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On the vanishing of local cohomology modules

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1. Introduction

Let A be a commutative ring, and let I be an ideal of A. The *cohomological dimension of I in A*, denoted by $cd(A, I)$, is the smallest integer n such that the local cohomology modules $H^q_I(M) = 0$ for all A-modules M, and for all $q > n$. (Actually if $H^q_I(A) = 0$ for all $q > n$, then $cd(A, I) \leqq n$, cf. [Hal, Sec. 1].) If no such n exists, we set $cd(A, I) = \infty$. However, if A is Noetherian of finite Krull dimension d, then $cd(A, I) \leqq d$; a well-known result due to Grothendieck [Gr1]. Another important vanishing result is that if I is defined set-theoretically by t equations, then $cd(A, I) \leqq t$; in particular $cd(A, I)$ provides a lower bound for the number of elements needed to define I set-theoretically.

The cohomological dimension has been studied by several authors; for example see Faltings [Fa1], Grothendieck [Gr1], Hartshorne [Ha1], [Ha2], Hartshorne and Speiser [H-S], Ogus [O], Peskine and Szpiro [P-S], and [S]. In this paper we will extend results from all of these articles. The main thrust of this paper is to give bounds on $cd(A, I)$, especially when I has small height (i.e., $\mathrm{Spec}(A/I)$ has small codimension) compared to the dimension of A.

Hartshorne [Ha1] proved an important result, widely known as the Hartshorne–Lichtenbaum vanishing theorem. This theorem basically states that if A is a complete local Noetherian domain, and I is an ideal of A then $\dim(A/I) \geqq 1$ iff $cd(A, I) < \dim(A)$. This theorem plays a crucial role in our proofs. In the same paper, Hartshorne proves a vanishing theorem for the cohomological dimension of the complement of a subvariety of projective space, which was later generalized by Ogus [O, Cor. 2.11] in equicharacteristic 0, and by Peskine and Szpiro [P-S, III, 5.5] in equicharacteristic $p > 0$. Combining their results gives:

Theorem 1.1. *Let (A, m) be a regular local ring of dimension d, and let I be an ideal of A. Assume that A is complete and has a separably closed residue field k which it*

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contains. Then the following are equivalent:

(*i*) $cd(A, I) \leqq d - 2$.
(*ii*) $\dim(A/I) \geqq 2$, *and* $\operatorname{Spec}(A/I) - \{m\}$ *is connected.*

Of course one can remove the assumption that A is complete with separably closed residue field if one assumes in (ii) that $\operatorname{Spec}(A/I) - \{m\}$ is "formally geometrically connected", which in essence reduces the problem to (ii) of Theorem 1.1.

Ogus proves this theorem in equicharacteristic 0 by using the theory of algebraic De Rham cohomology developed by Grothendieck and Hartshorne, while Peskine and Szpiro use the action of Frobenius in characteristic $p > 0$.

We are able (Theorem 2.9) to give a new proof of this result, independent of characteristic. The proof of Theorem 2.9 is based upon an induction theorem (2.5) which generalizes a result of Faltings [Fa1, Satz 1]. The need to have k separably closed in both Theorem A and in our later work necessitates some work, which discusses the splitting of primes in power series rings when the coefficient field is extended. Although we need this work in Sect. 2, we postpone the proofs until Sect. 4, since they are independent of the rest of this paper. Our main theorem in Sect. 4 is Theorem 4.2, which says that if a prime P in a formal power series ring with coefficients in a field k splits into t irreducible components in the power series ring over a field L containing k, then there is a finite separable extension K of k, contained in L, such that P already splits into t irreducible components in the power series ring with coefficients in K. This result generalizes some of the results of Chevalley's classic paper on power series rings [Ch].

The chief use of our induction theorem, however, is not to give a simpler and unified proof of Theorem 1.1, but is rather to give improved bounds for $cd(A, I)$ for arbitrary Noetherian rings A and ideals I of A.

Faltings [Fa1, Korollar 2] gave the most general vanishing result for ideals I in Noetherian rings A. Note that in general $cd(A, I) = \sup\{cd(A_P, I_P)\colon P \in \operatorname{Spec}(A)\}$, so there is no loss of generality in assuming A is local, which we will normally do throughout the paper. Let (A, m) be a local ring. In [Fa1] there is a bound for $cd(A, I)$ which in part depends upon the embedding dimension of A. Since this theorem, as well as many of the results in this paper are somewhat technical in nature (especially in the case that the base ring A is non-regular), we will restrict ourselves in this introduction to the case where A is a regular local ring.

Accordingly, let (A, m) be a regular local ring of dimension d, and let I be an ideal with $\operatorname{bight}(I) = b$. (Recall the *big height of* $I = \max\{ht(P)\colon P$ is a minimal prime of $I\}$.) Faltings shows that $cd(A, I) \leqq d - [(d - 1)/b]$. In general this is the best possible result without additional assumptions (see [Ly1]). We are able to improve this result in several directions. We show in (5.1) that:

(0.1) If I is formally geometrically irreducible, $cd(A, I) \leqq d - 1 - [(d - 2)/b]$, and this is best possible for all b, d, with $0 < b < d$.

and,

(0.2) If A/I is normal, $cd(A, I) \leqq d - [d/(b + 1)] - [(d - 1)/(b + 1)]$.

These theorems are based on work in Sect. 3. We also give a bound on $cd(A, I)$ if one assumes that there is an integer t such that I_P is generated up to radical by t elements for all primes P with $\dim(A/P) \geqq 3$. For instance if one knows that I is locally generated up to radical by t equations on the punctured spectrum, then

these results are applicable. We refer the reader to Theorem 3.10 for a precise statement of this bound.

In Sect. 5 we specialize our work to regular local rings, give some examples, and try to extend Theorem 1.1 to the case where $cd(A, I) \leqq \dim(A) - 3$. Let (A, m) be a regular local ring containing a field. We are able to show under the assumption that $2 \operatorname{bight}(I) + 2 \leqq \dim(A)$, and provided no three minimal primes of I add to an m-primary ideal, then $cd(A, I) \leqq \dim(A) - 3$. (The best previously known result in this direction was that provided $3 \operatorname{bight}(I) < \dim(A)$, then $cd(A, I) \leqq \dim(A) - 3$.) This condition is not necessary; however we give necessary and sufficient conditions for $cd(A, I) \leqq \dim(A) - 3$ in the case where the number of minimal primes of I is at most 5 (under the condition that $2 \operatorname{bight}(I) + 2 \leqq \dim(A)$). This work generalizes in this case the theorem of Faltings, and other results in this paper.

Finally in Sect. 6 we apply our theorems to give vanishing results for the relative singular homology groups with complex coefficients of subvarieties of projective space. The fact that there is a relationship between the vanishing of local cohomology and the vanishing of relative singular homology was developed by Grothendieck and Hartshorne (see [Ha3]). For example, we show that if $Y^d \subseteq \mathbb{P}^n_{\mathbb{C}}$ is irreducible then $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$ for $i \leqq [(n-1)/(n-\mathrm{d})]$. If Y is in addition normal, then $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$ for $i \leqq [(n+1)/(n-d+1)] + [n/(n-d+1)] - 1$ (cf. Theorem 6.2). If Y is locally set-theoretically defined by $n-d$ equations, except on a Zariski closed set of dimension at most 1, then $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$ for $i \leqq 2d - n + 1$, a generalization of the well-known result of Barth [Ba]. A little more work shows that in all of the above cases $\pi_i(\mathbb{P}^n, Y)$ are finite in the same range as $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$.

2. An induction theorem

Let (A, m) be a local ring containing a field, and let I be an ideal of A. We denote the completion of A by $A^\wedge$, and by B we denote the ring obtained from $A^\wedge$ by separably closing its residue field and completing. Specifically, set $B = ((A^\wedge)^{sh})^\wedge$, the completion of the strict Henselization of $A^\wedge$. We note that B is a faithfully flat extension of A.

If M is any finitely generated A-module, set $N = M \otimes_A B$. Since B is faithfully flat over A, we have that $H^q_I(M) \otimes_A B \cong H^q_{IB}(N)$, and $H^q_I(M) = 0$ if and only if $H^q_{IB}(N) = 0$. Thus, in deciding whether $H^q_I(M)$ is zero, it suffices to consider these modules when the ring A is complete with separably closed residue field. In fact, it is necessary to incorporate this case in any assumptions made concerning the ideal I; e.g., certain vanishing results on the local cohomology of I are true if I remains prime after passing to B, but it is not enough to assume I is prime in A to obtain the same results.

Our first result is a sharpening of [Fa1, Satz1]. It is rather technical, but supplies us with the necessary inductive tool to prove our main results. The main improvements over the result of [Fa1] come from the use of the Hartshorne–Lichtenbaum local vanishing theorem [Hal, Thm. 3.1] (which we refer to from now on as 'HLVT'). First, however, we need a couple of lemmas.

Definition 2.1. *Let A be a local ring. By* $\operatorname{mdim}(A)$ *we denote* $\min\{\dim(A/Q) | Q$ *is a minimal prime of* $A\}$. *If I is an ideal of A, we let* $c(I) = \operatorname{embdim}(A) - \operatorname{mdim}(A/IA)$.

Lemma 2.2. *Suppose that A is a universally catenary local ring containing a field, and*

I is an ideal of A. Then $\mathrm{mdim}(A) = \mathrm{mdim}(A^\wedge)$, *and* $c(I) = c(IA^\wedge)$. *If B is defined as above, then* $c(I) = c(IB)$.

