then we may construct maps $M_{p^i} \to M_{p^{i-1}}$, inducing reduction mod p^{i-1}, and can arrange a tower of fibrations

$$\ldots\ \Omega Q(X \wedge M_{p^i}) \to \Omega Q(X \wedge M_{p^{i-1}}) \to \ldots$$

by converting maps into fibrations in the standard way. Let $Z_i = \Omega Q(X \wedge M_{p^i})$.

Proposition III.4. *The inverse limit of the tower of fibrations $\{Z_i\}$ is weakly equivalent to $R_\infty QX$.*

Proof. It is easy to check that Z_i is an R-tower (see [7], p. 87). This gives the result. □

Corollary III.5. *Let X be any based G-complex with connected fixed point sets. Then $R_\infty Q^G X$ is weakly equivalent to $\varprojlim_i \Omega Q^G(X \wedge M_{p^i})$.*

Proof. This is straightforward from the preceding Proposition. □

Theorem III.6. *Let G be a p-group and H a normal subgroup. Let X be any based, finite dimensional G-complex with connected fixed point sets. Then the natural map $R_\infty Q^G X \to F(EG/H^+, R_\infty Q^G X)$ is a weak G-equivalence.*

Proof. One first readily reduces to the case $X = \bigvee_{\alpha\in A} S^i_\alpha$, where the S^i_α's are equipped with trivial G-action, and A is some indexing set. Using Corollary III.5, we reduce to the case $X = \bigvee_{\alpha\in A} S^i_\alpha \wedge M_{p^j}$. By Corollary III.2, the pro-group $\{\pi_n(F(EG/H^{(k)+}, R_\infty Q^G(S^i_\alpha \wedge M_{p^j}))^G)\}$ is pro-isomorphic to the constant pro-group $\{\pi^G_n(S^i_\alpha \wedge M_{p^j})\}$. Now, the pro-group $\{\pi_n(F(EG/H^{(k)+}, Q^G(\bigvee_{\alpha\in A} S^i_\alpha \wedge M_{p^j}))^G)\}$ is pro-isomorphic to the infinite direct *sum* of pro-groups $\bigoplus_{\alpha\in A}\{\pi_n(F(EG/H^{(k)+}, Q^G(S^i_\alpha \wedge M_{p^j}))^G)\}$, which in turn is pro-isomorphic to the constant pro-group $\{\bigoplus_{\alpha\in A}\pi^G_n(S^i_\alpha \wedge M_{p^j})\}$. This is the desired result. □

Proposition III.7. *Let G be a p-group, $H \lhd G$ a normal subgroup, and X a finite dimensional G-complex. Let $* \in X^G$ be a basepoint, K any subgroup of G/H, and let $\mathscr{F}(X, \varepsilon^K, *)$ denote the homotopy fibre of the map ε^K: $Q^{G/H}(X)^K \xrightarrow{\varepsilon^K} F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H X))^K$ at $*$. Then we have*
(A) $\pi_i(\mathscr{F}(X, \varepsilon^K, *))$ is uniquely p-divisible for each i (including $i = 0$)
(B) The map $R_\infty Q^{G/H}(E_G H^+ \wedge_H X) \to F(EG/H^+, R_\infty Q^{G/H}(E_G H^+ \wedge_H X))$ is a weak G/H-equivalence.

Proof. (A) We first reduce to the case $K = G/H$; this reduction is immediate using the fact that, K-equivariantly, $EG/H \cong EK/H$ and $Q^{G/H}(E_G H^+ \wedge_H X) = Q^{K/H}(E_K H^+ \wedge_K X)$. The second fact requires Proposition I.2. Proposition III.3 now asserts that the homotopy fibre of the map

$$Q^G(X)^G \xrightarrow{\varepsilon^G} F(EG/H^+, Q^G(X))^G$$

is connected, and has uniquely p-divisible homotopy groups. Note further that since H acts trivially on EG/H^+, $F(EG/H^+, Q^G(X))^G \cong F(EG/H^+, Q^G(X)^H)^{G/H}$ According to Proposition I.14, $Q^{G/H}(E_G H^+ \wedge_H X)$ is G/H-equivariantly a direct factor of $Q^G(X)^H$; consequently, $Q^{G/H}(E_G H^+ \wedge_H X)^{G/H}$ is a direct factor in $Q^G(X)^G$ and $F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H X))^{G/H}$. One sees readily that the product splitting is preserved by the map ε^G, and so $\mathscr{F}(X, \varepsilon^G, *)$ is a direct factor of the homotopy fibre ϕ of the map $Q^G(X)^G \to F(EG/H^+, Q^G(X))^G$ at $*$. Since ϕ is has uniquely p-divisible homotopy groups. So does $\mathscr{F}(X, \varepsilon^G, *)$. Similarly, $F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H X))^K$ is connected. The proof of (B) proceeds along the same lines, and we omit it. □

We wish to extend these results to the case where G/H is a p-group, but where G itself is not necessarily a p-group. We first record an elementary lemma.

Lemma III.8. *Let A be an abelian group, and let n be an integer such that $(n, p) = 1$. Suppose that there is a commutative diagram*

$$\begin{tikzcd} A \arrow[rr, "xn"] \arrow[dr] & & A \\ & B \arrow[ur] & \end{tikzcd}$$

where B is uniquely p-divisible. Then A is uniquely p-divisible.

Proof. The p-torsion subgroup A_p is trivial, because $A_p \xrightarrow{xn} A_p$ is an isomorphism, and $B_p = 0$ since B is uniquely p-divisible. Thus, $A \xrightarrow{xp} A$ is injective. Suppose $x \in A$. We wish to show that x is p^l-divisible. Select k so that $n^k \equiv 1 \pmod{p^l}$, i.e. $n^k = 1 + Np^l$. $n^k x$ is in the image of the homomorphism $B \to A$, so $n^k x$ is p^l-divisible, say $n^k x = p^l y$. Then we have $(Np^l + 1)x = p^l y$, or $x = p^l(y - Nx)$, so x is p^l-divisible. Hence $A \xrightarrow{xp^l} A$ is surjective, which gives the result. □

Proposition III.9. *Let G be a finite group (not necessarily a p-group), and let $H \trianglelefteq G$ be a normal subgroup of index p^i. Let X be a finite dimensional G-complex. If $* \in X^G$, let $\mathscr{F}(X, \varepsilon^K, *)$ denote the homotopy fibre of the map $Q^{G/H}(E_G H^+ \wedge_H X)^K \to F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H X))^K$ at $*$, for $K \subseteq G/H$. Then we have*
(A) $\pi_i(\mathscr{F}(X, \varepsilon^K, *))$ is uniquely p-divisible for each $i > 0$.
(B) The map $R_\infty Q^{G/H}(E_G H^+ \wedge_H X) \to F(EG/H^+, R_\infty Q^{G/H}(E_G H^+ \wedge_H X))$ is a weak G/H-equivalence.

Proof. (A) Let G_p denote the p-Sylow subgroup of G, $H_p = H \cap G_p$. Then, using Adams' transfer [1], we have a diagram

$$\begin{tikzcd} Q^{G/H}\left(E_G H^+ \underset{H}{\wedge} X\right) \arrow[rr, "x|G/G_p|"] \arrow[dr] & & Q^{G/H}\left(E_G H^+ \underset{H}{\wedge} X\right) \\ & Q^{G/H}\left(E_{G_p} H_p^+ \underset{H_p}{\wedge} X\right) \arrow[ur, "\tau"'] & \end{tikzcd}$$

If we let $\mathscr{F}'$ denote the homotopy fibre of the map $(\varepsilon')^K$: $Q^{G/H}(E_{G_p} H_p^+ \wedge_H X)^K \to F(EG/H^+, Q^{G/H}(E_{G_p} H_p^+ \wedge_{H_p} X))^K$, we obtain a diagram

$$\begin{tikzcd} \mathscr{F}(X, \varepsilon^K, *) \arrow[rr, "x|G/G_p|"] \arrow[dr] & & \mathscr{F}(X, \varepsilon^K, *) \\ & \mathscr{F}' \arrow[ur] & \end{tikzcd}$$

and the induced diagram of homotopy groups

$$\begin{tikzcd} \pi_i\mathscr{F}(X, \varepsilon^K, *) \arrow[rr, "x|G/G_p|"] \arrow[dr] & & \pi_i\mathscr{F}(X, \varepsilon^K, *) \\ & \pi_i\mathscr{F}' \arrow[ur] & \end{tikzcd}$$

The preceding lemma, together with III.7 (A) gives the result. The proof of (B) is entirely analogous and we omit it. □

We now wish to prove the following propositions.

