Werk

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group. The Peter-Weyl theorem yields a sequence of finite-rank *smooth* projections in $L^2(G)$ which converge weakly to the identity. Renormalization can be effected via these kernels, with slight modifications of the above arguments. The main differences arise first from replacing (6.4) with the projection law, and second the Følner condition (6.5) must be proven using fairly tedious representation theory of a compact group. For the circle case, $G=S^1$, this method of renormalization was carried out in [43], and can be viewed as approximating the longitudinal operator $\tilde{c}_\Pi(f_0,\ldots,f_q)$ of (3.23) by its finite Fourier sums in the S^1-parameter. The group methods thus correspond to lattice renormalization, while the approach of this section via an F.A.I. corresponds to heat kernel regularization. These ideas appear also in the recent works [33] and [57, 58].

7. Eta distributions

In this section we introduce the eta-distribution

$$\eta(D_M,\alpha): R_\infty(G)\to\mathbf{R} \tag{7.1}$$

associated to a geometric operator D_M on the odd-dimensional manifold M and a homomorphism $\alpha:\Gamma\to G$. We will give two definitions of (7.1); the first is a direct extension of the usual definition of the eta-invariant, and only mildly uses the structure theory of the representation ring of a compact Lie group. The second definition uses the lifted operator D_V acting on sections over the principal bundle V, where D_V is transversally elliptic to the G-action.

The main new result of this section is the observation that the eta-distribution is *tempered*. That is, there is a constant $c(D_M)$ such that for a smooth class function $\psi\in R_\infty(G)$,

$$|\eta(D_M,\alpha,\psi)|\leqq c(D_M)\cdot\sum_{\chi\in\hat{G}}|a(\psi,\chi)|\cdot N(x)^2 \tag{7.2}$$

with the notation explained below. This will follow from results of Cheeger and Gromov.

Distributional eta and zeta-invariants have appeared in the literature in several contexts. For a finite group G, the distributional eta invariant was defined by Atiyah, Patodi and Singer in (Formula (2.13), page 412, [5]) as part of their study of index formula for G-manifolds with boundary. Donnelly proved in [36] the relevant index theorem in this context. Note that for G finite, the manifold V is a finite covering of M so that the lift D_V is also elliptic, hence the analytic subtleties encountered for non-elliptic operators do not arise in their work.

For G a compact connected Lie group, Šubin [86] and Smagin and Šubin [84] introduced a distributional zeta function for D_V, which is now only G-transversally elliptic. Via the methods of Wodzicki [89], this can be used to define the distributional eta-invariant. Finally, Donnelly in [37] studied the multiplicative property of eta-invariants for the fibered manifolds $V_\alpha=\tilde{M}\times_\Gamma X$. The relation between the eta-invariants of V_α and the eta-distributions we define is a topic for further exploration.

For the rest of this section we fix a geometric operator D_M acting on sections $C^\infty(E_M)$, in the sense of [49]. We assume that M has odd dimension equal m. The Riemannian metric on TM is that obtained from the principal symbol of D_M, so that D_M has the customary nice properties (cf. § 1, [49]). In particular, D_M is symmetric

and has a unique self-adjoint closure in $L^2(E_M)$. The compact group G and homomorphism α are also fixed.

There is a unique lift of D_M to a leafwise operator D along the leaves of the foliation $\mathcal{F}_\alpha$ on the principal bundle $\pi: V \to M$. As before, we let D_V denote the first-order differential operator acting on sections of

$$E = \pi^*(E_M) \to V.$$

Using the bi-invariant Haar measure on G, we obtain a Riemannian volume form on V as in Sect. 2, and use this to give $C^\infty(E)$ an inner product. The lifted operator D_V is symmetric on $C^\infty(E)$, and it is shown in (Chap. 5, [56], cf. [22]) that D_V has a unique self-adjoint closure in $L^2(E)$. In fact, there is a more precise result.

