Werk

Titel: Appendix C: Some uniform eigenvalue estimates.

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S-tubes around the leaf diagonals. The spectral theorem and the Sobolev lemma, along with (Proposition 5.8, [78]) imply that these smooth kernels are uniformly locally trace class, hence Cauchy in the Tr_μ-norm for S large. (As a hint to the reader, we remark that we must allow the value of S to depend upon t, since P is not differential, its support increases in size as we take powers P^l.)

It is then standard, given (B38), that $\mathrm{Ind}_\mu(X(\infty))$ exists and is the limit of the right-hand side of (B37). More details for the case $P=\not{\partial}$ are given in [45].

Appendix C: Some uniform eigenvalue estimates

Let M denote a compact Riemannian manifold. For this appendix, we consider D, a first order, essentially self-adjoint elliptic ψDO acting on the smooth sections $C^\infty(E)$ of an Hermitian vector bundle $E\to M$. Under a mild hypothesis on the support of D, for each representation $\beta:\Gamma=\pi_1(M)\to U_N$ we can form the extended operator $D\underset{\beta}{\otimes} I_N$ on $E\otimes E(\beta)$. Introduce the spectral counting function

$$\text{(C1)}\qquad E(D,\beta,\lambda)=\dim \text{ span }\{\text{eigenvectors of } D\underset{\beta}{\otimes} I_N \text{ with eigenvalue of modulus } \leqq\lambda\}.$$

In this appendix, we adopt the method of Allard (cf. pages 78–79, [47]) to give an elementary uniform estimate on this function which is linear in N, and then derive two important consequences.

Proposition C1. *There exists a constant $c(D)$, depending only on D and the Hermitian structure on E and TM, such that*

$$\text{(C2)}\qquad E(D,\beta,\lambda)\leqq N\cdot c(D)\cdot(1+\lambda)^{2d}$$

where d is the least integer with $d>\dim M/2$.

Proof. First we explain the mild hypothesis on D that is required. Fix a good, finite cover of M by open coordinate charts which are geodesically convex, and thus their multiple intersections are all contractible. Let ε_0 be the Lebesgue number for this cover. Then we assume that D is represented by a distributional kernel whose support is contained in an $\varepsilon_0/4d$-neighborhood of the diagonal in $M\times M$.

The method of the proof is to adapt the estimate in Lemma 1.6.3 of [47], based on Allard's Trick, to an operator coupled to a flat bundle. For completeness, we reprove this result in our context, and indicate the one nuance needed to obtain a bound linear in the dimension N.

Replace D with the power $P=D^d$. This will have support in an $\varepsilon_0/4$-neighborhood of the diagonal. Choose a parametrix Q for P as in Sect. 3 such that Q has kernel supported in an $\varepsilon_0/2$-neighborhood, and $(PQ-I)$, $(QP-I)$ are smoothing.

For a flat bundle $E(\beta)\to M$, both P and Q admit canonical extensions to operators on the sections of $E\otimes E(\beta)\to M$. There are two ways to define these extensions, and they are easily seen to be equivalent. First, a section $\varphi\in C^\infty(E\otimes E(\beta))$ can be written $\varphi=\sum_{j=1}^{l}\varphi_j$, where each φ_j has support in a closed set of diameter at most $\varepsilon_0/4$. For each j choose an open set in the cover, U_{i_j}, containing an $\varepsilon_0/2$-neighborhood of the support of φ_j, and a trivialization

$$E(\beta)|_{U_{i_j}} \cong U_{i_j} \times \mathbf{C}^N$$

compatible with the flat Hermitian structure. Then define $P \underset{\beta}{\otimes} I_N(\varphi_j)$ and $Q \underset{\beta}{\otimes} I_N(\varphi_j)$ to be the sections over U_{i_j} obtained by applying the kernels of P and Q to the components of $\varphi_j = (\varphi_j^1, \ldots, \varphi_j^N)$.