Recall that a ring A is universally catenary if it is Noetherian and any A-algebra of finite type is catenary [Mat, p. 84]. Also if a local domain A is universally catenary, then $\hat{A}$ is equidimensional. [Ra]

Proof. Since $\mathrm{embdim}(A) = \mathrm{embdim}(A^\wedge)$, to prove that $c(I) = (IA^\wedge)$, it clearly suffices to prove the first statement of the lemma. (Note that A universally catenary implies that A/I is also.) Choose any minimal prime ideal P of A. Since A/P is a universally catenary domain, if follows that if Q is any minimal prime over $PA^\wedge$, then $\dim(A/P) = \dim(A^\wedge/PA^\wedge) = \dim(A^\wedge/Q)$. As minimal primes of $A^\wedge$ contract to minimal primes of A, it follows that $\mathrm{md\,m}(A^\wedge) = \mathrm{mdim}(A)$, as required. For the last statement it suffices to assume that $A = k[[x_1, \ldots, x_n]]$, $B = k^{\mathrm{sep}}[[x_1, \ldots, x_n]]$, and I is a prime ideal of A. Then $c(I) = ht(I)$, and $c(IB) = \mathrm{bight}(IB) = ht(I)$ by [Ch, Prop. 9a]. □

Lemma 2.3. *Let A be a universally catenary local ring, and suppose that P is a prime ideal of A and I is an ideal of A. Then,*

i) $\dim(A/P) + \mathrm{mdim}(A_P) \geqq \mathrm{mdim}(A)$.
ii) $c(I) \geqq c(I_P)$, *if P contains I.*

Proof. Choose a prime ideal Q', minimal in A and contained in P, such that $\mathrm{mdim}(A_P) = \dim(A_P/Q'A_P)$. Since A is catenary, we obtain that $\dim(A/P) + \mathrm{mdim}(A_P) = \dim(A/Q') \geqq \mathrm{mdim}(A)$ which proves (i).

By [V, Thm. 2.2], $\mathrm{embdim}(A) - \mathrm{embdim}(A_P) \geqq \dim(A/P)$. Hence we obtain from i) that $c(I) = \mathrm{embdim}(A) - \mathrm{mdim}(A/I) \geqq \mathrm{embdim}(A_P) + \dim(A/P) - (\dim(A/P) + \mathrm{mdim}(A_P/I_P)) = c(I_P)$. □

Lemma 2.4. *Let A be a Noetherian local ring, and let I be an ideal of A. Suppose that B is a faithfully flat local extension of A with* $\dim(B) = \dim(A)$. *If* $H^q_{IA_P}(M_P) = 0$ *for an A-module M and for all* $P \in \mathrm{Spec}(A)$ *with* $\dim(A/P) > r$, *then* $H^q_{IB_Q}(M \otimes_A B_Q) = 0$ *for all* $Q \in \mathrm{Spec}(B)$ *with* $\dim(B/Q) > r$.

Proof. As $\dim(B) = \dim(A)$, we have that $\dim(B/mB) = 0$ [Mat, 21.A], where m is the maximal ideal of A. Let Q be a prime of B with $\dim(B/Q) > r$, and set $P = Q \cap A$. Since B/PB is faithfully flat over A/P, and B/mB is 0-dimensional, $\dim(A/P) = \dim(B/PB) \geqq \dim(B/Q) > r$. Therefore $H^q_{IA_P}(M_P) = 0$, and hence $H^q_{IB_Q}(M \otimes_A B_Q) = H^q_{IA_P}(M_P) \otimes_{A_P} B_Q = 0$. □

Theorem 2.5. *Let A be a Noetherian local ring containing a field, and let I be an ideal of A. Let* $B = ((A^\wedge)^{sh})^\wedge$. *Set* $c = c(IB)$. *Let M be any finitely generated A-module, and let* $n > c$ *be an integer. Assume that for all integers s, with* $1 \leqq s \leqq c - 1$, *and for all* $q \geqq n - s$, *the following hold*:

i) $H^q_{IA_P}(M_P) = 0$ *for all* $P \in \mathrm{Spec}(A)$, *such that* $I \subseteq P$, *and* $\dim(A/P) > s + 1$.
ii) $H^q_{IB_P}(M \otimes_A B_P) = 0$, *for all* $P \in \mathrm{Spec}(B)$ *such that* $IB \subseteq P$, $\dim(B/P) = s + 1$, *and* $P + Q$ *is primary to the maximal ideal of B for some minimal prime ideal Q of IB.*

Then $H^q_I(M) = 0$ *for all* $q \geqq n$.

Proof. We first reduce to the case where $A = B$ and $I = IB$. Assume we have shown the theorem in the case where A is already complete with separably closed residue

field, i.e., for $A = B$. The hypothesis i) of the theorem implies that $H^q_{IB_P}(M \otimes_A B_P) = 0$ for all primes $P \in \mathrm{Spec}(B)$ with $\dim(B/P) > s + 1$, by Lemma 2.4. Hence i) and ii) hold in B, and thus $H^q_{IB}(M \otimes_A B) = 0$ for all $q \geqq n$. As B is faithfully flat over A, we obtain that $H^q_I(M) = 0$ for all $q \geqq n$, as required. Thus we may assume that A is complete with separably closed residue field $k \subset A$.

Set $e = \mathrm{embdim}(A)$. Since A is a quotient of $k[[T_1, \ldots, T_e]]$, upon replacing A by this ring, and I by its preimage, we may assume that A is regular. In this case c is just the big height of I (bight(I)), namely the largest height of any minimal prime of I. By descending induction on q we may assume that $H^q_I(M) = 0$ for all $q > n$, and we only need to show that $H^n_I(M) = 0$. We induct on the dimension of A/I, over regular local rings A, and ideals I. The starting point is when $\dim(A/I) = 0$. In this case $c = \dim(A)$, and $H^n_I(M) = 0$ for $n > \dim(A)$. We claim that this induction shows that the module $H^n_I(M)$ must have support only at m. If not, then localize at a prime P' minimal in the support of $H^n_I(M)$; we must show that the assumptions i) and ii) of the theorem are still satisfied for the ring $A_{P'}$ and for the ring $C = (((A_{P'})^\wedge)^{sh})^\wedge$. Observe that $c(I) \geqq c(IA_{P'})$ by (2.3 ii), and $c(IA_{P'}) = c(IC)$ by (2.2) (A is complete, hence universally catenary) Let Q be a prime ideal of C with $\dim(C/Q) = s + 1$. Set $Q' = Q \cap A$. Then $\dim(C/Q) \leqq \dim(A_{P'}/Q'A_{P'})$ (*as* $A_{P'} \to C$ is flat with closed fiber of dimension 0) $< \dim(A/Q')$. Hence $\dim(A/Q') > s + 1$, so that i) shows that $H^q_{IA_{Q'}}(M_{Q'}) = 0$ for all $q \geqq n - s$. It follows that $H^q_{IC_Q}(M \otimes_A C_Q) = 0$ for $q \geqq n - s$. By our choice of Q, it follows that condition ii) is satisfied. Let $Q_{P'} \in \mathrm{Spec}(A_{P'})$. If $\dim(A_{P'}/Q_{P'}) > s + 1$, then $\dim(A/Q) > s + 1$, and $H^q_{IA_Q}(M_Q) = 0$ for $q \geqq n - s$. Hence i) is also satisfied, which by induction shows that $H^q_{IA_{P'}}(M_{P'}) = 0$ for $q \geqq n$, contradicting the choice of P'. This proves our claim.

Let $D = (A/I) \hat{\otimes}_k A$ be the complete tensor product. The natural surjection of A onto A/I induces an (A/I)-linear surjection of D onto A/I whose kernel Δ is generated by the e elements $T_i \hat{\otimes} 1 - 1 \hat{\otimes} T_i$, where $A = k[[T_1, \ldots, T_e]]$. Set $N = D \otimes_A M = (A/I) \hat{\otimes}_k M$.

Since $ID = (A/I) \hat{\otimes}_k I \subset \Delta$, there is a spectral sequence for the composition of functors, $E_2^{p,q} = H^p_\Delta(H^q_{ID}(N)) \Rightarrow H^{p+q}_\Delta(N)$. Because D is a flat A-algebra, we have that $H^q_{ID}(N) \cong D \otimes_A H^q_I(M)$.

Set $d = \dim(A/I)$, and consider the term,

$$E_2^{d,n} = H^d_\Delta(D \otimes_A H^n_I(M)) \Rightarrow E_\infty^{d,n}(N) .$$

Since $d + n > d + c \geqq d + (e - d) = e$, and since Δ is generated by e elements, $H^{d+n}_\Delta(N) = 0$, so $E_\infty^{d,n}(N) = 0$.

All incoming differentials to $E_2^{d,n}$ are zero, since they come from $E_r^{d-r,n+r-1}$ for $r > 1$, and $H^{d-r}_\Delta(D \otimes_A H^{n+r-1}_I(M)) = 0$, as by assumption $H^q_I(M) = 0$ for $q > n$.

The outgoing differentials land in $E_r^{d+r,n-(r-1)}$, and we claim that $H^{d+r}_\Delta(D \otimes_A H^{n-(r-1)}(M)) = 0$ for $r \geqq 2$. Indeed, if $r > c$, then $d + r > d + c \geqq \dim(A) = e$, and since Δ is generated by e elements, H^{d+r}_Δ (any module) $= 0$. Suppose that $r \leqq c$, and set $s = r - 1$. In general $H^{n-s}_I(M)$ is the direct limit of modules L_α, where $\{L_\alpha\}$ is the collection of finitely generated submodules of $H^{n-s}_I(M)$. Hence $H^{d+r}_\Delta(D \otimes_A H^{n-s}_I(M)) = \varinjlim(H^{d+r}_\Delta(D \otimes_A L_\alpha)) = \varinjlim(H^{d+r}_\Delta((A/I) \hat{\otimes}_k L_\alpha))$, since direct limits commute with the tensor product and are exact [R].