Proposition 7.1. *The operator D_V has real pure-point spectrum* (p.p.s.), *and for each eigenvalue $\lambda \in \mathbf{R}$, the corresponding eigenspace $H_\lambda \subset L^2(V)$ is G-invariant with each irreducible character $\chi \in \hat{G}$ having finite multiplicity in H_λ.*

Proof. Let Δ_G denote the G-Laplacian acting along fibers of $\pi: V \to M$ constructed in Sect. 6 for $X = G$. There is a natural extension of Δ_G to a second-order operator on $C^\infty(E)$. The operators D_V^2 and Δ_G commute, and their sum $\Delta_G + D_V^2$ is a symmetric, elliptic second order operator on $C^\infty(E)$ which commutes with D_V. As $\Delta_G + D_V^2$ has real p.p.s. and D_V has a unique self-adjoint extension, this implies the same for D_V. The standard estimates on the growth rate of the eigenvalues (counted with multiplicity) of $\Delta_G + D_V^2$ implies that the multiplicity of $\chi \in \hat{G}$ in H_λ is finite. (Estimates on the growth rate of $\dim(H_\chi)$ can also be obtained, as in Sect. 2 of [1].) □

Let $\{\lambda_i\}$ be the set of eigenvalues of D_M, listed with multiplicity. The eta-function of D_M is the sum

$$\eta(D_M, s) = \sum_{\lambda_i > 0} \lambda_i^{-s} - \sum_{\lambda_i < 0} (-\lambda_i)^{-s} + \dim(\ker D_M). \tag{7.3}$$

By general facts from [4], this function is holomorphic in $s \in \mathbf{C}$ for $\mathrm{Re}(s) > m$, and admits a meromorphic extension to all of $\mathbf{C}$ with isolated simple poles at $s = m - l$, for natural numbers $l \geqq 0$. Moreover, for a geometric operator D_M, $\eta(D_M, s)$ is holomorphic for $\mathrm{Re}(s) > -2$ due to supersymmetric cancellations (cf. [13]). We define

$$\eta(D_M) = \eta(D_M, 0). \tag{7.4}$$

Given a unitary representation $\varrho: G \to U_N$, let $E(\varrho \circ \alpha) \to M$ be the Hermitian flat $\mathbf{C}^N$-bundle associated to the flat principal bundle $P(\varrho \circ \alpha)$ of (4.17). The differential operator D_M has a canonical extension to $C^\infty(E \otimes E(\varrho \circ \alpha))$ using the flat structure, which will be denoted by $D_M \otimes \nabla^{\varrho \circ \alpha}$. This extension is again symmetric and of geometric type, so it has a well-defined eta-invariant. We set

$$\eta(D_M, \alpha, \varrho) = \eta(D_M \otimes \nabla^{\varrho \circ \alpha}). \tag{7.5}$$

Introduce the vector subspace $Cl_f(G) \subset C^\infty(G)$ whose elements are finite linear combinations of characters of finite dimensional representations ϱ. Let $\hat{G}$ denote the set of irreducible representations of G, indexed by their characters χ. The dimension of the representation space $V(\chi)$ for χ will be denoted by $N(\chi)$. A typical element

$\psi \in Cl_f(G)$ can then be written

$$\psi = \sum_{\chi \in G} N(\chi) \cdot a(\psi, \chi) \cdot \chi \tag{7.6}$$

where at most finitely many of the "Fourier coefficients" $a(\psi, \chi)$ are non-zero. Note that we scale χ by $N(\chi)$ in (7.6), as the basic characters for this work will be those for the isotypical summands of the right regular representation of G on $L^2(G)$.

Recall that the left and right regular representations of G on $L^2(G)$ denoted by λ and ϱ, respectively, are given by

$$\begin{aligned} &\lambda(g)(f)(h) = f(g^{-1}h) \\ &\varrho(g)(f)(h) = f(hg) \end{aligned} \qquad \text{for } f \in L^2(G) \text{ and } g \in G. \tag{7.7}$$

If we consider $V(\chi)$ as a right G-space, then the dual $V(\chi)^* = \mathrm{Hom}(V(\chi), \mathbf{C})$ is a left G-space and $M(\chi) \equiv V(\chi)^* \otimes V(\chi)$ is a $G \times G$-module. By the Peter-Weyl Theorem, there is a $G \times G$-invariant subspace $L^2(G, \chi) \subset L^2(G)$ and an isomorphism of $G \times G$-modules

$$M(\chi) \cong L^2(G, \chi). \tag{7.8}$$

Let λ_χ and ϱ_χ denote the restrictions of λ and ϱ to $L^2(G, \chi)$, which intertwine via (7.8) with the fixed left and right G-actions on $M(\chi)$. Observe then that

$$\begin{aligned} \mathrm{Tr}(\varrho_\chi) &= N(\chi) \cdot \chi \\ \mathrm{Tr}(\lambda_\chi) &= N(\chi) \cdot \bar{\chi}. \end{aligned} \tag{7.9}$$