For the second definition of $P \underset{\beta}{\otimes} I_N$ and $Q \underset{\beta}{\otimes} I_N$, identify the lift of $E(\beta)$ to $\tilde{M}$

$$\widetilde{E(\beta)} \cong \tilde{M} \times \mathbf{C}^N \tag{C3}$$

so that the Γ-covering action on $\widetilde{E(\beta)}$ is transformed into the diagonal action on $\tilde{M} \times \mathbf{C}^N$. The contractibility of ε_0-balls in M implies there is a covering map

$$N(\varepsilon_0/2, \tilde{\Delta}) \xrightarrow{\pi} N(\varepsilon_0/2, \Delta)$$

from the $\varepsilon_0/2$-neighborhood of the diagonal $\tilde{\Delta} \subset \tilde{M} \times \tilde{M}$ to the $\varepsilon_0/2$-neighborhood of $\Delta \subset M \times M$. Use this covering map to lift the distributional kernels of P and Q to kernels on $\tilde{M} \times \tilde{M}$, defining operators $\tilde{P}$ and $\tilde{Q}$ on $C^\infty(\tilde{E})$, where $\tilde{E} \to \tilde{M}$ is the lift of $E \to M$. These operators extend naturally to $C^\infty(\tilde{E} \otimes \mathbf{C}^N)$, and via the isomorphism (C3) define Γ-equivariant operators on $C^\infty(\tilde{E} \otimes \widetilde{E(\beta)})$. For a section $\varphi \in C^\infty(E \otimes E(\beta))$, we lift it to a section $\tilde{\varphi} \in C^\infty(\tilde{E} \otimes \widetilde{E(\beta)})$, and observe that $\tilde{P} \otimes I_N(\tilde{\varphi})$ and $\tilde{Q} \otimes I_N(\varphi)$ are Γ-invariant, so descend to sections denoted by $P \underset{\beta}{\otimes} I_N(\varphi)$ and $Q \underset{\beta}{\otimes} I_N(\varphi)$. The reader can easily check that this definition agrees with the previous one.

Let $\phi \in C^\infty(E \otimes E(\beta))$ be an eigenfunction of $P \underset{\beta}{\otimes} I_N$. The lift $\tilde{\phi}$ is an eigenfunction for $\tilde{P} \otimes I_N$ on $\tilde{M}$, so in particular each of the component functions $\tilde{\phi}^l$ in $\tilde{\phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^N)$ is an eigenfunction of $\tilde{P}$. Thus, the second description of $P \underset{\beta}{\otimes} I_N$ realizes eigenfunctions of this operator as N (generalized) eigenfunctions for the lift of P to $\tilde{M}$.

As the bundle E is considered as part of the "fixed" data, we can without loss reduce to the case where P and Q act on $C^\infty(M)$ via the method of (page 45, [47]).

For $f \in C^\infty(M)$, let

$$|f|_{\infty,0} = \sup_{x \in M} |f(x)|$$

$$|f|_d \quad = d - \text{Sobolev } L^2\text{-norm}$$

For our choice of d, we have

Sobolev Lemma. *There exists a constant c_1 such that for all $f \in C^\infty(M)$,*

$$|f|_{\infty,0} \leqq c_1 \cdot |f|_d . \tag{C4}$$

From general elliptic theory we obtain a constant c_2 such that for $f \in C^\infty(M)$,

$$|Qf|_d \leqq c_2 \cdot |f|_0 \tag{C5}$$

$$|(QP - \mathrm{Id})f|_d \leqq c_2 \cdot |f|_0 \tag{C6}$$

where we use that $(QP - \mathrm{Id})$ and Q are ψDO's with order $\leqq d$. Finally, we need:

Gårdings Inequality. *There exists a constant c_3 such that for all $f \in C^\infty(M)$*

$$|f|_0 + |Pf|_0 \leqq c_3 \cdot |f|_d \,. \tag{C7}$$

Also note that (C5) and (C6) yield a converse estimate to (C7),

$$\begin{aligned} |f|_d &\leqq |(QP - \mathrm{Id})f|_d + |QPf|_d \\ &\leqq c_2 \cdot (|f|_0 + |Pf|_0) \,. \end{aligned} \tag{C8}$$