Let $P_1, \ldots, P_u$ be the minimal primes of the support of L_α in A for a fixed α, and let $Q_1, \ldots, Q_v$ be the minimal primes of A/I. We claim that the minimal primes of $(A/I) \hat{\otimes}_k L_\alpha$ are $\Omega_{ji} = \mathrm{nilrad}(Q_j \hat{\otimes}_k A + (A/I) \hat{\otimes}_k P_i) \subset D$. To prove this,

we first observe that by Corollary 4.5 the ideals Ω_{ji} are prime, since k is separably closed and is the residue field of A and A/I. Note that A and A/I embed into $D = (A/I)\hat{\otimes}_k A$. We have that $\Omega_{ji}(D/P_iD) = \text{nilrad}(Q_j(D/P_iD))$ is a prime ideal, and its contraction to A/I is Q_j since the map from A/I to $D/P_iD = (A/I)\hat{\otimes}_k(A/P_i)$ is faithfully flat. It follows that the ideals $\text{nilrad}(Q_j(D/P_iD))$ are exactly the minimal prime ideals of D/P_iD, since any minimal prime of D/P_iD must contract to one of the Q_j (by flatness). As D is flat over A we see that $\text{Ass}(D \otimes_A L_\alpha) = \bigcup_{P \in \text{Ass}(L_\alpha)} \text{Ass}(D/PD)$ by [Mat, 21.B]. Any minimal prime of $D \otimes_A L_\alpha$ is an associated prime of this module, and therefore is associated to D/PD for some $P \in \text{Ass}(L_\alpha)$. In particular, if N is a minimal prime of $D \otimes_A L_\alpha$, then N contains P_iD for some i. By our calculation above, this implies that N contains Ω_{ji} for some i. However, all of the Ω_{ji} are associated to $D \otimes_A L_\alpha$ by the formula above. Hence N must be equal to one of the Ω_{ji}, which proves our claim.

We know that $\dim(A/Q_j) \leqq d = \dim(A/I)$, and by i) $\dim(A/P_i) \leqq s + 1 = r$, and hence $\dim(D/\Omega_{ji}) \leqq d + r$. Suppose that $\dim(A/P_i) < r$ for every i. Then $\dim((A/I)\hat{\otimes}_k L_\alpha) < d + r$, and so $H^{d+r}_{\Delta}((A/I)\hat{\otimes}_k L_\alpha) = 0$, as we claimed. Thus we may assume that $\dim(A/P_i) = r$ for some i. Then ii) guarantees that $P_i + Q_j$ is not m-primary for every j. Hence in D, the ideal $\Delta + \Omega_{ji}$ is not η-primary for any j, where η is the maximal ideal of D. Thus $\Delta + \Omega_{ji}$ is not η-primary for any minimal prime Ω_{ji} of D having dimension equal to $d + r = \dim(C)$, where $C = (A/I)\hat{\otimes}_k(A/\text{ann}(L_\alpha))$. Now apply HLVT [Ha1, Thm. 3.1]; C is complete, so we may conclude that $H^{d+r}_{\Delta}(M) = 0$ for any module M. In particular, $H^{d+r}_{\Delta}((A/I)\hat{\otimes}_k L_\alpha) = 0$ in this case. Thus in any case the outgoing differentials are zero as well, and we may conclude that $H^d_{\Delta}(D \otimes_A H^n_I(M)) = 0$. Since $H^n_I(M)$ is supported only at the maximal ideal, it is the direct limit of modules L having finite length. For any such L, we have that $H^d_{\Delta}(D \otimes_A L) \cong H^d_{\Delta}((A/I)\hat{\otimes}_k L) \cong H^d_{\Delta}((A/I) \otimes_k L)$. The support of $(A/I) \otimes_k L$ can be identified with $\text{Spec}(A/I)$. The intersection of Δ with this support is the closed point $\{m/I\}$, where m/I is the maximal ideal of A/I. Hence, $H^d_{\Delta}((A/I) \otimes_k L) = H^d_{m/I}((A/I) \otimes_k L) \cong H^d_{m/I}(A/I) \otimes_k L$, where the latter isomorphism follows as L is flat over k. Passing to the direct limit, we obtain that $H^d_{\Delta}(D \otimes_A H^n_I(M)) \cong H^d_{m/I}(A/I) \otimes_k H^n_I(M) = 0$. However, since $H^d_{m/I}(A/I) \neq 0$ [Gr1, 6.4], and the tensor product is over a field, we conclude that $H^n_I(M) = 0$, as required. □

Remark 2.6. The hypotheses of Theorm 2.5 are fulfilled if for all s between 1 and $c - 1$ and for all $q \geqq n - s$, the module $H^q_{IA_P}(M_P) = 0$ for all primes P with $\dim(A/P) = s + 1$. This is Faltings' original theorem [Fa1, Satz 1].

Remark 2.7. Assume that $\text{nilrad}(IB)$ is prime, in the context of Theorem 2.5. Then no prime $P \supseteq IB$ such that $\dim(B/P) = s + 1 (1 \leqq s \leqq c - 1)$, and such that $H^q_{IB_P}(M \otimes_A B_P) \neq 0$, satisfies the condition that $\text{nilrad}(IB) + P$ is primary for the maximal ideal of B. Hence condition ii) of Theorem 2.5 is vacuous in this case, and so the hypotheses of Theorem 2.5 will be satisfied if $H^q_{IA_P}(M_P) = 0$ for all primes P with $\dim(A/P) = s + 2$, and for all $q \geqq n - s$. This observation will allow us to improve Faltings' estimate in case I is formally geometrically irreducible (i.e., $\text{nilrad}(IB)$ is prime).

Remark 2.8. If $c \leqq 1$ there are no integers s with $1 \leqq s \leqq c - 1$. Hence (i) and (ii) are vacously satisfied. Hence $H^q_I(M) = 0$ for $q > c$. This can also be seen directly. Indeed if $c = 0$ then A is regular and $I = 0$. If $c = 1$ then $A^\wedge$ is the surjective image

of a regular ring A' of dimension $e = \text{embdim}(A)$, and the preimage I' of $I\hat{A}$ in A' has bigheight 1. Hence $\sqrt{I'}$ is principal and the result follows.

To finish this section we wish to illustrate how Theorem 2.5 allows a new characteristic-free proof of theorems of Hartshorne and Ogus in characteristic 0, and Peskine and Szpiro in characteristic p [P-S]. These theorems give necessary and sufficient conditions for an ideal I in a complete regular local ring of dimension d with separably closed residue field to satisfy $H_I^{d-1}(M) = 0$ for all modules M.

Theorem 2.9. *Let (A, m) be a regular local ring of dimension d containing a field, and let I be an ideal of A. Then $H_I^q(M) = 0$ for all $q \geqq d - 1$ if and only if*

i) $\dim(A/I) \geqq 2$, *and*
ii) $\text{Spec}(A/I) - \{m\}$ *is formally geometrically connected, i.e., If* $B = ((A\hat{\ })^{sh})\hat{\ }$, then $\text{Spec}(B/IB) - \{mB\}$ *is connected.*

Proof. By the remarks in the beginning of this section, we may assume that $A = B = k[[x_1, \ldots, x_d]]$, with k separably closed. First we will show the necessity of conditions i) and ii). If I is m-primary, then $H_I^d(A) \neq 0$ [Gr1, 6.14]. If I has a minimal prime over-ideal P of dimension 1, then $H_{I_P}^{d-1}(A_P) \neq 0$, and therefore $H_I^{d-1}(A) \neq 0$. Thus $\dim(A/I) \geqq 2$. Assume that $\text{Spec}(A/I) - \{m\}$ is disconnected. Then there are two ideals J and K such that $J \cap K$ has the same radical as I, both J and K contain I but do not have the same radical, and $J + K$ is m-primary. By HLVT, $H_J^d(M) = H_K^d(M) = 0$ for any module M. The Mayer–Vietoris sequence of local cohomology, applied to J and K, shows that $H_I^{d-1}(A)$ maps onto $H_{J+K}^d(A) \neq 0$. Thus condition ii) is also necessary.

For the sufficiency of conditions i) and ii), we first consider the case where I is prime. Set $n = d - 1$. By Theorem 2.5 and Remark 2.7, to see that $H_I^q(M) = 0$ for all $q \geqq n$, it suffices to prove that $H_{IA_P}^q(M_P) = 0$ for all $q \geqq n - s$, and all P with $\dim(A/P) = s + 2$, for $1 \leqq s \leqq \text{bight}(I) - 1 = ht(I) - 1$, since I is prime. However, $\dim(A/P) = s + 2$ implies that $\dim(A_P) = d - s - 2 = n - s - 1$. Hence if $q \geqq n - s$, then $q > \dim(A_P)$, so that $H_{IA_P}^q(M_P) = 0$.

Now consider the general case. Let $P_1, \ldots, P_r$ be the minimal primes over I, and use induction on r to prove the theorem, the case $r = 1$ being done above. Every P_i must have dimension at least 2; if not, say P_1 has dimension 1. Then $P_1 + \bigcap_{i \neq 1} P_i$ is m-primary, and this gives a disconnection of the punctured spectrum of A/I, in contradiction to assumption ii). Consider the graph Γ with r vertices $1, \ldots, r$, where i and j are joined by an edge if and only if $P_i + P_j$ is not m-primary. Assumption ii) shows that Γ is connected. Any finite connected graph has a spanning tree [B, Cor. 5, p. 7]. The number of edges in a tree is smaller than the number of vertices, so there is a vertex from which only one edge emanates [B, Cor. 7, p. 8]. The original graph minus this vertex is connected. Hence we may assume that the graph $\Gamma - \{1\}$ is connected, and set $J = \bigcap_{i \neq 1} P_i$. The ideal J satisfies the assumptions of the theorem, so by induction, $H_J^q(M) = 0$ for all M, and for all $q \geqq d - 1$. Also $H_{P_1}^q(M) = 0$ for all $q \geqq d - 1$. The Mayer–Vietoris sequence applied to J and P_1 yields an exact sequence, $H_J^{d-1}(M) \oplus H_{P_1}^{d-1}(M) \to H_I^{d-1}(M) \to H_{J+P_1}^d(M)$. Since $J + P_1$ is not m-primary, $H_{J+P_1}^d(M) = 0$ by HLVT. Hence $H_I^{d-1}(M) = 0$. By HLVT, $H_I^d(M) = 0$, while $H_I^q(M) = 0$ for $q \geqq d + 1$ as $\dim(A) = d$. □