The $G \times G$-module $L^2(G)$ admits a Hilbert space decomposition into irreducible summands

$$L^2(G) \cong \hat{\bigotimes_{\chi}} L^2(G, \chi). \tag{7.10}$$

The orthogonal projection P_χ onto the summand $L^2(G, \chi)$ is given by convolution with $N(\chi) \cdot \chi$; that is, for $f \in L^2(G)$

$$P_\chi(f)(h) = f_\chi(h) = \int_G N(\chi) \cdot \chi(hg^{-1}) f(g) dg. \tag{7.11}$$

Note that χ is a smooth function, so that each $f \in L^2(G, \chi)$ is smooth on G.

We introduce the ring of central smooth functions on G,

$$R_\infty(G) = \{\psi \in C^\infty(G) | \psi(gh) = \psi(hg) \text{ all } h, g \in G\}. \tag{7.12}$$

Clearly, $Cl_f(G) \subset R_\infty(G)$ and the Peter-Weyl Theorem implies that each $\psi \in R_\infty(G)$ can be written as an infinite sum

$$\psi = \sum_{\chi} a(\psi, \chi) N(\chi) \cdot \chi \tag{7.13}$$

where the $|a(\psi, \chi)|$ are rapidly decreasing as a function of the weight of the representation χ. We let $w(\chi)$ denote the weight of χ.

For each $\psi \in R_\infty(G)$, define a kernel

$$k_\psi(h, g) = \psi(hg^{-1}), \tag{7.14}$$

with corresponding convolution operator on $L^2(G)$:

$$f_\psi(h)=k_\psi * f(h)=\int_G \psi(hg^{-1})f(g)dg\,. \tag{7.15}$$

Combining (7.11), (7.13) and (7.15) we have:

Lemma 7.2. *For $\psi\in R_\infty(G)$, $f\in L^2(G)$*

$$k_\psi * f=\sum_\chi a(\psi,\chi)\cdot P_\chi(f)\,. \quad \square \tag{7.16}$$

After these preliminaries, we can now introduce the eta-distribution. For $\psi\in Cl_f(G)$, define via (7.5):

$$\eta(D_M,\alpha,\psi)=\sum_\chi a(\psi,\chi)\cdot\eta(D_M,\alpha,\lambda_\chi)\,. \tag{7.17}$$

The following important property of the functional $\eta(D_M,\alpha)$ on $Cl_f(G)$ defined by (7.17) is a direct consequence of estimates of Cheeger and Gromov:

Proposition 7.3. *There exists a constant $c(D_M)$ such that (7.2) holds for all $\psi\in Cl_f(G)$. In particular, if each $a(\chi,\psi)$ is non-negative real, then*

$$|\eta(D_M,\alpha,\psi)|\leqq c(D_M)\psi(e)\,. \tag{7.18}$$

Proof. For the case when D_M is the signature operator $d*-*d$, Theorem 1.2 of [17] shows that there exists a constant $c(D_M)$ such that for each representation $\beta:\Gamma\to U_N$,

$$|\eta(D_M\otimes\nabla^\beta)|\leqq c(D_M)\cdot N\,. \tag{7.19}$$

Moreover, Remark 4.1 of [18] asserts that (7.19) holds for D_M an arbitrary geometric operator. A proof of this estimate for D_M a geometric operator is given more recently by Ramachandran [71]. Clearly (7.2) is a consequence of (7.19), and then (7.18) follows from noting that $\mathrm{Tr}(\lambda_\chi(e))=N(\chi)\cdot\chi(e)=N(\chi)^2$. $\square$

Corollary 7.4. *The functional $\eta(D_M,\alpha)$ extends to $R_\infty(G)$.*

Proof. There exists constants c_{10}, $r>0$ so that the function

$$\mu(N)=\#\{\chi\in\tilde{G}\,|\,w(\chi)\leqq N\}\leqq c_{10}\cdot N^r\,.$$

It then follows that the sum (7.17) is uniformly convergent for $|a(\psi,\chi)|$ rapidly decreasing in $w(\chi)$ by the estimate (7.2). $\square$

The functional $\eta(D_M,\alpha)$ on $R_\infty(G)$ can be reformulated in terms of the spectral theory of D_V acting on $C^\infty(E)$. It is this alternate formulation of $\eta(D_M,\alpha)$ that is used to identify the renormalized cyclic Chern character of Sect. 6 with a spectral invariant of D_M.