Proposition III.10. *Let G be a finite group, $H \lhd G$ a normal subgroup, with $|G/H| = p^i$. Further, let X be a finite dimensional G-complex, with connected fixed point sets. Let $X^{(k)}$ denote the skeletal filtration of X, and let $\tilde{C}_*$ denote the chain complex with chain groups $\tilde{C}_k = \pi_k^S(EH^+ \wedge_H (X^{(k)}/X^{(k-1)}))$, and boundary maps induced by the boundary maps in the cofibration sequence $EH^+ \wedge_H (X^{(k)}/X^{(k-1)}) \to EH^+ \wedge_H (X^{(k+1)}/X^{(k-1)}) \to EH^+ \wedge_H (X^{(k+1)}/X^{(k)})$ If $H_i(\tilde{C}_*) = 0$ for $i \leqq k$, then the map $\pi_i F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H X))^{G/H} \to \pi_i F(EG/H^+, R_\infty Q^{G/H}(E_G H^+ \wedge_H X))^{G/H}$ is an isomorphism for $i < k$ and a surjection for $i = k$.*

The proof proceeds in several steps and requires some preliminary lemmas. Before we launch into the proof, we note the corollary which will be of use to us.

Corollary III.11. *Let G be a p-group, and X a finite dimensional G-complex. Suppose further that $\pi_i(X) = 0$ for $i \leqq k$, and let $\Gamma \subseteq \sum_n$ be any subgroup. Then the map $\pi_i(F(EG^+, Q^G(E_{G\times\Gamma}\Gamma^+ \wedge_\Gamma X^{\wedge n}))^G) \to \pi_i(F(EG^+, R_\infty Q^G(E_{G\times\Gamma}\Gamma^+ {}_\Gamma X^n))^G)$ is an isomorphism for $i \leqq nk + n - 2$, and a surjection for $i = nk + n - 1$.*

Proof. According to III.10, we must show that the complex with chain groups $\pi_*^S(E\Gamma^+ \wedge_\Gamma ((X^{\wedge n})^{(k)}/(X^{\wedge n})^{(k-1)}))$ has homology vanishing through dimension $nk + n - 1$. Note that this condition does not involve the G-action, so we may view X as a CW-complex with no G-action. Since $\pi_i(X) = 0$ for $i \leqq k$, there is a cellular equivalence e: $X \to \bar{X}$, where $\bar{X}$ has cells in only in dimension $k + 1$ and higher. Further, there is a cellular inverse to e, and these two maps induce an equivalence of chain complexes $\{\pi_*^S(E\Gamma^+ \wedge_\Gamma ((X^{\wedge n})^{(k)}/(X^{\wedge n})^{(k-1)}))\} \to \{\pi_*^S(E\Gamma^+ \wedge_\Gamma ((\bar{X}^{\wedge n})^{(k)}/(\bar{X}^{\wedge n})^{k-1}))\}$. Since the first cell of $\bar{X}$ appears in dimension $k + 1$, the first cell of $\bar{X}^{\wedge n}$ appears in dimension $nk + n$, so $\pi_*^S(E\Gamma^+ \wedge_\Gamma ((\bar{X}^{\wedge n})^{(i)}/(\bar{X}^{\wedge n})^{(i-1)})) = 0$ for $i \leqq nk + n - 1$. Therefore, $H_i(\tilde{C}_*) = 0$ for $i \leqq nk + n - 1$, which gives the result. □

In order to prove III.10, we will need a preliminary lemma about chain complexes over the Burnside ring $A(G)$.

Lemma III.12. *Let G be a p-group, let C be an $A(G)$ chain complex, and suppose $C_n = 0$ for $n < 0$. Suppose further that for some j, $p^j[G]H_i(C) = 0$ for all $i \leqq N$, where $[G] \in A(G)$ denotes the element represented by a free G-set on one generator. Then the map $H_i((C)_I^\wedge) \to H_i((C)_p^\wedge)$ is an isomorphism for $i \leqq N$, where ${}_I^\wedge$ and ${}_p^\wedge$* denote $I(G)$-adic and p-adic completion, respectively.

Proof. The proof is given in Appendix II. □

Proof of III.10. We first define chain complexes $\tilde{B}(X)$, $\tilde{B}^f(X)$, and $\tilde{B}^0(X)$ of graded $A(G/H)$-modules, as follows. We let

$$\tilde{B}_k(X) = \pi_*^{G/H}(E_G H^+ \underset{H}{\wedge} (X^{(k)}/X^{(k-1)}))$$

$$\tilde{B}_k^f(X) = \pi_*^{G/H}(EG/H^+ \wedge (E_G H^+ \underset{H}{\wedge} (X^{(k)}/X^{(k-1)})))$$

$$\cong \pi_*^S(EG^+ \underset{G}{\wedge} (X^{(k)}/X^{(k-1)}))$$

$$\tilde{B}_k^0(X) \cong \pi_*^S(EH^+ \underset{H}{\wedge} (X^{(k)}/X^{(k-1)}))$$

and define ∂ to be the boundary map associated to the cofibration sequences

$$E_G H^+ \underset{H}{\wedge} (X^{(k)}/X^{(k-1)}) \to E_G H^+ \underset{H}{\wedge} (X^{(k+1)}/X^{(k-1)}) \to E_G H^+ \underset{H}{\wedge} (X^{(k+1)}/X^{(k)})$$

$$EG^+ \underset{G}{\wedge} (X^{(k)}/X^{(k-1)}) \to EG^+ \underset{G}{\wedge} (X^{(k+1)}/X^{(k-1)}) \to EG^+ \underset{G}{\wedge} (X^{(k+1)}/X^{(k)})$$

$$EH^+ \underset{H}{\wedge} (X^{(k)}/X^{(k-1)}) \to EH^+ \underset{H}{\wedge} (X^{(k+1)}/X^{(k-1)}) \to EH^+ \underset{H}{\wedge} (X^{(k+1)}/X^{(k)}) .$$

We first remark that $H_i(\tilde{B}^0(X)) = 0$ for $i \leqq N$ by hypothesis. By the naturality of Tom Dieck's splitting of stable equivariant homotopy theory, $\tilde{B}^f_*(X)$ is a $\mathbb{Z}$-direct summand in $\tilde{B}_*(X)$, and $[G/H]\cdot \tilde{B}_*(X) \subseteq \tilde{B}^f_*(X)$, where $[G/H] \in A(G/H)$ is the element represented by a free G/H-set on one generator. Also, the transfer map $\sum^\infty EG^+ \wedge_G X \to \sum^\infty EH^+ \wedge_H X$ induces a chain map $\tilde{B}^f_*(X) \xrightarrow{\tau} \tilde{B}^0_*(X)$, so that the composite $\tilde{B}^f_*(X) \xrightarrow{\tau} \tilde{B}^0_*(X) \xrightarrow{\alpha} \tilde{B}^f_*(X)$ is multiplication by $|G/H|$, where α is induced by the natural map $EH^+ \wedge_H X \to EG^+ \wedge_G X$. We conclude that $|G/H|$ annihilates $H_i(\tilde{B}^f_*(X))$ for $i \leqq N$ and hence that $|G/H|\cdot[G/H]$ annihilates $H_i(\tilde{B}_*(X))$ for $i \leqq N$. By Lemma III.12, we find that the map

$$\tilde{B}_*(X)\hat{}_{I(G/H)} \to \tilde{B}_*(X)\hat{}_p$$

induces an isomorphism in dimensions less than or equal to N.

We now define new chain complexes D_* and E_*, where $D_k = \pi_*(F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H (X^{(k)}/X^{(k-1)})))^{G/H})$ and $E_k = \pi_*(F(EG/H^+, R_\infty Q^{G/H}(E_G H^+ \wedge_H (X^{(k)}/X^{(k-1)})))^{G/H})$ D_* and E_* have the following properties

(1) There is a natural map i: $D_* \to E_*$, arising from the map $Q^{G/H} \to R_\infty Q^{G/H}$.