3. Vanishing theorems

In this section we prove our main vanishing results for local cohomology. They are fairly easy inductive consequences of our Theorem 2.5. Let (A, m) be a local noetherian ring containing a field, let I be an ideal of A with $c = c(IA^\wedge) > 0$, and M be a finitely generated A-module of dimension d. G. Faltings [Fa1, Korollar 2] proved that

$$H_I^q(M) = 0 \qquad \text{for } q > d - [(d-1)/c] . \tag{3.1}$$

Faltings also proved (cf. [Fa1, Korollar 4]) that if I_P is generated set-theoretically by r elements for every prime $P \supseteq I$ with $\dim(A/P) \geqq 2$, then

$$H_I^q(M) = 0 \qquad \text{for } q > c + r - 1 . \tag{3.2}$$

It is known that without further restrictions, (3.1) is best possible [Ly1]. In this section we sharpen these results by putting additional assumptions on the ideal I. If I is formally geometrically irreducible (see Definition 3.6 below), we are able to show (cf. Theorem 3.8) that provided $0 < c(IA^\wedge) < \dim(A)$ then with $d = \dim(M)$,

$$H_I^q(M) = 0 \qquad \text{for } q > d - 1 - [(d-2)/c] . \tag{3.3}$$

In Sect. 5 we will show that (3.3) is best possible. In Theorem 3.9 we generalize the vanishing theorems further by assuming more about I. Let A be an excellent local ring containing a field, let I be formally geometrically irreducible with $c(IA^\wedge) = c$, and assume that $(A/I)_P$ is normal for all primes P containing I with $\dim(A/P) \geqq 3$. If M is a finitely generated A-module of dimension d then,

$$H_I^q(M) = 0 \qquad \text{for } q > d - [(d/(c+1)] - [(d-1)/(c+1)] . \tag{3.4}$$

We are also able to generalize Faltings' result (3.2). We show in Theorem 3.10 below that if A is a local ring containing a field and I is a formally geometrically irreducible ideal such that I_P is set-theoretically generated by r elements for all primes P containing I with $\dim(A/P) \geqq 3$, then

$$H_I^q(M) = 0 \qquad \text{for all } q > c + r - 1, \quad \text{for all } A\text{-modules } M . \tag{3.5}$$

We begin with a definition.

Definition 3.6. We say that an ideal I in a local ring A is formally geometrically irreducible if B/IB has a unique minimal prime, where $B = ((A^\wedge)^{sh})^\wedge$, the completion of the strict Henselization of the completion of A.

Our main technical tool of this section is (3.7) below which is a consequence of our induction theorem (2.5).

Lemma 3.7. *Let A be a local noetherian ring containing a field and let I be a formally geometrically irreducible ideal of A. Let $n > c = c(IA^\wedge)$ be an integer. Let M be any finitely generated A-module. Assume for all integers s, $1 \leqq s \leqq c - 1$, and all primes $P \supseteq I$ with $\dim(A/P) = s + 2$ that the local cohomology modules $H^q_{IA_P}(M_P) = 0$ for all $q \geqq n - s$. Then $H_I^q(M) = 0$ for all $q \geqq n$.*

Proof. Set $B = ((A^\wedge)^{sh})^\wedge$. By assumption nilrad$(IB)$ is prime. Hence there does not exist any prime P in Spec(B) such that P contains IB, $\dim(B/P) > 0$, and $P + Q$ is primary to the maximal ideal of B for some minimal prime Q of IB. This gives that condition (ii) of Theorem 2.5 is vacuously satisfied. Hence all of the hypotheses of (2.5) are satisfied and the conclusion of (2.5) gives the result. □

Theorem 3.8. *Let A be a local ring containing a field, let I be a formally geometrically irreducible ideal in A with $0 < c(IA^\wedge) = c < d$, where d is the dimension of a finitely generated A-module M. Then $H^q_I(M) = 0$ for $q > d - 1 - [(d-2)/c]$. In particular if $0 < c < \dim(A)$, then $cd(A, I) \leqq \dim(A) - 1 - [(\dim(A) - 2)/c]$.*

Proof. Set $B = ((A^\wedge)^{sh})^\wedge$. Since $A^\wedge$ is universally catenary, (2.2) shows that $c = c(IB)$, and the remarks at the beginning of Sect. 2 then show that we may assume that $A = B$. If $c = 1$, then $\sqrt{IB}$ is principal (see (2.8)), and the result is trivial. Henceforth we assume that $c \geqq 2$. Set $n(t) = t - [(t-2)/c]$. Then n is a nondecreasing function of t. In particular for $t = d$, using that $d \geqq c + 1$, we see that $n = n(d) > c$. We apply (3.7) with this n. Choose any s, $1 \leqq s \leqq c - 1$, and let P be a prime containing I with $\dim(A/P) = s + 2$. A is complete, hence universally catenary, so (2.3) shows that $c(I_P) \leqq c$. As $\dim(M_P) \leqq d - s - 2$, (3.1) implies that $H^q_{I_P}(M_P) = 0$ for $q \geqq d - s - 1 - [(d - s - 3)/c]$. To apply (3.7) it suffices to show that $n - s \geqq d - s - 1 - [(d - s - 3)/c]$, which reduces to $1 \geqq [(d-2)/c] - [(d - s - 3)/c]$. As $s + 1 \leqq c$ the inequality follows. We can now apply (3.7) to obtain the first conclusion of the theorem. The last statement follows from setting $M = A$ and using [Ha1, Sec. 1]. □

Theorem 3.9. *Let A be an excellent local ring containing a field. Let I be a formally geometrically irreducible ideal of A with $0 < c = c(IA^\wedge)$, and assume that $(A/I)_P$ is normal for all primes $P \supseteq I$ with $\dim(A/P) \geqq 3$. Let M be a finitely generated A-module of dimension d. Then $H^q_I(M) = 0$ for $q \geqq d + 1 - [d/(c+1)] - [(d-1)/(c+1)]$. In particular if $n = \dim(A)$ and $0 < c$ then*

$$cd(A, I) \leqq n - [n/(c+1)] - [(n-1)/(c+1)] .$$

Proof. We induct upon d to prove the theorem. If $d \leqq c + 1$, the result follows from (3.1). Henceforth we assume that $d > c + 1$, and the result holds for modules of dimension $< d$. If $c = 1$, the conclusion holds (by (2.8)). Thus we also assume that $c \geqq 2$. We again apply (3.7). Set $n = d + 1 - [d/(c+1)] - [(d-1)/(c+1)]$. To apply (3.7) we first show that $n > c$. This reduces to proving that $d + 1 > (2d - 1)/(c+1) + c$. Clearing the denominator yields the equivalent inequality $c^2 - cd + d - 2 < 0$, i.e. that $(c-1)(c+1-d) < 1$ which holds by our constraints upon c. Thus $n > c$.

Let $1 \leqq s \leqq c - 1$, and let $P \supseteq I$ be a prime with $\dim(A/P) = s + 2$. By (2.2) and (2.3) $c(I(A_P)^\wedge) = c(I_P) \leqq c$. The dimension of $M_P \leqq d - s - 2$. As $\dim(A/P) \geqq 3$, A/P is normal. Since A is excellent $(A_P/I_P)^\wedge$ is also normal and (4.4) shows that I_P is formally geometrically irreducible. We may apply the induction to A_P, I_P, and M_P to obtain that $H^q_{IA_P}(M_P) = 0$ for $q \geqq (d - s - 2) + 1 - [(d - s - 2)/(c+1)] - [(d - s - 3)/(c+1)]$. Now (3.7) will finish the proof if we show that $n - s \geqq (d - s - 2) + 1 - [(d - s - 2)/(c+1)] - [(d - s - 3)/(c+1)]$. This reduces to $2 \geqq (([d/(c+1)] - [(d - s - 2)/(c+1)]) + ([(d-1)/(c+1)] - [(d - s - 3)/(c+1)]))$ which is true since $s + 2 \leqq c + 1$. The last statement of the proof follows by setting $M = A$. □

Finally we give our last vanishing result.

Theorem 3.10. *Let A be a local ring containing a field, let I be a formally geometrically irreducible ideal of A with $c = c(IA^\wedge)$, and assume that I_P is set-theoretically generated by r elements for all primes $P \supseteq I$ with $\dim(A/P) \geqq 3$. Then $H^q_I(M) = 0$ for all $q > c + r - 1$, and for all A-modules M, i.e. $cd(A, I) \leqq c + r - 1$.*

Proof. Set $n = r + c$, and apply (3.7). For the last statement set $M = A$. □

Remark 3.11. Let A be a universally catenary local ring containing a field, let I be an ideal of A with $c = c(IA^{\wedge})$, and let M be a finitely generated A-module of dimension d. Set $c(v) = \max\{c(I_P)\colon P \supseteq I, \dim(A/P) \geqq v\}$. By (2.3) the function $c(v)$ is non-increasing. All of the results of this section were proven assuming only that $c(0)$ is known. However, if more information of $c(v)$ is available, stronger results are possible. For example:

(i) Assume that there is a positive integer $t \leqq c = c(0)$ such that $c(t+1) \leqq t$. Then $H^q_I(M) = 0$ for $q > d - 1 - [(d-1-c)/t]$.

(ii) Assume that I is formally geometrically irreducible and there is a positive integer $t \leqq c = c(0)$ such that $c(t+2) \leqq t$. Then $H^q_I(M) = 0$ for $q > d - 2 - [(d-c-2)/t]$.

(iii) Assume that A is excellent, I is formally geometrically irreducible, A_P/I_P is normal for every prime $P \supseteq I$ with $\dim(A/P) \geqq 3$, and there exists a positive integer $t \leqq c$ such that $c(t+2) \leqq t$. Then $H^q_I(M) = 0$ for $q - 2 - [(d-c-1)/(t+1)] - [(d-c-2)/(t+1)]$.

We omit the proofs since no new ideas are involved. Of course, if precise values of $c(v)$ are known, even sharper estimates are possible. Also, if in addition to information about $c(v)$ it is known that I_P is set-theoretically generated by r elements for primes $P \supseteq I$ with $\dim(A/P) \geqq m$, where r and m are some fixed integers, even sharper results are possible, but the formulas are quite cumbersome. We conclude this section with several examples to illustrate the strengths of the various estimates.