The Hilbert space $L^2(E)$ is a right G-module, and there is an orthogonal decomposition into G-spaces

$$L^2(E)\cong\hat{\bigoplus_\chi} L^2(E,\chi) \tag{7.20}$$

where $L^2(E,\chi)$ is the closed subspace which transforms by the right representation ϱ_χ. Let

$$e(\chi):L^2(E)\to L^2(E,\chi) \tag{7.21}$$

be the orthogonal projection, which is defined by convolution (5.5) with the smooth kernel $N(\chi)\cdot k_\chi^* \in C^\infty(\pi)$,

$$k_\chi^*(y, y\cdot g)=\chi(g^{-1}); \quad y\in V, \quad g\in G. \tag{7.22}$$

The operator D_V is G-invariant, thus its closure commutes with each projection $e(\chi)$, and the compressions $D(\chi)=D_V\circ e(\chi)$ are densely defined on $L^2(E,\chi)$.

Proposition 7.5. *For each $\chi\in\hat{G}$, there is a natural isomorphism*

$$L^2(E,\chi)\cong L^2(E_M\otimes E(\lambda_\chi\circ\alpha)) \tag{7.23}$$

which identifies $D(\chi)$ with $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}$.

Proof. The quotient map $\tilde{M}\times G\to V$ identifies pairs $(\tilde{x},g)\sim(\tilde{x}\circ\gamma^{-1},\alpha(\gamma)\cdot g)$ for $\gamma\in\Gamma$, and commutes with the right G-action. Let $\tilde{E}_M\to\tilde{M}$ be the lift of E_M, and $\tilde{E}\to\tilde{M}\times G$ the lift of E. There is an isomorphism of right G-modules

$$\begin{aligned} C^\infty(\tilde{E},\chi)&\cong C^\infty(\tilde{E}_M)\otimes L^2(G,\chi)\\ &\cong C^\infty(\tilde{E}_M)\otimes M(\chi). \end{aligned} \tag{7.24}$$

For a smooth section $s\in C^\infty(E,\chi)$, the lift $\tilde{s}\in C^\infty(\tilde{E},\chi)$ satisfies

$$\begin{aligned} \tilde{s}(\tilde{x}\cdot\gamma,g)&=\tilde{s}(\tilde{x},\alpha(\gamma)g)\\ &=\lambda_\chi\circ\alpha(\gamma^{-1})\cdot\tilde{s}(\tilde{x},g) \end{aligned} \tag{7.25}$$

via the first isomorphism of (7.24). A section of the flat bundle $E(\lambda_\chi\circ\alpha)\to M$ is equivalent to a map $\psi:\tilde{M}\to\mathbf{C}^{N(\chi)}$ such that

$$\psi(\tilde{x}\cdot\gamma)=\lambda_\chi\circ\alpha(\gamma^{-1})\cdot\psi(\tilde{x}),$$

since we require that

$$(\tilde{x}\cdot\gamma,\psi(\tilde{x}\cdot\gamma))\sim(\tilde{x},\lambda_\chi(\gamma)\cdot\psi(\tilde{x}\cdot\gamma))=(\tilde{x},\psi(\tilde{x})).$$

By the second isomorphism of (7.24), we can thus identify $\tilde{s}$ with the lift of a section $\hat{s}\in C^\infty(E_M\otimes E(\lambda_\chi\circ\alpha))$. The linear map $s\to\hat{s}$ is an isomorphism of smooth section spaces

$$C^\infty(E,\chi)\cong C^\infty(E_M\otimes E(\lambda_\chi\circ\alpha)), \tag{7.26}$$

which extends to the isometry (7.23) as (7.8) and (7.10) are isometries.

The isomorphism (7.26) becomes an identification when lifted to $\tilde{M}\times G$, so to show $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}$ and $D(\chi)$ agree under (7.26), it will suffices to show they agree on $\tilde{M}\times G$. The first operator, $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}$, which is defined via the flat Hermitian connection on $E(\lambda_\chi\circ\alpha)$, lifts to the operator $\tilde{D}_M\otimes I_{M(\chi)}$. Here, $\tilde{D}_M$ is the lift of D_M to $C^\infty(\tilde{E}_M)$, then we extend via the identity map on $M(\chi)$.