(2) $D_* \cong \tilde{B}_*(X)\hat{}_{I(G/H)}$ and $E_* \cong \tilde{B}_*(X)\hat{}_p$. This follows from III.1 and III.9 (B). Furthermore, the map $\tilde{B}_*(X)\hat{}_{I(G/H)} \to \tilde{B}_*(X)\hat{}_p$ is compatible with i above.

(3) D_* and E_* are E_2-terms for spectral sequences converging to $\pi_*(F(EG/H^+, Q^{G/H}(E_G H^+ \wedge_H X))^{G/H})$ and $\pi_*(F(EG/H^+, R_\infty Q^{G/H}(E_G H^+ \wedge_H X))^{G/H})$. The spectral sequence is simply the one based on the filtration $\{X^{(k)}\}$. Since the map $\tilde{B}_*(X)\hat{}_{I(G/H)} \to \tilde{B}_*(X)\hat{}_p$ induces an isomorphism in dimensions less than or equal to N, the map i: $D_* \to E_*$ induces an isomorphism on homology in dimensions less than or equal to N. A standard spectral sequence argument now shows the result. □

IV. The universal covering space

Let X be any based G-complex, with $* \in X^G$ as a basepoint. $\tilde{X}$, the universal cover of X^G, as a set is $\{\phi: I \to X \mid \phi(0) = *\}/\approx$, where $\phi \approx \psi$ if and only if $\phi(1) = \psi(1)$ and ϕ is homotopic to ψ rel $[0, 1]$. It is topologized in such a way that the map $\phi \to \phi(1)$ becomes a local homomorphism. The construction is completely natural, so $\tilde{X}$ becomes a G-complex in a natural way. We wish to describe the inverse image of the fixed point set X^G in $\tilde{X}$.

Since X is a based G-complex, G acts on $\pi_1(X)$. Of course, $\pi_1(X)$ acts freely on $\tilde{X}$ via deck transformations and taken together the G and $\pi_1(X)$ actions give a $G \tilde{\times} \pi_1(X)$ action on $\tilde{X}$, where $G \tilde{\times} \pi_1(X)$ denotes the semidirect product defined by

the G-action on $\pi_1(X)$. Now suppose $x \in X^G$, and let $\tilde{x} \in \tilde{X}$ be a lifting of x. Then, for each $g \in G$, there exists a unique $\gamma(g) \in \pi_1(X)$ so that $g\tilde{x} = \gamma(g)\tilde{x}$, since $p(g\tilde{x}) = p(\tilde{x})$, where $p\colon \tilde{X} \to X$ is the natural projection. The conclusion is that the homomorphism $(G\tilde{x}\pi_1(X))_{\tilde{x}} \to G$ is surjective. On the other hand, $(G\tilde{x}\pi_1(X))_{\tilde{x}} \cap \pi_1(X) = \{e\}$, since $\pi_1(X)$ acts freely on $\tilde{X}$. The choice of lifting $\tilde{x}$ alters the subgroup $((G\tilde{x}\pi_1(X))_{\tilde{x}}$ only by conjugation by an element of $\pi_1(X)$. As in the proof of Proposition I.3, we see that the $\pi_1(X)$-conjugacy classes of such subgroups are in bijective correspondence with the $G\tilde{x}\pi_1(X)$-conjugacy classes of such subgroups. Given $\tilde{x} \in p^{-1}(X^G)$, let $K(\tilde{x}) \subseteq G\tilde{x}\pi_1(X)$ denote the stabilizer of $\tilde{x}$. Given a point $x \in X^G$, let $C(x)$ denote the conjugacy class of $K(\tilde{x})$ for any lifting $\tilde{x}$ of x to $\tilde{X}$. For a conjugacy class C of subgroups of $G\tilde{x}\pi_1(X)$, which surject to G, and which intersect $\pi_1(X)$ trivially, we set $X^G(C) = \{x \in X^G \mid C(x) = C\}$. We now have the following descriptions of X^G and $p^{-1}X^G$.

Proposition IV.1.
(a) $X^G = \coprod_C X^G(C)$.
(b) $p^{-1}X^G = \coprod_K \tilde{X}^K$
K ranges over all subgroups $K \subseteq G\tilde{x}\pi_1(X)$ so that $K \cap \pi_1(X) = \{e\}$, and which surject to G. C ranges over all conjugacy classes of such subgroups. We further obtain the description $X^G(C) = \tilde{X}^K / N_{G\tilde{x}\pi_1(X)}(K) \cap \pi_1(X)$. The group $N_{G\tilde{x}\pi_1(X)}(K) \cap \pi_1(X)$ of course acts freely on $\tilde{X}^K$.

We now will describe the fundamental groups of the various components $\tilde{X}^K$. Let $C(K)$ denote the conjugacy class of K.

Proposition IV.2. *Let $\tilde{x} \in \tilde{X}^K$, and let $x = p(\tilde{x}) \in X^G(C(K))$. Then $\pi_1(\tilde{X}^K, \tilde{x}) \cong$* $\operatorname{Ker}(\pi_1(X^G(C(K)), \tilde{x}) \to \pi_1(X, x)$. $\square$

Proof. This is standard covering space theory.

We now define a "partial completion" of X in the sense of [7], which will be useful later. Let $T = \mathbb{F}_p$ or Q^G. We define the "total space T-completion" of X to be the balanced product $E\pi_1(X) \times_{\pi_1(X)} (T_\infty \tilde{X})$. Note that $G\tilde{x}\pi_1(X)$ acts on $T_\infty \tilde{X}$ since T_∞ is a functorial construction on the category of unbased G-spaces. We write $T_\infty^{\mathrm{tot}} X$ for the completion. We describe $(T_\infty^{\mathrm{tot}} X)^G$ in non-equivariant terms. Let $S = \mathbb{F}_p$ or $\mathbb{Z}$, viewed as triples on the category of spaces. If X is a space, $* \in X$, and $N \subseteq \pi_1(X, *)$ is a normal subgroup, we define the partial S-completion of X with respect to N to be the space $E(\pi_1(X)/N) \times_{\pi_1(X)/N} (S_\infty X(N))$, where $X(N)$ denotes the cover of X corresponding to the subgroup N. We write $S_\infty(X, N)$ for this construction. Let G be a p-group.

Proposition IV.3. *Let $\{X_i^G\}_{i \in I}$ be the connected components of X^G, and let $p_i \in X_i^G$ be a basepoint. Let $N_i \subseteq \pi_1(X_i^G, p_i)$ denote the kernel of the map $\pi_1(X_i^G, p_i) \to \pi_1(X, p_i)$. Then $((\mathbb{F}_p)_\infty^{\mathrm{tot}} X)^G \cong \coprod_{i \in I} (\mathbb{F}_p)_\infty (X_i^G, N_i)$ and $(Q_\infty^{G, \mathrm{tot}} X)^G \cong \coprod_{i \in I} \mathbb{Z}_\infty (X_i^G, N_i)$.*

Proof. This is immediate from the fact that $(\mathbb{F}_p)_\infty(X^G) = ((\mathbb{F}_p)_\infty(X))^G$ and $(Q_\infty^G X)^G \cong \mathbb{Z}_\infty(X^G)$, which follow from II.6, II.7, and II.8, and IV.2. $\square$

We also state the following straightforward fact.

Proposition IV.4. *Let X be any simply connected free G-space, and Y any G-space. Then the $G\tilde{x}\pi_1(Y)$ space $F(X, \tilde{Y})$ is free as a $\pi_1(Y)$-space, and there is an equivalence of G-spaces $F(X, \tilde{Y})/\pi_1(Y) \cong F(X, Y)$.*

Proof. One needs only check this for $X = G$, viewed as a discrete space, for which it is obvious. □

V. Applying the Snaith splitting

In §I, we stated an equivariant version of the Snaith splitting. We now apply it to study the homotopy fixed point sets in $(Q^G)^k(X)$ and $(\mathbb{F}_p)_\infty(Q^G)^k(X)$.