With these preliminaries, we can now give the proof of the proposition. Fix $a \geqq 0$, and let $\{\phi_1, \ldots, \phi_{n(a)}\}$ be an orthonormal set in $C^\infty(M, E(\beta))$ which are eigenfunctions for $P \underset{\beta}{\otimes} I_N$ having eigenvalues

$$-a \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_{n(a)} \leqq a \,.$$

For $f \in \mathrm{Span}\{\phi_1, \ldots, \phi_{n(a)}\}$, the estimates (C4) to (C8) yield

$$|f|_{\infty,0} \leqq c_1 \cdot |f|_d \leqq c_4(1+a)|f|_0 \tag{C9}$$

where $c_4 = c_1 c_2$. For any set of constants $\{c_j^l \mid 1 \leqq j \leqq n(a)\}$, where the dependence on l will be made explicit in a moment, (C9) implies

$$\begin{aligned} \left\| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j(x) \right\| &\leqq \left| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j \right|_{\infty,0} \\ &\leqq c_4(1+a) \left| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j \right|_0 \\ &= c_4(1+a) \left[\sum_{j=1}^{n(a)} (c_j^l)^2 \right]^{1/2} . \end{aligned} \tag{C10}$$

Let $x \in U_i$, and use the Hermitian trivialization of $E(\beta)|_{U_i}$ to express in coordinates

$$\phi_j = (\phi_j^1, \ldots, \phi_j^N) \,.$$

As the local trivialization is fiberwise Hermitian, we obtain by (C10) the pointwise estimate, for each $1 \leqq l \leqq N$,

$$\left| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j^l(x) \right| \leqq c_4(1+a) \left[\sum_{j=1}^{n(a)} (c_j^l)^2 \right]^{1/2} . \tag{C11}$$

Fix x and set $c_j^l = \overline{\phi_j^l(x)}$, so that (C11) yields the pointwise estimate

$$\sum_{j=1}^{n(a)} |\phi_j^l(x)|^2 \leqq c_4(1+a) \cdot \left[\sum_{j=1}^{n(a)} |\phi_j^l(x)|^2 \right]^{1/2} \tag{C12}$$

hence

$$\sum_{j=1}^{n(a)} |\phi_j^l(x)|^2 \leqq c_4^2(1+a)^2 \,. \tag{C13}$$

Now sum (C13) over l and integrate over x to obtain

$$n(a)=\int_M \sum_{j=1}^{n(a)} \|\phi_j(x)\|^2 \leqq N\cdot c_4^2(1+a)^2\cdot \mathrm{vol}(M). \tag{C14}$$

Substitute $a=\lambda^{m+1}$ to obtain

$$E(D,\beta,\lambda)=N\cdot c(D)\cdot(1+\lambda)^{2d}$$

for a new constant $c(D)$ depending only on c_4, $\mathrm{vol}(M)$ and the dimension m of M. □

We draw two corollaries of the estimate (C2). First, introduce the "estimated eigenvalues" for $D\underset{\beta}{\otimes} I_N$:

$$\begin{cases}\lambda_n^*=f_N^{-1}(n), & n>N\cdot c(D)\\ \lambda_n^*=0 \text{ for } 0\leqq n\leqq N\cdot c(D)\end{cases} \tag{C15}$$

where $f_N(x)=N\cdot c(D)\cdot(1+x)^{2d}$.

Lemma C2. *Order the eigenvalues of* $D\underset{\beta}{\otimes} I_N$:

$$\leqq\lambda_{-2}\leqq\lambda_{-1}<0\leqq\lambda_0\leqq\lambda_1\leqq\dots.$$

Then

$$\begin{cases}\lambda_n\geqq\lambda_n^*, & n=0,1,2,\dots\\ \lambda_{-n}\leqq-\lambda_n^*, & n=1,2,\dots\end{cases} \tag{C16}$$

Proof. The function $f_N(x)$ is monotone increasing for $x\geqq 0$, so that (C16) for $n>N\cdot c(D)$ follows from the estimate based on (C2),

$$n\leqq E(D,\beta,\lambda_n)\leqq f_N(\lambda_n).$$

For $n\leqq -N\cdot c(D)$ a similar estimate holds. (C16) is evident for $|n|\leqq N\cdot c(D)$. □

Taking $N=1$, we define

$$\begin{cases}\mu_n=f_1^{-1}(n), & n>c(D)\\ \mu_n=0, & n\leqq c(D).\end{cases} \tag{C17}$$

Introduce the heat operator $\exp\left(-t\cdot\left\{D\underset{\beta}{\otimes} I_N\right\}^2\right)$ on the Hilbert space $\mathscr{H}_\beta=L^2(E\otimes E(\beta))$.