Example 3.12. Let k be a field of characteristic 0, and let $X = (x_{ij})$ be a generic 3×6 matrix. Consider the ideal I generated by the 3×3 minors of X in the ring $B = k[x_{ij}]$. Let $A = B_{(x_{ij})}$. Then $\dim(A) = 18$, $\mu(I) = 20$, and $ht(I) = c(I) = 5$. Furthermore (e.g. see [B-V]), A/I is normal, Cohen–Macaulay, and if $P \neq (x_{ij})$, $\mu(I_P) \leqq 10$. We apply our various estimates to compute $cd(A, I)$. HLVT gives that $cd(A, I) \leqq 19$. Faltings estimate (3.1) gives that $cd(A, I) \leqq 18 - [17/5] = 15$. Since I is homogeneous it is formally geometrically irreducible so that our Theorem 3.8 gives $cd(A, I) \leqq 18 - 1 - [16/5] = 14$. Since A/I is normal and I is formally geometrically irreducible we may also apply Theorem 3.9 to conclude that $cd(A, I) \leqq 18 - [18/6] - [17/6] = 13$. Finally we can apply (3.10) to obtain that $cd(A, I) \leqq 5 + 10 - 1 = 14$. Obviously (3.9) gives the best estimate.

The actual cohomological dimension of I, however, is 10, by the remark after this example. This can be obtained by our methods, or even Faltings' original theorem by using induction upon the size of the matrix. Clearly, though, the theorems above are not always the best possible—this is because information concerning localizations of an ideal will not in general tell if I is defined up to radical by a small number of equations.

Remark 3.13. Let k be characteristic 0 and let I be generated by the $r \times r$ minors of an $r \times s$ generic matrix X $(r \leqq s)$. Let $B = k[x_{ij}]$, and $A = B_{(x_{ij})}$. Then $cd(A, I) = cd(B, I) = r(s-r) + 1$.

Proof. We sketch the proof, which was probably known. If I is ideal in a polynomial ring generated by homogeneous equations of the same degree, then the analytic spread of I is the dimension of $k[I]$ as a subring of the polynomial ring. In the case of this example, $k[I]$ is just the homogeneous coordinate ring of the

Grassman variety of r-planes in s-space, which is known to have dimension $r(s-r)+1$. Therefore, I is generated up to radical by $r(s-r)+1$ equations, so that $cd(A, I) \leqq r(s-r)+1$.

On the other hand, as Hochster pointed out to us, the map from $C = k[I]$ to $B = k[x_{ij}]$ splits as a C-module since C is the ring of invariants of a linearly reductive group acting linearly on B. Hence the local cohomology $H^q_{IC}(C)$ injects into $H^q_{IC}(B)$ for all q. In particular, $H^q_I(B)$ cannot be zero for $q = \dim(C) = r(s-r)+1$, and consequently $H^q_I(A) \neq 0$. Thus $r(s-r)+1 \leqq cd(A, I)$, which proves the remark. □

Suppose that we assume in this example that $\operatorname{char} k = p > 0$. Since A/I is Cohen–Macaulay, [P-S] shows that $cd(A, I) = ht(I) = 5$.

Example 3.14. To obtain a better idea of the various estimates, let us consider a complete regular local ring (A, m) of dimension 60 containing its separably closed residue field, and let $I \subseteq A$ be an ideal with $\operatorname{bight}(I) = 5$. HLVT shows that $cd(A, I) \leqq 59$, while Faltings' result (3.1) shows that $cd(A, I) \leqq 60 - [59/5] = 49$. If I is formally geometrically irreducible then (3.8) may be applied to obtain that $cd(A, I) \leqq 60 - [58/5] - 2 = 48$. If in addition $(A/I)_P$ is normal for all $P \supseteq I$ with $\dim A/P \geqq 3$, then we can use (3.9) to get $cd(A, I) \leqq 60 - [60/6] - [59/6] = 41$. Finally suppose that I_P is set-theoretically generated by r elements for all primes $P \supseteq I$ with $\dim(A/P) \geqq 3$, and I is formally geometrically irreducible. Then (3.10) applies to show that $cd(A, I) \leqq r + 4$, which is < 41 if $r < 37$.

Remark 3.15. If we judge the efficacy of our estimates by the size of $\dim(A) - cd(A, I)$, then our results show that when I is formally geometrically irreducible, we can do one better than in general, while if A/I is normal, we can do approximately twice as well. These results suggest that if one assumes that A/I satisfies Serre's properties S_i and R_{i-1}, then better estimates can be obtained as i increases. However, as far as we can tell, our proofs can not be extended to prove any such result.

Remark 3.16. This example shows that, at least in some cases, the bound given by (3.9) is best possible. Let $X = (x_{ij})$ be a $2 \times n$ matrix of indeterminates. Let A be the polynomial ring in the $2n$ variables x_{ij} localized at the irrelevant ideal. Set I equal to the ideal generated by the 2×2 minors of X. Then A/I is normal. In 3.13 we showed that $cd(A, I) = 2(n-2)+1 = 2n-3$. We get from (3.9) that $cd(A, I) \leqq 2n - 2 - [(2n-(n-1)-1)/n] - [(2n-(n-1)-2)/n] = 2n-3$. Hence the estimate is exact in this case. However, we have not in general found examples for all d and c showing that (3.9) is best possible.

4. Prime ideals in power series rings

In this section we prove several results about the extension of prime ideals in power series rings which were needed in Sect. 2. These results, particularly Corollary 4.4, were crucial to some of our theorems. In particular, Corollary 4.4 is needed to prove the vanishing theorems for normal ideals in characteristic $p > 0$. The main result is Theorem 4.2 which extends results in Chevalley's paper [Ch], and Gilmer's paper [G] (see also [Gr2, IV, 7.5]) and is of independent interest. It says that if a prime in a power series ring splits into several components after a field extension

of the coefficients, then it must already split in some finite separable subextension. Our first lemma shows this for polynomial rings—a result which is probably well-known, but for which we could find no reference.

Lemma 4.1. *Let P be a prime in $R = k[x_1, \ldots, x_n]$, where k is a field and R is a polynomial ring over k. Let L be an extension field of k, and set $S = L[x_1, \ldots, x_n]$. If $\{Q_1, \ldots, Q_t\}$ are distinct minimal primes over PS, then there is a finite separable extension W of k, $W \subseteq L$, such that $PW[x_1, \ldots, x_n]$ has t distinct minimal primes over it, namely $Q_i \cap W[x_1, \ldots, x_n]$, $1 \leqq i \leqq t$.*

Proof. Let F be the separable closure of k in L. First we show that PR_F has t distinct minimal primes over it in R_F, where we write $R_G = G[x_1, \ldots, x_d]$ for any extension ring G of k. It is enough to see that any prime Q in R_F cannot have more than 1 minimal prime when extended to R_L. Let N be the quotient field of R_F/Q. Then $R_L/QR_L = (R_F/Q) \otimes_F L \subset N \otimes_F L$, so it suffices to see that every zero-divisor in $N \otimes_F L$ is nilpotent. As F is separably algebraically closed in L, this follows at once from [Ja, Thm. 24, pg. 202].

Thus we may assume that L is separable algebraic to prove the lemma. In this case choose a generating set of each Q_i. Adjoin all the coefficients of these polynomials to F, and call that field W. Note that W is a finite separable extension of F, and PR_W must have t distinct minimal primes, namely $Q_i' = Q_i \cap R_W$. For Q_i' clearly is a prime containing PR_W, and $\dim(R_W/Q_i') = \dim(R_L/Q_i)$ (by integrality) $= \dim(R/P)$, since $Q_i \cap R = P$. Hence Q_i' is minimal over PR_W, and they are necessarily distinct by construction. □

We now will prove the same result for power series rings. First we establish some notation. Let $R = k[[x_1, \ldots, x_n]]$, where k is a field, and R is a power series ring over k. If G is any extension ring of k, let R_G denote the ring $G[[x_1, \ldots, x_n]]$. Let P be a prime in R, and by S denote R/P. By S_G denote R_G/PR_G, if G is an extension of k. If a ring C is a domain, by $F(C)$ we will denote the quotient field of C.

Theorem 4.2. *Adopt the notation of the above paragraph, and assume that L is a field extension of k. If PR_L has exactly t distinct minimal primes $Q_1, \ldots, Q_t$ lying over it in R_L, then there is a finite separable extension K of k, $K \subseteq L$, such that PR_K has exactly t distinct minimal primes lying over it in R_K, which are the contractions of the Q_i. In particular, there is a one-to-one correspondence between the minimal primes of PR_L and PR_K.*

Proof. Set $d = \dim(S)$. After a change of variables, if necessary, we may assume that $A = k[[x_1, \ldots, x_d]] \subset S$, and S is a finite A-module. Let $P' = P \cap A[x_{d+1}, \ldots, x_n]$. Note that $A[x_{d+1}, \ldots, x_n]/P'$ maps onto S, has the same dimension, and is a domain; thus $A[x_{d+1}, \ldots, x_n]/P' = S$. Observe also that $P' \cap A = P \cap A = (0)$.

Let $B = A_L = L[[x_1, \ldots, x_d]]$. We claim that $Q_i \cap B = (0)$ for all i. Suppose not, set $q_i = Q_i \cap B$, with $q_i \neq (0)$. We have that $R_L/PR_L = R_L/P'R_L = B[x_{d+1}, \ldots, x_n]/P'B[x_{d+1}, \ldots, x_n]$ is integral over B, by the fact that the x_i, with $d + 1 \leqq i \leqq n$, satisfy integral equations over A. Hence R_L/Q_i is integral over $B/Q_i \cap B$, which in turn implies that $\dim(R_L/Q_i) = \dim(B/q_i) < \dim(B) = d$. However, [Ch, Prop. 9] shows that $\dim(R_L/Q_i) = \dim(R/P) = d$. This contradiction shows that $Q_i \cap B = (0)$.

As $Q_i \cap B = (0)$ and $P' \cap A = (0)$, we must have that there are t minimal primes

over $P'F(B)[x_{d+1}, \ldots, x_n]$, namely $Q_i \cap F(B)[x_{d+1}, \ldots, x_n]$. By Lemma 4.1, this implies that the same is true in a finite separable extension W of $F(A)$ contained in $F(B)$. Consider the integral closure of A in W; call this C. Choose a primitive element $\gamma \in F(B)$ which generates W over $F(A)$. We can write γ as a quotient f/g of two elements of C, and from elementary commutative algebra, we may assume that $g \dot{\in} A$. Since f is integral over A, we see that $f \in B$, as B is integrally closed and $f \in F(B)$. Thus, f satisfies a separable integral equation over A, and $W = F(A)(f)$.