Theorem V.1. *Let G be a p-group, and let X be a finite G-complex with connected fixed point sets. Then the map $(\mathbb{F}_p)_\infty(Q^G)^k(X) \to F(EG^+, (\mathbb{F}_p)_\infty(Q^G)^k(X))$ is a weak G-equivalence.*

Proof. For $k = 1$, this was Proposition III.7(B). The equivariant Snaith splitting (I.11) asserts that

$$(Q^G)^r(X) \cong \prod_{\alpha\in A} Q^G(E_{G\times_{\Gamma_\alpha}}\Gamma_\alpha \underset{\Gamma_\alpha}{\wedge} X^{n_\alpha}),$$

and the similar decomposition

$$(\mathbb{F}_p)_\infty(Q^G)^r(X) \cong \prod_{\alpha\in A} (\mathbb{F}_p)_\infty Q^G(E_{G\times_{\Gamma_\alpha}}\Gamma_\alpha \underset{\Gamma_\alpha}{\wedge} X^{n_\alpha})$$

follows directly.

The result now follows from III.9. □

Theorem V.2. *Let G and X be as in V.1. Then every component of the homotopy fibre of the map $(Q^G)^k(X) \to F(EG^+, (Q^G)^k(X))^H$ has uniquely p-divisible homotopy groups, where $H \subseteq G$ is a subgroup.*

Proof. Entirely similar to the proof of V.1.

VI. Proofs of the theorems

We apply the discussion in §II and §V to obtain a proof of Theorem (A) in the introduction. Throughout this section, let G be a p-group and let $R = \mathbb{F}_p$.

Theorem VI.1. *Let X be a finite-dimensional G-complex. Then the map $R_\infty X^G \to F(EG^+, R_\infty X)^G$ is a weak equivalence.*

Proof. By Corollary II.11, the map $R_\infty X \to \operatorname{Tot} R_\infty \underset{\sim}{Q}^G\underset{\sim}{N}X$ is a weak G-equivalence. Now for any l, $\mathbf{N}_l X = \{0, 1, \ldots, l\} * X$ has connected H-fixed point set for all n and all $H \subseteq G$. By V.1, the map $R_\infty(Q^G)^k(\mathbf{N}_l X) \to F(EG^+, R_\infty(Q^G)^k(\mathbf{N}_l X))$ is a weak equivalence for all k and l. Now, we have the commutative diagram

$$\begin{array}{ccc} R_\infty X & \longrightarrow & F(EG^+, R_\infty X) \\ \downarrow & & \downarrow \\ \operatorname{Tot} R_\infty \underset{\sim}{Q}^G\underset{\sim}{N}X & \longrightarrow & \operatorname{Tot} F(EG^+, R_\infty \underset{\sim}{Q}^G\underset{\sim}{N}X) \end{array}$$

in which the left hand vertical arrow and the bottom horizontal arrow are weak G-equivalences by the above remarks, and where the right hand vertical arrow is an

isomorphism of simplicial sets. We conclude that $R_\infty X \to F(EG^+, R_\infty X)$ is a weak G-equivalence. □

Theorem VI.2. *Let X be a finite dimensional G-space, which is simply connected. Then there is a natural map θ: $F(EG^+, X)^G \to R_\infty X^G$, where $R = F_p$, such that*
(a) θ is a bijection on components.
(b) Let A denote the set of components of X^G, and if $\alpha \in A$, let X^G_α denote the corresponding component of X^G. Then the homotopy fibre $\mathscr{F}_\alpha$ of the map θ_α: $X_\alpha \to R_\infty(X^G_\alpha)$ has abelian fundamental group and uniquely p-divisible homotopy groups.
(c) θ induces an isomorphism on $H_*(\ ;\mathbb{F}_p)$.

Proof. This follows directly from VI.1 by the results of [21]. □

Corollary VI.3. *Let X be any finite dimensional G-complex. Then there is a natural map θ: $F(EG^+, X)^G \to (\mathbb{F}_p)^{\mathrm{tot}}_\infty(X)^G$ which satisfies the following conditions*
(a) θ is a bijection on components
(b) Let A and $\mathscr{F}_\alpha$ be as in VI.2(b), then $\pi_1(\mathscr{F}_\alpha)$ is abelian, and $\pi_i(\mathscr{F}_\alpha)$ is uniquely p-divisible for all i.
(c) θ induces an isomorphism on $H_*(\ ;\mathbb{F}_p)$

Proof. Apply IV.4. □

We finally wish to prove Theorem B(d) of the introduction. Recall from 1.10 that for any k, if X is a G-space with connected fixed point sets for all $H \subseteq G$, then $(Q^G)^k(X)$ admits a natural decomposition.

$$(Q^G)^k(X) \cong \prod_{\alpha\in A} Q^G(E_{G\times\Gamma_\alpha}\Gamma_\alpha^+ \underset{\Gamma_\alpha}{\wedge} X^{\wedge n\alpha}),$$

where $\Gamma_\alpha \subseteq \sum_{n\alpha}$ and where for given l, all but finitely many of the factors have l-connected fixed point sets for all $H \subseteq G$. Let $\underset{\sim}{N}X$ be the cosimplicial G-space defined in §III. Due to the naturality of the above splitting, we have corresponding splitting of the cosimplicial G-spaces $Q^G(\underset{\sim}{N}X)$ and $(F_p)_\infty Q^G(\underset{\sim}{N}X)$, and consequently of their total spaces.

Thus, let

$$\mathrm{Tot}(Q^G)^k(\underset{\sim}{N}X) \cong \prod_{\alpha\in A_k} \mathrm{Tot}\,F_\alpha(\underset{\sim}{N}X) \text{ and}$$

$$\mathrm{Tot}(F_p)_\infty(Q^G)^k(\underset{\sim}{N}X) \cong \prod_{\alpha\in A_k} \mathrm{Tot}\,G_\alpha(\underset{\sim}{N}X),$$

where $F_\alpha(X) \cong Q^G(E\Gamma_\alpha^+ \wedge_{\Gamma_\alpha} X^{\wedge n_\alpha})$, and $G_\alpha(X) \cong (F_p)_\infty Q^G(E\Gamma_\alpha^+ \wedge_{\Gamma_\alpha} X^{\wedge n_\alpha})$, and A_k denotes the indexing set of the Snaith decomposition of $(Q^G)^k(\underset{\sim}{N}X)$. We wish to examine the connectivity of the maps $F(EG^+, \mathrm{Tot}_k\mathrm{Tot}_l(Q^G)^k(N_lX))^G \to F(EG^+, \mathrm{Tot}_k\mathrm{Tot}_l(F_p)_\infty(Q^G)^k(\underset{\sim}{N}_lX))^G$. But by the definition of the total space, these can be identified with $\mathrm{Tot}_k\mathrm{Tot}_l F(EG^+, (Q^G)^k(N_lX))^G$ and $\mathrm{Tot}_k\mathrm{Tot}_l F(EG^+, (F_p)_\infty(Q^G)^k(\mathrm{N}_lX))^G$, respectively. For simplicity, let

$$T^kX = \underset{l}{\mathrm{Tot}}(Q^G)^{k+1}(\mathrm{N}_lX) \quad\text{and}\quad \hat{T}^kX = \underset{l}{\mathrm{Tot}}(F_p)_\infty(Q^G)^{k+1}(\mathrm{N}_lX).$$

Theorem VI.4. *Let X be a finite dimensional G-complex, and suppose $\pi_i X = 0$ for $i \leqq N$. Then the map θ: $F(EG^+, X)^G \to F(EG^+, F_{p\infty}^{\text{tot}} X)^G$ is N-connected, i.e. $\pi_i\theta$ is an isomorphism for $i < N$ and a surjection for $i = N$.*

Proof. The statement follows from VI.1 for $i = 1$, so we assume $i \geqq 2$. We consider both the cosimplicial spaces $T\cdot(X)$ and $\hat{T}\cdot(X)$. Recall from above that $T^k X \equiv \prod_{\alpha \in A_k} \operatorname{Tot} F_\alpha((\underset{\sim}{N}X)$ and $T^k X \cong \prod_{\alpha \in A_k} \operatorname{Tot} G_\alpha((\underset{\sim}{N}X)$. Let $A_k^0 \subseteq A_k$ be the set of those α so that $n_\alpha \leqq k + 1$, and let $A_k^1 = A_k - A_k^0$. It is easy to check from the Snaith decomposition that the map $s^0 \times s^1 \times \ldots \times s^k$: $T^k X \to \prod_{i=0}^k T^{k-1} X$ factors as a composite

$$T^k X \xrightarrow{\pi} Z_k \xrightarrow{i} T^k X \cong \prod_{i=0}^{k} T^{k-1} X ,$$

where π is a projection on a summand corresponding to a subset B_k of A_k containing A_k^0, and where i is the inclusion on a wedge summand. The analogous decomposition holds for $\hat{T}^k X$, with the summand being referred to as $\hat{Z}_k$.