Corollary C3. *For any representation* $\beta:\Gamma\to U_N$,

$$\mathrm{Tr}_{\mathscr{H}_\beta}\left\{\exp\left(-t\cdot\left\{D\underset{\beta}{\otimes} I_N\right\}^2\right)\right\}\leqq 2N\cdot\sum_{n=0}^{\infty} e^{-t(\mu_n)^2}. \tag{C18}$$

Proof. The left-hand side of (C18) is estimated by

$$\sum_{n\in Z} e^{-t\lambda_n^2}\leqq 2\cdot\sum_{n=0}^{\infty} e^{-t(\lambda_n^*)^2}, \tag{C19}$$

so that (C18) follows from (C19) and the observations that $f_1(x)$ is monotone for $x>0$, and

$$\lambda_n^* = f_1^{-1}(n/N) \quad \text{for} \quad n > N \cdot c(D). \quad \square$$

The estimate (C18) is a weak (global) form of Kato's Inequality for the heat operators of the ψDO's $D \underset{\beta}{\otimes} I_N$. The standard Kato Inequality compares the pointwise trace of the kernel for $\exp\left(-t \cdot \left\{D \underset{\beta}{\otimes} I_N\right\}^2\right)$ with $c \cdot N \cdot \mathrm{Tr}_x \exp(-tD^2)$, where the trace is pointwise and c is a geometric constant. Integrating the pointwise trace over M yields (C18), as the $\{\mu_n\}$ dominate the eigenvalues of D. The pointwise estimate depends upon the existence of a "Weizenbock Formula" for $D \underset{\beta}{\otimes} I_N$, which requires that D be a geometric operator. The point of the corollary is that more standard elliptic estimates suffice to prove the weaker global estimate (C18).

The proof of the Cheeger-Gromov estimate (7.19) depends upon two estimates. The first is a pointwise estimate for the heat kernel for small t. As discussed in (§2, [71]), this is a consequence of the Bismut-Freed condition

$$\mathrm{Tr}_{\mathscr{H}_\beta} \exp\left(-t \cdot \left\{D \underset{\beta}{\otimes} I_N\right\}\right) = o(t^{1/2}) \tag{C20}$$

for $t \to 0$, and the uniformity of the coefficients in an asymptotic expansion. The second is an estimate of the trace of the heat kernel of $D \underset{\beta}{\otimes} I_N$ for $t \geqq 1$ which is linear in N. This second estimate can be derived from Kato's inequality, as in [71], or in the context of flat bundles, where we apply the Cheeger estimate, from (C18).

We next derive the second Corollary of Proposition C1, which was cited in Sect. 8. Recall that D_V is the densely-defined self-adjoint operator on $L^2(V, E)$ considered in Sect. 8. For each $\chi \in \hat{G}$, convolution with the kernel k_χ^* defines projection onto the χ-subspace of $L^2(V, E)$. Moreover, it was proven in Proposition 7.5 that the restriction $D_V \circ k_\chi^*$ is unitarily conjugate to $D \underset{\beta}{\otimes} I_N$ where $\beta = \lambda_\chi \circ \alpha$. For a bounded measurable real function ψ on $\mathbf{R}$, the analysis of $\psi(D_B) \circ k_\chi^*$ thus reduces to that of $\psi\left(D_M \underset{\beta}{\otimes} I_N\right)$.

Introduce the functions

$$\mathrm{sign}(x) = \begin{cases} +1, & x \geqq 0 \\ -1, & x < 0 \end{cases}$$

$$\varphi(x) = \begin{cases} +1, & x \geqq 1 \\ -1, & x \leqq -1 \\ & \text{monotone increasing} \\ & \text{for } -1 < x < 1 \end{cases}$$

$$\varphi_\varepsilon(t) = \varphi(t/\varepsilon)$$

$$\psi(\xi) = \begin{cases} 1, & \xi \leqq 1/2 \\ 0, & \xi \geqq 1 \\ & \text{monotone decreasing} \\ & \text{for } 1/2 < \xi < 1 \end{cases}$$