By [G, Thm. 4.1], the field $K' = k(\text{coefficients of } f) \subseteq L$ has the property that $[K' : k]_{\text{sep}} < \infty$, and has finite exponent e over k. Let $\beta = f^q$, where $q = p^e$. Since W is separable over $F(A)$, we have that $W = F(A)(\beta)$. Let K be the field $k(\text{coefficients of } \beta)$. Then K is finite separable over k, and $W \subset F(A_K)$. (Note that this shows that $f \in F(A_K)$, which in turn means that the coefficients of f are in K; thus $K' = K$ and K' was actually separable. However, we shall not need this comment.)

We claim that PR_K has t distinct minimal primes over itself. It suffices to see that $Q_i \cap R_K$ are distinct (by flatness they are necessarily minimal over PR_K), and for this it suffices to see that $Q_i \cap F(A_K)[x_{d+1}, \ldots, x_n]$ are distinct. However, $F(A_K) = K((x_1, \ldots, x_d))$ contains W, and by the above argument, $Q_i \cap W[x_{d+1}, \ldots, x_n]$ are distinct. Again by faithful flatness, every minimal prime of PR_K is the contraction of one of the Q_i, which finishes the proof. □

Corollary 4.3. *Adopt the notation of Theorem* 4.2. *Suppose that k is separably closed in L. Then nilrad*(PR_L) *is a prime ideal.*

Proof. As k is separably closed in L, there is only one minimal prime above PR_L, by Theorem 4.2. This gives the statement of the Corollary. □

Corollary 4.4. *Adopt the notation of Theorem* 4.2. *Assume that R/P is normal, and that L is any field extension of k. Then nilrad*(PR_L) *is a prime ideal.*

Proof. By Theorem 4.2 it suffices to prove this corollary when L is a finite separable extension field of k. Set $S = R/P$. We claim that k is algebraically closed in $F(S)$. If not, let K be the algebraic closure of k in $F(S)$. Since K is integral over k, and S is integrally closed, $K \subseteq S$, and hence $K \subseteq S/M$, where M is the maximal ideal of S. Since $S/M = k$, we get a contradiction.

By flatness, $S \otimes_k L \subset F(S) \otimes_k L$, so to prove the claim it suffices to see that $F(S) \otimes_k L$ is a domain. However, since k is algebraically closed in $F(S)$, and L is a separable extension of k, we may apply [Ja, Cor. 1, pg. 203] to obtain this conclusion. □

Corollary 4.5. *Let A and B be complete local rings containing a separably closed field k, which is the residue field of both A and B. Let* $P \in \text{Spec}(A)$, *and* $Q \in \text{Spec}(B)$. *Then nilrad*$(P \hat{\otimes}_k B + A \hat{\otimes}_k Q)$ *is a prime ideal in* $A \hat{\otimes}_k B$.

Proof. Both A and B are homomorphic images of power series rings over k; to prove the corollary we may assume A and B are equal to $k[[x_1, \ldots, x_d]]$, since the Corollary only depends on the quotients A/P and B/Q. Since k is a field, $(A/P) \hat{\otimes}_k (B/Q) \subset (A/P) \hat{\otimes}_k F(B/Q) = F(B/Q)[[x_1, \ldots, x_d]]/PF(B/Q)\ [[x_1, \ldots, x_d]]$, so to prove the statement, it is enough to show that $\text{nilrad}(F(B/Q)[[x_1, \ldots, x_d]]/PF(B/Q)[[x_1, \ldots, x_d]])$ is prime. Since k is separably closed, Corollary 4.3 proves this. □

5. Vanishing in regular local rings

The purpose of this section is twofold. First we simply collect our results of the previous section in the case that the base ring is regular; this is for the convenience of readers. We give a result and examples to show that some of the results are the best possible. The main point of this section, however, is to prove a theorem which gives a sufficient (but not necessary) criterion for $cd(A, I) \leqq d - 3$, where A is a regular local ring of dimension d. Recall that in Theorem 2.9 we gave a new proof of theorems of Hartshorne and Ogus (in char. 0) and Peskine and Szpiro (in char. p) which gave necessary and sufficient conditions for $cd(A, I) \leqq d - 2$. Our criterion in this section chiefly says that if every component of I has height at most $(d - 2)/2$, and no three minimal primes of I add to an m-primary ideal, then $cd(A, I) \leqq d - 3$. This criterion allows us to give a complete answer to the question if the number of components of I is at most 5 and we assume $\text{bight}(I) \leqq (d - 2)/2$. We begin, however, by collecting previous work.

Theorem 5.1. *Let A be an excellent regular local ring of dimension d containing a field, and I an ideal of A of $\text{bight}(I) = b$. Then:*

i) $cd(A, I) \leqq d - [(d - 1)/b]$.

ii) If, moreover, I is formally geometrically irreducible, then

$$cd(A, I) \leqq d - 1 - [(d - 2)/b] .$$

iii) If I is formally geometrically irreducible and $(A/I)_P$ is normal for all $P \supseteq I$ such that $\dim(A/P) \geqq 3$ then.

$$cd(A, I) \leqq d - [d/(b + 1)] - [(d - 1)/(b + 1)] .$$

Proof. Statement i) is due to Faltings [Fa1]. Statement ii) follows at once from (3.8), and iii) follows from (3.9). □

Theorem 5.2. *Let (A, m) be an excellent regular local ring of dimension d containing a field. Let $b < d$ be a positive integer. Set $s = [(d - 1)/b]$. Let $J_1, \ldots, J_s$ be ideals of A of big height b such that $\text{Spec}(A/(J_1 + \ldots + J_s)) - \{m\}$ is disconnected. Set $I = J_1 \cap \ldots \cap J_s$. Then $cd(A, I) = d - [(d - 1)/b]$.*

Proof. Set $\alpha_i = (\cap_{t \leqq i} J_t) + J_{i+1} + \ldots + J_s$. We will prove by induction on i that $cd(A, \alpha_i) = d - i$. The theorem follows by setting $i = s$. If $i = 1$, the claim follows from (2.9). Assume that $i > 1$, and the claim has been shown for $i - 1$. Set $J' = J_i + \ldots + J_s$ and $J'' = (\bigcap_{t \leqq i-1} J_t) + J_{i+1} + \ldots + J_s$. Then $\text{nilrad}(\alpha_i) = \text{nilrad}(J' \cap J'')$, and $\alpha_{i-1} = J' + J''$. By the lemma of [Ly1], $cd(A, J') < d - i + 1$, and $cd(A, J'') < d - i + 1$. Now the claim and the theorem follow from the Mayer–Vietoris sequence for the ideals J'' and J'', considering the induction hypothesis. □

Corollary 5.3 [Ly1]. *Let (A, m) be an excellent regular local ring containing a field. Let $I_0, \ldots, I_s$ be ideals of bigheight b such that $I_0 + \ldots + I_s$ is primary to the maximal ideal. Set $I = \bigcap_{j \leqq s} I_j$. Then $cd(A, I) = d - s$, where s is as in Theorem 5.2.*

Proof. Set $J_1 = I_0 \cap I_1$, and $J_i = I_i$ for $i \geqq 2$. Apply Theorem 5.2. □

Theorem 5.2 covers examples not covered by (5.3). Such an example is:

Example 5.4. Let (A, m) be an excellent regular local ring containing a field.

Suppose $P_1, \ldots, P_5$ are primes of height b such that $P_i + P_k + P_j$ is primary to the maximal ideal iff $\{i, j, k\} \in \Lambda = \{(1, 3, 4), (1, 3, 5), (1, 2, 5), (2, 3, 4), (2, 4, 5)\}$. Set $I = P_1 \cap \ldots \cap P_5$. Note that Λ has the following property. Each i, $1 \leqq i \leqq 5$, appears no more than three times, and each pair $\{i, j\}$ appears no more than twice. This implies that we cannot split $\{1, \ldots, 5\}$ into three disjoint groups C, D, and E in such a way that the sum of $P_i + P_j + P_k$ is m-primary whenever $i \in C, j \in D$, and $k \in E$. Hence we cannot apply Corollary 5.3. However, we can split the ideal $I = J \cap K$ in a such a way that $\operatorname{Spec}(A/(J + K)) - \{m\}$ is disconnected. Namely let $J = P_1 \cap P_2 \cap P_3$, and $K = P_4 \cap P_5$. We leave it to the reader to prove this claim.

Remark 5.5. The results of Theorem 5.1(i) and (ii) are exact for all d and b. Indeed, set $s = [(d - 1)/b]$, and let $J_0, \ldots, J_s$ be ideals of A of bigheight $\leqq b$ such that $J_0 + \ldots + J_s$ is primary to the maximal ideal of A. Set $I = \cap J_i$. Then $cd(A, I) = d - s$. This follows from Corollary 5.3, and shows that (i) is the best possible.

To show that ii) is the best possible for all b and d with $0 < b < d$ requires more work. We separate this in:

Theorem–Example 5.6. Let k be an algebraically closed field. Set $R = k[x_1, \ldots, x_{d-b}]$, and set $s = [(d - 1)/b]$. Let J be the defining ideal in R of $(s + 1)$ closed points in general position in $\mathbb{A}_k^{d-b}$. By W denote the $(d - b)$-dimensional affine variety whose coordinate ring is $B = k \oplus J$. Embed W in $\mathbb{A}_k^n$ for some n, and let W^- denote the closure of W in $\mathbb{P}_k^n$. Let V be the image of W^- after some generic projection of W^- on $\mathbb{P}_k^d$ in $\mathbb{P}_k^n$. Let A be the homogeneous coordinate ring of $\mathbb{P}_k^d$ localized at the irrelevant ideal (of course $\dim(A) = d + 1$). Finally, let I be the defining ideal of V in A. Then I is formally geometrically irreducible of height b, and $cd(A, I) = d - s = \dim(A) - 1 - [(\dim(A) - 2)/b]$, i.e., the bound of (5.1) (ii) is exact for all such d and b.