We now consider the cosimplicial spaces $F(EG^+, T\cdot X)^G$ and $F(EG^+, \hat{T}\cdot(X))^G$. Let x be any point of X^G, we consider the homotopy spectral sequences for the corresponding cosimplicial subspaces $F(EG^+, T\cdot X)_x^G$ and $F(EG^+, \hat{T}\cdot X)_x^G$. The E_2-terms are $\pi^s\pi_t(F(EG^+, T\cdot X)^G, x)$ and $\pi_s\pi_t(F(EG^+, \hat{T}\cdot X)^G, x)$. Now, recall that E_2-terms can be computed using the complex of normalized cochains for $\pi_t(F(EG^+, T\cdot X)^G, x)$ and $\pi_t(F(EG^+, ((\hat{T}\cdot X)^G, x)$. From the above description of the degeneracy maps, we see that in codimension t, the normalized cochains are given by $\prod_{\alpha \in A_k - B_k} \pi_t(F(EG^+, \operatorname{Tot}_l F_\alpha(\mathbf{N}_l X))^G, x)$ and $\prod_{\alpha \in A_k - B_k} \pi_t(F(EG^+, \operatorname{Tot}_l G_\alpha(\mathbf{N}_l X))^G, x)$. Since for all $\alpha \in A_k - B_k$, $n_\alpha \geqq k + 1$, we find by III.11 that the map $\pi_t(F(EG^+, \operatorname{Tot}_l F_\alpha(N_l X)), x) \to \pi_t(F(EG^+, \operatorname{Tot}_l G_\alpha(N_l X)), x)$ is an isomorphism for $t \geqq (k + 1)N + k - 1$ and a surjection for $t = (k + 1)N + k$. Consequently, if $N_{s,t}$ and $\hat{N}_{s,t}$ denote the normalized cochain groups of these two cosimplicial groups in codimension s, then $N_{s,t} \to \hat{N}_{s,t}$ is an isomorphism if $t \leqq (s + 1)N + s - 1$, and a surjection if $t = (s + 1)N + s$. This shows that the map induced by $T^\bullet \to \hat{T}^\bullet$ on E_2 is an isomorphism if $t \leqq (s + 1)N + s - 1$ and a surjection if $t = (s + 1)N + s$. Since the groups $\pi^s\pi_{k+s}$ are the part of the E_2-term corresponding to the k-th homotopy group of the total space, we see that $T^\bullet \to \hat{T}^\bullet$ induces an isomorphism on the associated graded groups of π_k for $k < N$, and a surjection for $k = N$. □

Appendix I

Let A denote the category whose objects are the nonnegative integers, and for which there is a unique morphism $n \to m$ whenever $n \leqq m$, and so that $\operatorname{Mor}_a(m, n) = \varnothing$ if $m > n$. Recall from [7, Ch. I] the notion of a cochain functor on a category $\mathscr{C}$, and also, if $T: \mathscr{C} \to \mathscr{C}$ is an endofunctor, the notion of T-representability and T-acyclicity.

Lemma A.I.1. *Let C be any category, and let $T: \mathscr{C} \times A \to \mathscr{C} \times A$ denote the functor $T(X, n) = (X, n + 1)$. Suppose K^* is a T-representable, T-acyclic cochain functor on $\mathscr{C} \times A$, and suppose further that $K^{-1}(X, 0) = K^0(X, 0) = 0$ for all $X \in ob\, \mathscr{C}$. Then K^* is naturally contractible as a cochain functor on $\mathscr{C} \times A$.*

Proof. T-acyclicity guarantees that there is a natural contracting homotopy s^* for K^* restricted to $T(\mathscr{C} \times A)$; for, every object and morphism in $T(\mathscr{C} \times A)$ can be written uniquely as lying in the image of T. Consider s^0: $K^0 | T(\mathscr{C} \times A) \to K^{-1} | T(\mathscr{C} \times A)$; s^0 extends to a natural transformation on all of $\mathscr{C} \times A$, since $K^0(X, 0) = 0 = K^{-1}(X, 0) \forall X \in ob\mathscr{C}$. Let σ^*: $K^* \to K^{*-1}$ be defined by $\sigma^0 = s^0$, $\sigma^i = 0$ for $i > 0$. σ^* is now a natural transformation from K to itself viewed as a graded abelian group valued-functor on $\mathscr{C} \times A$. Consider natural cochain transformation $f = Id - (\sigma\delta + \delta\sigma)$. It is chain homotopic to the identity, and note moreover that it induces the zero map $K^{-1}(X, n) \to K^{-1}(X, n)$. According to the generalized acyclic model theorem of Barr and Beck [4], f is naturally null-homotopic, which is the result. □

Corollary A.I.2. *Let K^* and L^* be two T-representable, T-acyclic cochain functors on $\mathscr{C} \times A$, and let f^*: $K^* \to L^*$ be a natural transformation of cochain functors preserving the s_i's and ε_s's, so that the maps f^{-1}: $K^{-1}(X, 0) \to L^{-1}(X, 0)$ and f^0: $K^0(X, 0) \to L^0(X, 0)$ are isomorphisms. Then f^* is a natural chain equivalence.*

Proof. Simply note that $\ker(f)$ and $\operatorname{coker}(f)$ are T-acyclic and T-representable, and apply Lemma A.I.1. □

Let $\underset{\sim}{N}X$ be defined as in §II, and define T^kX to be $\operatorname{Tot} S^{k+1}(\underset{\sim}{N}X)$, for $k \geqq 0$, where $S = Q_\bullet$, $\mathbb{F}_p$, or $(\mathbb{F}_p)_\infty Q$. For $k = -1$, set $T^{-1}X = X$. Then $T^\bullet X$ becomes an augmented cosimplicial space. There is a natural map of augmented cosimplicial spaces $\underset{\sim}{S}X \to T^\bullet X$, which we must study. Each point $x_0 \in X$ determines points in S^kX and T^kX, hence, we have a map of cochain functors ϕ: $\pi_i(\underset{\sim}{S}X, x_0) \to \pi_i(T^\bullet X, x_0)$.

Proposition A.I.3. *The transformation ϕ is a natural cochain equivalence.*

Proof. We first define some auxiliary functors. Let $T(k)^\bullet X$ denote the cosimplicial space defined by $T(k)^j(X) = \operatorname{Tot} S^j(S^k(\underset{\sim}{N}X))$. Note that we have natural maps ε_k: $T(k)^\bullet X \to T(k+1)^\bullet X$, $\forall k > 0$, and η_k: $T(k)^\bullet X \to T(k-1)^\bullet X$, $k > 2$, arising from the triple structure of S. Of course, $\eta\varepsilon = id$. We now define a cochain functor L_i on $\mathscr{T} \times \mathscr{A}$, where $\mathscr{T}$ denotes the category of based spaces, by $L_i(X, k) = \pi_i(T(k)^\bullet X)$ and let $L((X, k) \to (X, k+1))$ be the map $\pi_i(\varepsilon_k)$. We readily check that L_i is a cochain functor. Moreover, if T: $\mathscr{T} \times \mathscr{A} \to \mathscr{T} \times \mathscr{A}$ is the above defined shift operator, L_i is also T-acyclic and T-representable. Both are consequences of the existence of the maps η_k. We define a second cochain functor K_i by $K_i(X, k) = \pi_i(\underset{\sim}{S}(S^kX))$. K_1 is also readily checked to be a cochain functor, T-acyclic, and T-representable. The augmentation $X \to \underset{\sim}{N}X$ induces a natural transformation f_i^*: $K_i^* \to L_i^*$, which we will show is an equivalence. Since $K_i^{-1}(X, 0) = \pi_i(X) = L_i^{-1}(X, 0)$, f^{-1} is an isomorphism on objects of the form $(X, 0)$.