The operator D_V on $C^\infty(E)$ lifts to $\tilde{D}_V=\tilde{D}_M\otimes I_{C^\infty(G)}$ on $C^\infty(\tilde{E})\cong C^\infty(\tilde{E}_M)\hat{\otimes}C^\infty(G)$. The restriction $D(\chi)$ thus lifts to $\tilde{D}(\chi)=\tilde{D}_M\otimes I_{M(\chi)}$, which concludes the proof. □

The operator D_V has p.p.s. by Proposition 7.1, so for $s\in\mathbf{C}$ define

$$D_V^{-s}=\begin{cases}\lambda^s;\ \lambda>0, & \text{on } \lambda\text{-eigenspace of } D_V\\ -(\lambda)^s;\ \lambda<0, & \text{on } \lambda\text{-eigenspace of } D_V\\ 1; & \text{on nullspace of } D_V.\end{cases} \tag{7.27}$$

Let Tr_E denote the usual Hilbert space trace on $L^2(E)$. An immediate application of Proposition 7.3 yields

Corollary 7.6. *For each* $\chi \in \hat{G}$,

$$\mathrm{Tr}_E(D_V^{-s} \circ e(\chi)) = \eta(D_M \otimes \nabla^{\lambda_\chi \circ \alpha}, s). \tag{7.28}$$

The second formulation of the distributional trace is suggested by formula (7.28). For each $\psi \in R_\infty(G)$, let $k_\psi^* \in C^\infty(\pi)$ be defined by $k_\psi^*(y \cdot g, y) = \psi(g)$. Then set

$$\tilde{\eta}(D_M, \alpha, \psi) = \mathrm{Tr}_E(D_V^{-s} \circ k_\psi^*)|_{s=0}. \tag{7.29}$$

This is well-defined by (7.28) and Proposition 7.3, and by (7.28) we have:

Corollary 7.7. *For* $\psi \in R_\infty(G)$

$$\tilde{\eta}(D_M, \alpha, \psi) = \eta(D_M, \alpha, \psi). \quad \square \tag{7.30}$$

For our applications in Sect. 8, we need to extend the definition of $\eta(D_M, \alpha)$ to include an auxillary flat bundle. Let $\varrho : G \to U_N$ be a representation; then $D_M \otimes \nabla^{\varrho \circ \alpha}$ acting on $C^\infty(E_M \otimes E(\varrho \circ \alpha))$ is again a geometric operator. We can thus define

$$\eta(D_M, \alpha, \varrho, \psi) = \eta(D_M \otimes \nabla^{\varrho \circ \alpha}, \alpha, \psi) \tag{7.31}$$

where the right-hand-side of (7.31) is defined as in (7.17). The estimate (7.18) is correspondingly scaled by the dimension N of $E(\varrho \circ \alpha)$:

Corollary 7.8. *Let* $\psi \in R_\infty(G)$ *have all* $a(\psi, \chi)$ *real and non-negative. Then*

$$|\eta(D_M, \alpha, \varrho, \psi)| \leqq c(D_M) \cdot N \cdot \psi(e). \quad \square \tag{7.32}$$

When the representation α defines a topologically trivial bundle $P(\alpha) = V$, then the choice of a trivializing G-map $\Theta : V \to M \times G$ yields $\bar{\alpha} = (\alpha, \Theta)$. We use the definition (7.3) to define a linear functional $\eta(D_M, \bar{\alpha}, \varrho)$ on $R_\infty(G)$.

The trivialization Θ induces a product structure,

$$\Theta^* : E(\varrho \circ \alpha) \cong M \times \mathbf{C}^N \equiv \varepsilon^N, \tag{7.33}$$

which carries the Hermitian flat connection on $E(\varrho \circ \alpha)$ over to a connection $\overline{\nabla^{\varrho \circ \alpha}}$ on ε^N. The product flat connection on ε^N will be denoted by ∇^0.