Proof. Clearly B is a domain. We claim B is a finitely generated k-algebra. Let $M_1, \ldots, M_{s+1}$ be the maximal ideals of R such that $J = \cap M_i$. Since $R/M_i \cong k$ for each i, there exist $c_{ij} \in k$ with $x_j - c_{ij} \in M_i$. Then the elements $y_i = \Pi_{j=1,\ldots,s+1}(x_j - c_{ij}) \in J$, and R, and therefore B, is a finite module over $k[y_1, \ldots, y_{d-b}]$. Thus B is a finitely generated k-algebra of dimension $d - b$. By the same argument, R is integral over B, and is also clearly birational, with conductor J. It follows that B has an isolated singularity at the maximal ideal m of B generated by J, and that R is the integral closure of B.

There exist exactly $s + 1$ prime ideals in R lying over mR, namely $M_1, \ldots, M_{s+1}$. Hence the completion of B with respect to m (which we denote by $B^\wedge$) has $s + 1$ minimal prime ideals $J_1, \ldots, J_{s+1}$.

We claim that $B^\wedge/J_i$ is regular for each i, and that $J_i + J_j$ is $\hat{m}$ primary if $i \neq j$. Indeed $B^\wedge$ has an isolated singularity at $\hat{m}$, so nilrad$(J_i + J_j)$ must be equal to $\hat{m}$. The extension of B to R is unramified, and therefore so is the extension of $(B^\wedge/J_i) \to (R_{M_i})^\wedge$. As the residue field is algebraically closed, the extension is etale and must be an isomorphism. Thus $B^\wedge/J_i$ is regular, as claimed.

Let $G = k[[Z_1, \ldots, Z_n]]$ be the completion of the coordinate ring of $\mathbb{A}_k^n$ at the maximal ideal defining the closed point $\{m\}$ of $\operatorname{Spec}(B)$. Let $L \subset G$ be the extension to G of the defining ideal of W in $\mathbb{A}_k^n$. From our calculations above, it follows that

$L = L_1 \cap \ldots \cap L_{s+1}$, where G/L_i is regular of dimension $d - b$, and $L_i + L_j$ is equal to the maximal ideal of G if $i \neq j$.

Let $Y_1, \ldots, Y_{d-b}$ be $d - b$ generic linear combinations of $Z_1, \ldots, Z_n$. Then $G/L_i \cong k[[Y'_1, \ldots, Y'_{d-b}]]$, where the $'$ means taking images in G/L_i. After a change of variables, we may assume that $Y_i = Z_i$, for $1 \leqq i \leqq d - b$.

Let $C = k[[Z_1, \ldots, Z_d]] \subset G$. We have that $k[[Z_1, \ldots, Z_{d-b}]] \subseteq C/(L_i \cap C) \subseteq G/L_i = k[[Z_1, \ldots, Z_{d-b}]]$, which forces $C/(L_i \cap C) \cong k[[Z_1, \ldots, Z_{d-b}]]$. Set $N_i = L_i \cap C$. From linear algebra it is clear that if $Z_{d-b+1}, \ldots, Z_d$ are chosen generically, then the fact that $L_i + L_j$ is to equal the maximal ideal of G forces $N_1 + \ldots + N_{s+1}$ to equal the maximal ideal of C.

Thus $I \subset A$ is formally geometrically irreducible (since I is a homogeneous prime), but upon localizing and completing at a prime ideal of height d, it splits as in Remark 5.5. This shows that $cd(A, I) = d - s$, as required. □

We now prove a lemma needed in the proof of the main theorem. (see [Fa2] and [Br–R]).

Lemma 5.7. *Let (A, m) be a complete regular local ring containing its residue field k, which is separably closed. If P and Q are two prime ideals of A, with* $\dim(A/P) + \dim(A/Q) > d + 1 = \dim(A) + 1$, *then* $\operatorname{Spec}(A/(P + Q)) - \{m\}$ *is connected.*

Proof. Write $A = k[[x_1, \ldots, x_d]]$. We may identify $\operatorname{Spec}(A/(P + Q))$ with $\operatorname{Spec}((A/P) \hat{\otimes}_k (A/Q)/\Delta)$, where Δ is the ideal generated by the diagonal in $A \hat{\otimes}_k A$. By Corollary 4.5, the ideal $\operatorname{nilrad}(P \hat{\otimes}_k A + A \hat{\otimes}_k Q) = N$ in $B = A \hat{\otimes}_k A$ is a prime ideal, and $\dim(B/N) = \dim(A/P) + \dim(A/Q) \geqq d + 2$. Since d is the number of generators of Δ, by [Fa2, Thm.6], we get the desired conclusion, since $\operatorname{Spec}((A/P) \hat{\otimes}_k (A/Q)/\Delta) - \{m\}$ is connected iff $\operatorname{Spec}(B/(N, \Delta)) - \{m\}$ is connected. □

Theorem 5.8. *Let (A, m) be a complete regular local ring containing its residue field, which is separably closed. Let I be an ideal of A, with $\{P_1, \ldots, P_n\}$ the minimal primes of I, bight$(I) = b$. Then $cd(A, I) \leqq d - 3$ if*

i) $2b + 2 \leqq d$ *and,*
ii) $P_i + P_j + P_k$ *is not m-primary for any triple $\{i, j, k\}$.*

Proof. We may assume that I is reduced, hence $I = \cap P_i$. We induct upon n to prove the theorem. First we consider the case $n = 1$. In this case, by Theorem 5.1 ii), $cd(A, I) \leqq d - 1 - [(d - 2)/b] \leqq d - 3$ as $2b + 2 \leqq d$.

Let $J = P_1 \cap \ldots \cap P_{n-1}$. By induction we may assume that $cd(A, J) \leqq d - 3$. Consider the Mayer–Vietoris sequence for local cohomology with respect to $J, P = P_n$, and $I = J \cap P$. We obtain that $H^q_I(M) \cong H^{q+1}_{J+P}(M)$ for all $q \geqq d - 2$ and for any A-module M, since $cd(A, J) \leqq d - 3$. Hence $cd(A, I) \leqq d - 3$ iff $cd(A, J + P) \leqq d - 2$. By Theorem 2.9, this is true iff $\dim(A/(J + P)) \geqq 2$, and $\operatorname{Spec}(A/(J + P)) - \{m\}$ is connected. First of all, $\dim(A/(J + P)) \geqq \dim(A) - ht(J) - ht(P) \geqq 2$ by our assumptions.

Suppose that $\operatorname{Spec}(A/(J + P)) - \{m\}$ is disconnected. Then there are ideals F and G such that $F + G$ is m-primary, $FG \subseteq \operatorname{nilrad}(J + P)$, but neither F nor G is contained in $\operatorname{nilrad}(J + P)$. Choose any P_i for $i \leqq n - 1$. By Lemma 5.7, $\operatorname{Spec}(A/(P_i + P)) - \{m\}$ is connected (since $\dim(A/P_i) + \dim(A/P) \geqq 2d - 2b \geqq d + 2$). Hence for each i, either F or G is contained in $\operatorname{nilrad}(P_i + P)$. If one of them, say F, is contained in $\operatorname{nilrad}(P_i + P)$ for every i, $1 \leqq i \leqq n - 1$, then $F \subseteq \cap(\operatorname{nilrad}$

$(P_i + P)) \subseteq \text{nilrad}(\cap (P_i + P)) \subseteq \text{nilrad}(J + P)$. This contradiction shows that $F \subseteq \text{nilrad}(P_i + P)$ for some i, and $G \subseteq \text{nilrad}(P_j + P)$ for some $j \neq i$. Then $m \subseteq \text{nilrad}(F + G) \subseteq \text{nilrad}(P_i + P_j + P_n)$, which contradicts ii). □

Remark 5.9. Theorem 5.8 recovers the special case of a general ideal given in Theorem 5.1 i). For $cd(A, I) \leqq d - [(d-1)/b]$, shows that $cd(A, I) \leqq d - 3$ provided $3b \leqq d - 1$, a condition that forces condition ii) of Theorem 5.8 to be true since $ht(P_i + P_j + P_k) \leqq 3b$ in general. Of course, (5.8) is more general than (5.1 i) since it only requires that $2b + 2 \leqq d$. One might expect similar better results for ideals of smaller codimension, but there are serious problems to extend the results further. Observe giving a necessary and sufficient criterion for $cd(A, I) \leqq d - 3$ is practically impossible. For let P be a prime with $\dim(A/P) = 3$. Then if P is a set-theoretic complete intersection, $cd(A, P) \leqq d - 3$, yet the vanishing of the obstructions to being a set-theoretic complete intersection are not sufficient to guarantee that $cd(A, I) \leqq d - 3$, as they are for the inequality $cd(A, I) \leqq d - 2$, where essentially being connected on the punctured spectrum guarantees the correct cohomological dimension.

Remark 5.10. It is certainly possible to have some triples of minimal primes add up to be m-primary, but none-the-less have the estimate $cd(A, I) \leqq d - 3$. For instance, the proof shows that at the inductive stage, not only must some triple add to be m-primary if the proof breaks down, but in fact one can divide the primes $P_1, \ldots, P_{n-1}$ into two disjoint sets, Λ = those containing F and Π = those containing G, such that given any $P_i \in \Lambda$, and any $P_j \in \Pi$, $P_i + P_j + P_n$ is m-primary. Although this phenomena is suseptible to analysis, we have not yet found a definitive statement.

To illustrate some of the problems with further analysis, a small amount of work yields a complete answer if the number of primes minimal over I is at most 5.

Corollary 5.11. *Let (A, m) be a complete regular local ring containing its residue field, which is separably closed. Let I be an ideal of A, with $\{P_1, \ldots, P_n\}$ the minimal primes of I, $bight(I) = b$. Assume that $2b + 2 \leqq d$.*

i) If $n \leqq 2$, then $cd(A, I) \leqq d - 3$.
ii) If $n = 3$, then $cd(A, I) \leqq d - 3$ iff $P_1 + P_2 + P_3$ is not m-primary.
iii) If $n = 4$ or 5, then $cd(A, I) \leqq d - 3$ iff whenever we divide $\{P_1, \ldots, P_n\}$ into two disjoint sets C and D, and let $J = \bigcap_{P \in C} P_j$ and $K = \bigcap_{P \in D} P$, $\text{Spec}(A/(J + K)) - \{m\}$ is connected. If $n = 4$, then $cd(A, I) \leqq d - 3$ iff there is at most one triple $\{i, j, k\}$ such that $P_i + P_j + P_k$ is m-primary.