We check that the augmentation $SX \to \operatorname{Tot}_s S(\underset{\sim}{N}X)$ is an equivalent for $s \geqq 1$. Note that all the functors S convert wedges to products and satisfy $S = \Omega S \Sigma$. $\operatorname{Tot}_1 S(\underset{\sim}{N}X)$ is by definition $\Omega S \Sigma X$, and the augmentation evidently induces the equivalence $SX \to \Omega S \Sigma X$. We claim that the homotopy fibre of the map $\operatorname{Tot}_s S\underset{\sim}{N}X \to \operatorname{Tot}_{s-1} S\underset{\sim}{N}X$ is contractible. It is standard that the homotopy groups of this fibre are precisely $\ker(s^i$: $\pi_i(S(\{0, 1, \ldots, s\} * X)) \to \pi_i(S(\{0, 1, \ldots, s-1\} * X)))$, and $S(\{0, 1, \ldots, s\} * X) = \prod_{i=1}^{s} S(\Sigma X)$. Moreover, the degeneracies are the projections on products over subsets of cardinality $s - 1$, so the kernel is trivial. This shows the fibre to be contractible. □

Now let X be a G-space, and let $T^k(X) = \mathrm{Tot}(Q^G)^{k+1}(\underset{\sim}{N}X)$, and $\hat{T}^k(X) = \mathrm{Tot}(F_p)_\infty(Q^G)^{k+1}(\underset{\sim}{N}X)$. Also, let $S^k(X)$ and $\hat{S}^k(X)$ denote the G-spaces $(Q^G)^{k+1}(X)$ and $(F_p)_\infty(Q^G)^{k+1}(X)$. These fit together to form cosimplicial G-spaces $S^\bullet X$, $\hat{S}\cdot X$, $T^\bullet X$, and $\hat{T}\cdot X$, and we have natural maps

$$S^\bullet X \to T^\bullet X \text{ and } \hat{S}^\bullet X \to \hat{T}^\bullet X$$

of cosimplicial G-spaces. Let $H \subseteq G$ be a subgroup, and X a point of X^H, which determines basepoint in all the G-spaces $(S^kX)^H$, $(T^kX)^H$, $(\hat{S}^kX)^H$, and $(\hat{T}^kX)^H$. Consequently, we obtain cosimplicial abelian groups $\pi_i((S^\bullet X)^H, x)$, $\pi_i((\hat{S}^\bullet X)^H, x)$, $\pi_i(T^\bullet X)^H, x)$ and $\pi_i((\hat{T}^\bullet X)^H, x)$, for all i.

Proposition A.I.4. *The maps $\pi_i((S^\bullet X)^H, x) \to \pi_i((T^\bullet X)^H, x)$ and $\pi_i((\hat{S}^\bullet X)^H, x) \to \pi_i((\hat{T}^\bullet X)^H, x)$ are cochain equivalences for all i.*

Proof. This follows easily by the same argument used in the proof of A.I.3, using Q^G instead of Q. We omit the details. □

To prove that the maps $\mathrm{Tot}\, S^\bullet X \to \mathrm{Tot}\, T^\bullet X$ and $\mathrm{Tot}\, \hat{S}^\bullet X \to \mathrm{Tot}\, \hat{T}^\bullet X$ are weak G-equivalences, it will suffice to prove that the maps

$$\pi_0(X^H) \to \{\alpha \in \pi_0((T^0X)^H) | \delta^0\alpha = \delta^1\alpha \text{ in } \pi_0((T^1X)^H)\}$$

$$\text{and } \pi_0(X^H) \to \{\alpha \in \pi_0((\hat{T}^0X)^H) | \delta^0\alpha = \delta^1\alpha \text{ in } \pi_0((\hat{T}^1X)^H)\}$$

are bijections. To prove this, we must describe $\pi_0(\mathrm{Tot}(Q^GQ^G\underset{\sim}{N}X)^H)_\bullet$ and $\pi_0(\mathrm{Tot}(F_p)_\infty Q^GQ^G\underset{\sim}{N}X)^H)_\bullet$. We need some terminology. For any group G, let $\mathbf{Z}[G]$ denote the group ring of G. We have an augmentation $\mathbf{Z}(G) \xrightarrow{\varepsilon} \mathbf{Z}$, and we let $\hat{\mathbf{Z}}[G]$ denote the completion of $Z[G]$ at the kernel of the augmentation. Note that if G is free abelian, then the map $Z[G] \to \hat{\mathbf{Z}}[G]$ is an injection. We also let $\hat{\mathbf{Z}}[G]_{(p)}$ denote the completion of $\mathbf{Z}[G]$ at the ideal $(p) + \ker(\varepsilon)$. Suppose that a group K acts by automorphisms on G. Then $Z[K]$ acts by automorphisms on $\hat{\mathbf{Z}}[G]$ and $\hat{\mathbf{Z}}[G]_{(p)}$, and we let $Z \hat{\otimes}_{Z[K]} \hat{Z}[G]$ and $Z \hat{\otimes}_{Z[K]} \hat{\mathbf{Z}}[G]_{(p)}$ denote the corresponding completed tensor products.

Proposition A.I.5. *Let K be a finite group, and suppose that G is the free abelian group on a K-set X. Then the maps $\mathbf{Z} \otimes_{\mathbf{Z}[K]} \mathbf{Z}[G] \to \mathbf{Z} \hat{\otimes}_{\mathbf{Z}[K]} \hat{\mathbf{Z}}[G]$ and $\mathbf{Z} \otimes_{\mathbf{Z}[\mathbf{K}]} \mathbf{Z}[G] \to \mathbf{Z} \hat{\otimes}_{\mathbf{Z}[\mathbf{K}]} \hat{\mathbf{Z}}[G]_{(p)}$ are injective.*

Proof. Consider the first map, and let $\mathbf{Z}[G]^K \subseteq \mathbf{Z}[G]$ be the fixed point subring, which we denote by A. $\mathbf{Z}[G]$ is clearly an integral extension of A. Let I_A be the intersection of the augmentation ideal in $\mathbf{Z}[G\}$ with A. Then I_A is a prime of A, and it is easily seen to induce the I-adic topology on $\mathbf{Z}[G]$, where I denotes the augmentation ideal in $\mathbf{Z}[G]$. Therefore; $\hat{\mathbf{Z}}[G] = \mathbf{Z}[G]^{\wedge}_{I_A}$, where on the right $\mathbf{Z}[G]$ is viewed as a A-module.

Since the functor $\mathbf{Z} \otimes_{Z[K]} M$ can be identified with cokernel $(\oplus_{k \in K} M \xrightarrow{\oplus 1-k} M)$, and since I_A-adic completion is exact for finitely generated A-modules, we see that $\mathbf{Z} \hat{\otimes}_{\mathbf{Z}[K]} \mathbf{Z}[G] \cong \hat{A} \otimes_A (\mathbf{Z} \otimes_{\mathbf{Z}[K]} \mathbf{Z}[\mathbf{G}])$. Note that $\mathbf{Z} \otimes_{\mathbf{Z}[K]} \mathbf{Z}[G]$ is an A-module, since the A-action commutes with the K-action, and that $\hat{A}$ denotes the I_A-adic completion of A. The case of $\mathbf{Z} \otimes_{\mathbf{Z}[K]} \hat{\mathbf{Z}}[G]_{(p)}$ works similarly. □

In these terms, the decomposition of $\pi_0(Q^G(Q^G(X))^H)$ given in the proof of II.5 can be written as

$$\pi_0 Q^G(Q^G(X))^H \cong \bigoplus_{K \subseteq H} \mathbf{Z} \underset{\mathbf{Z}[W_G(K)]}{\otimes} Z[\pi_0((Q^G X)^k)]$$

The following is easy to verify using the Snaith splitting; we omit the proof.