The differential operator D_M has two essentially self-adjoint extensions to $C^\infty(E_M \otimes \varepsilon^N)$ defined in conjunction with $\bar{\alpha}$. The first is $D_0 = D_M \otimes \nabla^0$, obtained by using the product structure to extend the coefficients. The second is $D_1 = D_M \otimes \overline{\nabla^{\varrho \circ \alpha}}$, obtained by letting D_M act on the coefficients of parallel sections for $\overline{\nabla^{\varrho \circ \alpha}}$ of $C^\infty(E_M \otimes \varepsilon^N)$. Define a smooth 1-parameter family

$$D_t = t \cdot D_1 + (1-t) D_0 \tag{7.34}$$

of essentially self-adjoint operators on $C^\infty(E_M \otimes \varepsilon^N)$. The 1-parameter family of eta-invariants $\eta(D_t)$ varies smoothly in t, except for a finite number of integer jumps [6]. Thus, there is a well-defined derivative $\dot{\eta}(D_t)$ which depends continuously on t, and moreover $\dot{\eta}(D_t)$ is a local invariant of the family $\{D_t\}$. Define

$$\eta(D_M, \bar{\alpha}, \varrho) = \int_0^1 \dot{\eta}(D_t) dt. \tag{7.35}$$

As discussed in (§6, page 90, [6]) the correspondence

$$(\bar{\alpha}, \varrho) \to \eta(D_M, \bar{\alpha}, \varrho) \tag{7.36}$$

defines a linear functional on the odd-degree, real-valued K-theory of M. It is this topological invariant that is naturally related to the foliation index invariants of Sect. 3.

In the definition (7.35), we can replace D_M with $D_M \otimes \nabla^{\lambda_\chi \circ \alpha}$ acting on $C^\infty(E_M \otimes E(\lambda_\chi \circ \alpha))$, for any $\chi \in \hat{G}$. The resulting family will be denoted D_t^χ, and we set

$$\eta(D_M, \bar{\alpha}, \varrho, \chi) = \int_0^1 \eta(D_t^\chi) dt\,. \tag{7.37}$$

This is a purely formal device by the next result.

Lemma 7.9.

$$\eta(D_M, \bar{\alpha}, \varrho, \chi) = N(\chi)^2 \cdot \eta(D_M, \bar{\alpha}, \varrho)\,. \tag{7.38}$$

Proof. The derivative $\dot{\eta}(D_t)$ is local, so extending the operators D_t to $D_t^\chi = D_t \otimes \nabla^{\lambda_\chi \circ \alpha}$ multiplies this derivative by $\dim E(\lambda_\chi \circ \alpha) = N(\chi)^2$, as $\dot{\eta}(D_t^\chi)$ depends only on the local coefficients of D_t^χ. Integrating then gives (7.38). □

Now extend (7.37) to all of $R_\infty(G)$ by setting

$$\begin{aligned} \eta(D_M, \bar{\alpha}, \varrho, \psi) &= \sum_\chi a(\psi, \chi) \cdot \eta(D_M, \bar{\alpha}, \varrho, \chi) \\ &= \left\{ \sum_\chi N(\chi)^2 \cdot a(\psi, \chi) \right\} \cdot \eta(D_M, \bar{\alpha}, \varrho) \\ &= \psi(e) \cdot \eta(D_M, \bar{\alpha}, \varrho)\,. \end{aligned} \tag{7.39}$$

The functional $\eta(D_M, \bar{\alpha}, \varrho)$ can be reformulated in terms of D_V, and the formal extension (7.39) is then seen to be surprisingly natural. Given a smooth map $u: V \to U_N$, we obtain a multiplication operator on $C^\infty(E \otimes \varepsilon^N)$, denoted again by u, by letting u act pointwise on the left on the coefficients ε^N. Let u^* be the multiplication operator by the pointwise adjoint matrix to u. Extend D_V to $D_V \otimes I_N$ on $C^\infty(E \otimes \varepsilon^N)$, then set

$$D^u = u \circ (D_V \otimes I_N) \circ u^* \tag{7.40}$$

$$D_t^u = t \cdot D^u + (1-t) \cdot D_V \otimes I_N\,. \tag{7.41}$$

The representation ϱ induces a continuous map $u(\varrho)$, defined as the composition

$$V \xrightarrow{\theta} M \times G \longrightarrow G \xrightarrow{\varrho} U_N\,. \tag{7.42}$$

Then set

$$D^\varrho \equiv D^{u(\varrho)}\,; \quad D_t^\varrho = D_t^{u(\varrho)}\,. \tag{7.43}$$

Proposition 7.10. *D^ϱ is right G-invariant, with essentially self-adjoint restrictions $D^\varrho(\chi)$ to the invariant subspaces $C^\infty(E \otimes \varepsilon^N, \chi)$. There is an identification*

$$L^2(E \otimes \varepsilon^N, \chi) \cong L^2(E_M \otimes E(\lambda_\chi \circ \alpha) \otimes \varepsilon^N) \tag{7.44}$$

under which $D^\varrho(\chi)$ is unitarily conjugate to $D_M \otimes \nabla^{\lambda_\chi \circ \alpha} \otimes \nabla^{\overline{\varrho \circ \alpha}}$, hence they have the same spectrum.