Proposition A.I.6.

$$\pi_0(\mathrm{Tot}(Q^G Q^G(\underset{\sim}{\mathbb{N}}X)^H)) \cong \bigoplus_{K \subseteq H} \mathbf{Z} \underset{\mathbf{Z}[W_G(K)]}{\hat{\otimes}} \hat{\mathbf{Z}}[\pi_0((Q^G X)^K)] \text{ and}$$

$$\pi_0(\mathrm{Tot}((F_p)_\infty Q^G Q^G(\underset{\sim}{\mathbb{N}}X)^H) \cong \bigoplus_{K \subseteq H} \mathbf{Z} \underset{\mathbf{Z}[\mathbf{W}_G(\mathbf{K})]}{\hat{\otimes}} \hat{\mathbf{Z}}[\pi_0((Q^G(X)^K)]_{(p)}$$

Further the maps $\pi_0(Q^G Q^G X^H) \to \pi_0 \mathrm{Tot}(Q^G Q^G(\underset{\sim}{\mathbb{N}}X)^H)$ *and* $\pi_0(Q^G Q^G X^H) \to \pi_0 \mathrm{Tot}((F_p)_\infty(Q^G Q^G(\underset{\sim}{\mathbb{N}}X)^H))$ *are simply the inclusions of the uncompleted abelian groups into the completed ones; in particular, they are injective, by A.I.5.*

Corollary A.I.7. *The maps* $\pi_0(X^H) \to \{\alpha \in \pi_0((T^0 X)^H) | \delta^0 \alpha = \delta^1 \alpha\}$ *and* $\pi_0(X^H) \to \{\alpha \in \pi_0((\hat{T}^0 X)^H) | \delta^0 \alpha = \delta^1 \alpha\}$ *are bijections. Hence, the maps* $S^\bullet X \to T^\bullet X$ *and* $\hat{S}^\bullet X \to \hat{T}^\bullet X$ *induce weak G-equivalences on total spaces.*

Corollary A.I.8. *There is a weak G-equivalence* $(F_p)_\infty X \to \mathrm{Tot}\, \hat{T}^\bullet(X)$, *for any G-space X.*

Proof. We have $(\mathbb{F}_p)_\infty X \to \hat{S}^\bullet X \to \hat{T}^\bullet X$, where $(\mathbb{F}_p)_\infty X \to \hat{S}^\bullet X$ is simply the augmentation. $\mathrm{Tot}\, \hat{S} \cdot X \to \mathrm{Tot}\, \hat{T} \cdot X$ is a weak G-equivalence by A.I.6, and $(\mathbb{F}_p)_\infty X \to \mathrm{Tot}\, \hat{S} \cdot X$ can easily be shown to be a G-equivalence by an argument along the lines of the proof of II.7. The point is that from §II, there is a natural transformation of triples $Q^G \to F_p$. □

Appendix II

We will prove Lemma III.12 through a series of steps. Throughout, G will be a p-group.

Lemma A.II.1. *Let* $A(G) \xrightarrow{\varepsilon} \mathbb{Z}$ *be the augmentation, let* $I = I(G)$ *be the augmentation ideal, and* $J = J(G) = I(G) + p \cdot A(G)$. *Then the I-adic and J-adic topologies on* $I(G)$ *coincide. There is a Cartesian square of rings*

$$\begin{tikzcd} A(G)^{\wedge}_{I(G)} \arrow[r] \arrow[d, "\varepsilon"] & \hat{\mathbb{Z}}_p \underset{\mathbb{Z}}{\otimes} A(G) \arrow[d, "\mathrm{id} \otimes \varepsilon"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}_p \end{tikzcd}$$

and for any $A(G)$-module M a corresponding Cartesian square

$$\begin{tikzcd} M_I^\wedge \arrow[r] \arrow[d] & M_p^\wedge = M_J^\wedge \arrow[d] \\ M/IM \arrow[r] & (M/IM)_p^\wedge \end{tikzcd}$$

Moreover, I-adic and J-adic completion are exact on the category of finitely generated $A(G)$-modules.

Proof. The first statement as well as the Cartesian squares are immediate consequences of the result of [16]. The exactness statement follows since $A(G)$ is Noetherian, by Proposition 10.12 of [3]. □

Lemma A.II.2. *Let M be any $A(G)$-module. Then $|G|$ annihilates $\mathrm{Tor}_i^{A(G)}(M, \mathbb{Z})$ for $i > 0$.*

Proof. This is immediate from the existence of the following diagram of $A(G)$-module homomorphisms.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\theta"] \arrow[dr, "x|G|"'] & A(G) \arrow[d, "\varepsilon"] \\ & \mathbb{Z} \end{tikzcd}$$

Here $\theta(1) = [G]$, where $[G]$ is the element of $A(G)$ represented by a free G-set on one generator. □

Lemma A.II.3. *Let C be any bounded below chain complex over $\mathbb{Z}$, so that*
(*a*) $\mathrm{Hom}(\mathbb{Z}/p^\infty, C_i) = 0 \forall i$.
(*b*) $\mathrm{Hom}(\mathbb{Z}/p^\infty, H_iC) = 0 \forall i$.

Then the map $C_0 \to (C_0)_p^\wedge$ induces an isomorphism $H_i((C_0)_p^\wedge) \to H_i(C_0)_p^\wedge$.

Proof. From [7, Ch. VI], associated to any short exact sequence $0 \to L \to M \to N \to 0$ of abelian groups there is an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, L) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, M) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, N) \to L_p^\wedge \to M_p^\wedge \to N_p^\wedge \to 0$$
(I)

Let n be the smallest integer for which $C_n \neq 0$. Then we have the short exact sequence

$$0 \to \partial(C_{n+1}) \to C_n \to H_nC \to 0$$

The exact sequence (I) and hypotheses (a) and (b) permit us to conclude that $(H_nC)_p^\wedge \cong H_n((C_\bullet)_p^\wedge)$, and that $\mathrm{Hom}(\mathbb{Z}/p^\infty, \partial(C_{n+1})) = 0$. Next, we have the exact sequence

$$0 \to Z_{n+1}C \to C_{n+1} \xrightarrow{\partial} \partial C_{n+1} \to 0 .$$

Again, (I) and our hypotheses shows that $Z_{n+1}(C_p^\wedge) \cong (Z_{n+1}C_\bullet)_p^\wedge$, and that $\mathrm{Hom}(\mathbb{Z}/p^\infty, Z_{n+1}C_\bullet) = 0$. Finally, we have the exact sequence

$$0 \to \partial C_{n+2} \to Z_{n+1}C \to H_{n+1}C \to 0$$

Again, applying (I), we find that $(H_{n+1}C)_p^\wedge \cong H_{n+1}(C_p^\wedge)$, and that $\mathrm{Hom}(\mathbb{Z}/p^\infty, \partial C_{n+2}) = 0$. Proceeding inductively this way gives the result. □

Corollary A.II.4. *Let M be an $A(G)$-module so that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M/IM) = 0 = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, IM)$. Let R_0 denote an $A(G)$-free resolution of M. Then the natural map $H_i(R_0)_{I(G)}^\wedge \to H_i((R_0)_{I(G)}^\wedge)$ is an isomorphism.*

Proof. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M/IM) = 0 = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, IM)$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M) = 0$. For any free $A(G)$-module F, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, F) = 0$, so A.II.3 shows that the natural map $H_i(R_\bullet)_p^\wedge \to H_i((R_\bullet)_p^\wedge)$ is an isomorphism. A.II.1 shows that we have an exact sequence of complexes $0 \to (R_\bullet)_I^\wedge \to (R_\bullet)_p^\wedge \oplus R_\bullet/IR_\bullet \to (R_\bullet/IR_\bullet)_p^\wedge \to 0$ and hence a long exact sequence $\ldots \to H_i((R_\bullet)_I^\wedge) \to H_i(R_{\bullet p}^\wedge) \oplus H_i(R_\bullet/IR_\bullet) \to H_i((R_\bullet/IR_\bullet)_p^\wedge) \xrightarrow{\partial} \ldots$ Note that $H_i(R_\bullet/IR_\bullet) \cong \mathrm{Tor}_i^{A(G)}(M, \mathbb{Z})$, so $|G| \cdot H_i(R_\bullet/IR_\bullet) = 0$ by A.II.2, hence $\mathrm{Hom}(\mathbb{Z}/p^\infty, H_i(R_\bullet/IR_\bullet)) = 0$. Therefore, the map $H_i(R_\bullet/IR_\bullet) \to H_i((R_\bullet/IR_\bullet)_p^\wedge)$ is an isomorphism for $i > 0$. Therefore, we have that $H_i((R_\bullet)_I^\wedge) = H_i((R_\bullet)_p^\wedge) = 0$ for $i > 0$, by A.II.3, and that there is a short exact sequence $0 \to H_0((R_\bullet)_I^\wedge) \to M_p^\wedge \oplus M/IM \to (M/IM)_p^\wedge \to 0$. From A.II.1, we see that $H_0((R_\bullet)_I^\wedge) \cong M_I^\wedge$. □