Proof. The differential operator $D_V \otimes I_N$ commutes with the constant matrix multiplier $\varrho(g)$ for all $g \in G$, so

$$\begin{aligned} R_g^*(D^\varrho) &= u(\varrho) \cdot \varrho(g) \cdot R_g^*(D_V) \cdot \varrho(g)^* u(\varrho)^* \\ &= u(\varrho) \cdot D_V \cdot \varrho(g)\varrho(g)^* u(\varrho)^* \\ &= D^\varrho . \end{aligned}$$

The projection onto the closed subspace $L^2(E \otimes \varepsilon^N, \chi)$ commutes with D^ϱ so the restrictions are essentially self-adjoint. The isomorphism (7.44) is a consequence of (7.23), as G acts trivially on the coefficients ε^N. Finally, to identify $D^\varrho(\chi)$, we first identify the lift $\tilde{D}^\varrho$ to $C^\infty(\tilde{E} \otimes \varepsilon^N)$ on $\tilde{M} \times G \to V$. Our method will basically be a repeat of the idea of Theorem 4.4.

The lift of Θ to coverings gives a right G-map $\tilde{\Theta}$, commuting with the two Γ-actions in the diagram

(7.45)
$$\begin{array}{ccc} \Gamma & & \Gamma \\ \downarrow & & \downarrow \\ \tilde{M} \times G & \xrightarrow{\tilde{\Theta}} & \tilde{M} \times G \\ \downarrow & & \downarrow \\ V & \xrightarrow{\Theta} & M \times G \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & M & \end{array}$$

The identity $\pi_1 \circ \Theta(x) = \pi(x)$ for all $x \in V$ implies that

$$\tilde{\Theta}(\tilde{x}, g) = (\tilde{x}, \varphi(\tilde{x} \cdot g)) ; \quad (\tilde{x}, g) \in \tilde{M} \times G \tag{7.46}$$

where $\varphi : \tilde{M} \to G$ satisfies

$$\varphi(\tilde{x} \cdot \gamma) = \varphi(\tilde{x}) \cdot \alpha(\gamma) . \tag{7.47}$$

The lift $\tilde{u}(\varrho)$ to $\tilde{M} \times G$ of $u(\varrho) : V \to U_N$ is given by $\tilde{u}(\varrho) = \varrho \circ \pi_2 \circ \tilde{\Theta}$, so that

$$\tilde{u}(\varrho(\tilde{x}, g)) = \varrho(\varphi(\tilde{x})g) = \varrho(\varphi(\tilde{x})) \cdot \varrho(g) . \tag{7.48}$$

Thus, D^ϱ lifts to

$$\tilde{u}(\varrho) \cdot (\tilde{D}_V \otimes I_N) \cdot \tilde{u}(\varrho)^* = \varrho(\varphi(\tilde{x})) \cdot (\tilde{D}_M \otimes I_{C^\infty(G)} \otimes I_N) \cdot \varrho(\varphi(\tilde{x}))^*$$

and the restriction $\tilde{D}^\varrho(\chi)$ is given by

$$\tilde{D}(\chi) = \varrho \circ \varphi \cdot (\tilde{D}_M \otimes I_{M(\tilde{\chi})} \otimes I_N) \cdot \varrho \circ \varphi^* \tag{7.49}$$

acting on $C^\infty(\tilde{E}_M \otimes M(\chi) \otimes \varepsilon^N)$. Introduce an automorphism A of $C^\infty(\tilde{E}_M \otimes M(\chi) \otimes \varepsilon^N)$, defined as pointwise multiplication on the left on ε^N by $\varrho(\varphi(\tilde{x}))$. Thus

$$\tilde{D}^\varrho(\chi) = A \circ (\tilde{D}_M \otimes I_{M(\chi)} \otimes I_N) \circ A^* \tag{7.50}$$

induces a unitary conjugation between $\tilde{D}^\varrho(\chi)$ and a Γ-invariant operator acting on sections of the form $A^* \tilde{s}$, $\tilde{s} \in C^\infty(\tilde{E}_M \otimes M(\chi) \otimes \varepsilon^N)$. A section $A^* \tilde{s}$ transforms under