Proposition A.II.5. *Let $R_\bullet$ be a bounded below chain complex of $A(G)$-modules, such that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, R_i) = 0 \forall i$. Then there is an $A(G)$-free chain complex $\bar{R}_\bullet$ and a map $\bar{R}_\bullet \to R_\bullet$ inducing homology isomorphisms $H_i(\bar{R}_\bullet) \to H_i(R_\bullet)$, $H_i(\bar{R}_{\bullet I}) \to H_i(R_{\bullet I})$, and $H_i(\bar{R}_{\bullet p}) \to H_i(R_{\bullet p})$ for all i.*

Proof. One first constructs free resolutions $S_\bullet^{(i)} \to R_i$, with differential $d^{(i)}$. It is a standard construction in homological algebra that one can construct a free $A(G)$-complex $(T_\bullet, \partial)$, equipped with a filtration F_i, so that $\partial(F_i T_\bullet) \subseteq F_i T_\bullet$, so that $F_i T_\bullet / F_{i-1} T_\bullet \cong S_\bullet^{(i)}$, and the induced map

$$\partial\colon F_i T_\bullet / F_{i-1} T_\bullet \to F_i T_\bullet / F_{i-1} T_\bullet$$

is identified with $d^{(i)}$. Moreover, $(T_\bullet, \partial)$ is equipped with a chain map $\phi\colon T_\bullet \to R_\bullet$, so that $\phi(F_i T_\bullet) \subseteq \oplus_{s \le i} R_s$, and so that the map $\phi\colon F_i T_\bullet / F_{i-1} T_\bullet \to R_i$ is identified with the augmentation $S_\bullet^{(i)} \to R_i$. Consideration of the spectral sequence based on the filtration $F_i T_\bullet$ shows that ϕ is a homology isomorphism. Application of Corollary A.II.4 shows that the corresponding spectral sequences based on $F_i T_{\bullet I}^\wedge$ and $F_i T_{\bullet p}^\wedge$ also induce isomorphisms on homology, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, R_i) = 0$. □

Corollary A.II.6. *Let M be any $A(G)$-module so that for some j, $p^j[G]M = 0$, where $[G]$ is the element of $A(G)$ corresponding to a free G-set on one generator. Let $R_\bullet$ be any $A(G)$-free resolution of M. Then the map $(R_\bullet)_I^\wedge \to (R_\bullet)_p^\wedge$ induces an isomorphism on homology.*

Proof. Consider $(R_\bullet)_I^\wedge$ and $(R_\bullet)_p^\wedge$. We have a diagram

$$0 \to (IR_\bullet)_I^\wedge \to (R_\bullet)_I^\wedge \to R_\bullet/IR_\bullet \to 0$$

$$0 \to (IR_\bullet)_p^\wedge \to (R_\bullet)_p^\wedge \to (R_\bullet/IR_\bullet)_p^\wedge \to 0$$

where the rows are exact. By Lemma A.II.1, the map $(IR_\bullet)_I^\wedge \to (IR_\bullet)_p^\wedge$ is an isomorphism, so we must only show that the map $(R_\bullet/IR_\bullet) \to (R_\bullet/IR_\bullet)_p^\wedge$ induces an isomorphism on homology. Of course, $H_i(R_\bullet/IR_\bullet) \cong \mathrm{Tor}_i^{A(G)}(M, \mathbb{Z})$, so $|G| \cdot H_i(R_\bullet/IR_\bullet) = 0$ for $i > 0$, by A.II.2. For $i = 0$, $H_0(R_\bullet/IR_\bullet) \cong M/IM$. The element $p^j \cdot |G| - p^j[G]$ is an element of I, hence it acts trivially on M/IM. On the other hand, $p^j|G|$ acts trivially on M/IM since it acts trivially on M. Therefore, $p^j|G|$ acts trivially on M/IM, so $p^j|G| \cdot (M/IM) = 0$, so $p^j|G| \cdot H_0(R_\bullet/IR_\bullet) = 0$. Consequently, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty; H_i(R_\bullet/IR_\bullet) = 0 \forall i$. Of course, R_i/IR_i is a free $\mathbb{Z}$-module with the same basis as R_i, so $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty; R_i/IR_i) = 0 \forall i$. Applying Lemma A.II.3, we see that $H_i((R_\bullet/IR_\bullet)_p^\wedge) \cong H_i(R_\bullet/IR_\bullet)_p^\wedge \cong H_i(R_\bullet/IR_\bullet)$. □

Proof of Lemma III.12. Suppose first that the complex C is free. It is a standard construction in homological algebra that there is a "Postnikov-type" decomposition for $C_\bullet$, i.e. a family of free complexes $C_\bullet(n)$, surjective chain maps $C_\bullet(n) \xrightarrow{f_n} C_\bullet(n-1)$, so that the kernel of f_n is an $A(G)$-free resolution of the module $H_n(C)$, with degrees increased by n, and maps $C_\bullet \xrightarrow{\phi_n} C_\bullet(n)$ so that $f_n \phi_n = \phi_{n-1}$, and for which $H_i(\phi_n)$ is an isomorphism. The ϕ_n's give a chain equivalence $C_\bullet \to \varprojlim_n C_\bullet(n)$. Now, $C_\bullet(0)$ is an $A(G)$-free resolution of $H_0(C_\bullet)$, and $p^j[G] \cdot H_0(C_\bullet) = 0$ by hypothesis, so A.II.6 shows that the map $C_\bullet(0)_I^\wedge \to C_\bullet(0)_p^\wedge$ induces an isomorphism on homology. We will prove that the map $C_\bullet(N)_I^\wedge \to C_\bullet(N)_p^\wedge$ induces an isomorphism on homology. Suppose $C_\bullet(k)_I^\wedge \to C_\bullet(k)_p^\wedge$ induces an isomorphism on homology and $k < N$. We have the exact sequence of free chain complexes $0 \to \mathrm{Ker}(f_{k+1}) \to C_\bullet(k+1) \xrightarrow{f_{k+1}} C_\bullet(k) \to 0$. Since $C_\bullet(k)$ is a free complex, this exact sequence is split as a sequence of graded $A(G)$-modules. Consequently, we have a diagram of exact sequences

$$0 \to \mathrm{Ker}(f_{k+1})_I^\wedge \to C_\bullet(k+1)_I^\wedge \to C_\bullet(k)_I^\wedge \to 0$$
$$0 \to \mathrm{Ker}(f_{k+1})_p^\wedge \to C_\bullet(k+1)_p^\wedge \to C_\bullet(k)_p^\wedge \to 0$$

Since $\mathrm{Ker}(f_{k+1})$ is an $A(G)$-free resolution of $H_{k+1}(C_\bullet)$ and $p^j[G]$ annihilates $H_{k+1}(C_\bullet)$, A.II.6 shows that the left vertical arrow induces an isomorphism on homology, and the right vertical arrow induces an isomorphism on homology by hypothesis. The long exact homology sequence now shows the result for $C_\bullet(k+1)$, so the result is true for $C_\bullet(N)$ by induction. The kernel of the map $\varprojlim C_\bullet(n) \to C_\bullet(N)$, K_0, satisfies $K_i = 0$ for $i \leq N$, so the result follows from the corresponding result for $C_\bullet(N)$. If $C_\bullet$ is not free, A.II.5 permits us to reduce to this case. □

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