Proof. If $n \leqq 2$, then the condition ii) of Theorem 5.8 is vacuously satisfied, so $cd(A, I) \leqq d - 3$. If $n = 3$, and $P_1 + P_2 + P_3$ is not m-primary then we can again apply Theorem 5.8 to obtain that $cd(A, I) \leqq d - 3$. This condition is also necessary by 5.3 with $s = 2$.

To prove (iii), first suppose that there exist two sets C and D as in the statement of (iii) such that $\text{Spec}(A/(J + K)) - \{m\}$ is disconnected. Then $H^{d-1}_{J+K}(A) \neq 0$. Since $H^{d-1}_J(A) = H^{d-1}_K(A) = 0$ (for dimensional reasons neither $\text{Spec}(A/J) - \{m\}$ nor $\text{Spec}(A/K) - \{m\}$ can be disconnected, so we may apply (2.9)) it follows from the Mayer–Vietoris sequence that $H^{d-2}_I(A) \neq 0$, so that $cd(A, I) = d - 2$. Hence the stated condition is necessary.

Assume the conditions in (iii). Certainly there must be some triple, say $\{1, 2, 3\}$,

such that $P_1 + P_2 + P_3$ is not m-primary (else the condition in (iii) is easily seen to fail). Set $J = P_1 \cap P_2 \cap P_3$, and $K = P_4$ if $n = 4$, and $K = P_4 \cap P_5$ if $n = 5$. By (i) and (ii) of this corollary, $cd(A, J) \leqq d - 3$, and $cd(A, K) \leqq d - 3$. By assumption $\mathrm{Spec}(A/(J + K)) - \{m\}$ is connected, so that $cd(A, J + K) \leqq d - 2$ by (2.9). Now the Mayer–Vietoris sequence shows that $cd(A, I) \leqq d - 3$. The second statement of iii) for $n = 4$ is pure combinatorics and is left to the reader. □

Example 5.12. Unfortunately, once the number of components of I is at least 6, the situation becomes much more complicated. A good case to consider is the following: let I be an ideal of $\mathrm{bight}(I) = b$, with $2b + 2 \leqq d$, and suppose that I has six minimal primes, $P_1, \ldots, P_6$. Set $\Lambda = \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 4, 5\}, \{2, 4, 6\}, \{1, 5, 6\}, \{3, 5, 6\}, \{2, 3, 5\}, \{3, 4, 6\}, \{1, 2, 6\}, \{1, 4, 5\}\}$. Suppose that $\{i, j, k\} \in \Lambda$ iff $P_i + P_j + P_k$ is m-primary. Is $cd(A, I) \leqq d - 3$? This example is arranged so that *no matter how one divides* I into two components J and K, either $cd(A, J)$ or $cd(A, K)$ is $> d - 3$. This prevents use of the Mayer–Vietoris sequence to compute $cd(A, I)$.

6. Topological applications

In this section we present some topological applications of our vanishing theorems, applied mainly to subschemes of projective space. Most of the translation from cohomological information to topological information was done by Grothendieck, Hartshorne [Ha1], [Ha2], and Ogus [O]. In particular we are able to show that for subvarieties of projective space, the homotopy groups are finite up to an integer depending upon the dimension of the ambient projective space, and the codimension of the variety. We begin by proving a proposition which is probably well-known to experts.

Proposition 6.1. *Let $X \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed equidimensional algebraic set, and let $Y \subseteq X$ be a closed subset of X such that $X - Y$ is smooth. Let A be the homogeneous coordinate ring of X, and $I \subseteq A$ the defining ideal of Y. Then $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq \dim(X) - cd(A, I)$.*

Proof. The local cohomology exact sequence implies that the cohomological dimension of $\mathrm{Spec}(A) - V(I)$ is $cd(A, I) - 1$. Hence the cohomological dimension of $X - Y$ is also $cd(A, I) - 1$, since $\mathrm{Spec}(A) - V(I)$ is a $\mathbb{C}^*$ bundle over $X - Y$. Now [Ha2, p. 148, Thm 7.4] implies that $H^i(X; \mathbb{C}) \to H^i(Y; \mathbb{C})$ is an isomorphism if $i < \dim(X) - cd(A, I)$, and an injection if $i = \dim(X) - cd(A, I)$. By the universal coefficient theorem [Sp, pg. 244] this implies that the map $H_i(Y; \mathbb{C}) \to H_i(X; \mathbb{C})$ is an isomorphism for $i < \dim(X) - cd(A, I)$, and surjective if $i = \dim(X) - cd(A, I)$. This implies the result. □

Theorem 6.2. *Let $Y \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed algebraic set consisting of irreducible components of codimension $\leqq b$. Then $H_i(\mathbb{P}^e(\mathbb{C}), Y; \mathbb{C}) = 0$ provided,*

i) (cf. [P]) $i \leqq [e/b] - 1$, *if no additional assumptions on Y are made,*
ii) $i \leqq [(e - 1)/b]$, *if Y is irreducible,*
iii) $i \leqq [(e + 1)/(b + 1)] + [e/(b + 1)] - 1$, *if Y is normal.*

Proof. Let A be the homogeneous coordinate ring of $\mathbb{P}^e(\mathbb{C})$, localized at the irrelevant ideal, and let $I \subseteq A$ be the defining ideal of Y. Then $\dim(A) = e + 1$. We

see that $c = c(I) = \text{embdim}(A) - \text{mdim}(A/I) \leqq e - b$. We may apply (3.8) to obtain that $cd(A, I) \leqq (e + 1) - [e/b]$, and hence by Proposition 6.1 the result (i) follows.

Consider (ii), using the same notation as in the proof of (i). Since I is formally geometrically irreducible, we can apply (3.8) to conclude that $cd(A, I) \leqq e + 1 - [(e - 1)/b]$. Applying Proposition 6.1 yields (ii).

Finally we prove (iii). Since I is formally geometrically irreducible, and A/I is normal on the punctured spectrum, we may apply (3.9) to obtain that $cd(A, I) \leqq (e + 1) - [(e + 1)/(b + 1)] - [e/(b + 1)]$. Applying Proposition 6.1 gives (iii). □

Remark 6.3. The result of (6.2 i) is exact for all e and b if b does not divide e. Let A be the homogeneous coordinate ring of $\mathbb{P}^e(\mathbb{C})$, and let $V \subseteq \mathbb{P}^e(\mathbb{C})$ be the union of $[e/b] + 1$ coordinate subspaces of codimension b in general position. Let I be the defining ideal of V in A. By (5.5) $cd(A, I) = (e + 1) - [e/b]$. By (5.1) i) for any prime P not equal to the irrelevant maximal ideal m, $cd(A_P, I_P) < e + 1 - [e/b]$. Thus $H^q_I(A)$ with $q = e + 1 - [e/b]$ is supported at m only. It follows from the proof of [O, Thm 4.4] that in this situation $H^q_I(A) \neq 0$ is equivalent to the fact that $H^{[e/b]}(\mathbb{P}^e\mathbb{C}) \to H^{[e/b]}(V; \mathbb{C})$ is not an isomorphism. Since this map is known to be injective, we conclude that $H^{[e/b]}(\mathbb{P}^e, V; \mathbb{C}) \neq 0$. By the universal coefficient theorem, $H_{[e/b]}(\mathbb{P}^e, V; \mathbb{C}) \neq 0$. In fact 6.2i) is exact even if b divides e [Ly2].

Theorem 6.4. *Let $Y \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed algebraic set consisting of irreducible components of codimensions $\leqq b$. Assume that Y can be cut out set-theoretically by t equations along every subvariety of $\mathbb{P}^e(\mathbb{C})$ of dimension $\geqq m$. Then $H_i(X, Y; \mathbb{C}) = 0$ for*

i) $i \leqq e - t - b + 1$ if $m = 1$,
ii) $i \leqq e - t - b + 1$, if Y is irreducible and $m = 2$.

Proof. The proof follows at once from (6.1) and (3.2) in case i), and from (6.1) and (3.10) in case ii). □

Remark 6.5. Assume that Y is locally set-theoretically a complete intersection of codimension b except at finitely many closed points. From (6.4 i) we get that $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq e - 2b + 1$. Similarly, if Y is irreducible and the set of closed points at which Y is not a set-theoretic complete intersection has dimension $\leqq 1$, we get from (6.4 ii) that $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq e - 2b + 1$. This is a generalization of the famous result of Barth [Ba].

Before we are able to apply these results to show finiteness of relative homotopy groups, we need to know that varieties of low codimension are simply connected. The results of Fulton and Lazersfeld allow one to show:

Proposition 6.6. *Let $Y \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed algebraic set consisting of irreducible components of codimension $\leqq e/3$. Then Y is simply connected.*

Proof. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y. We use induction on r to prove the proposition. If $r = 1$ the result follows from [F-L, Cor. 5.3, p. 47]. By induction $Y' = Y_1 \cap \ldots \cap Y_{r-1}$ is simply connected. The intersection $Y' \cap Y_r$ is the union of $(r - 1)$ algebraic sets $Y_i \cap Y_r$, each one of which is connected by [F-H, Cor. 1], since $\text{codim}(Y_i) + \text{codim}(Y_r) \leqq e - 2$. The intersection of any two of them,

say $(Y_i \cap Y_r) \cap (Y_j \cap Y_r) = Y_i \cap Y_j \cap Y_r$ is nonempty as the sum of the codimensions of any three of the Y_k is at most e. Therefore $Y' \cap Y_r$ is connected, and the proposition follows from the Van Kampen theorem. □

Theorem 6.7. *Under the hypotheses of* (6.2) *and* (6.4), $\pi_i(\mathbb{P}^e(\mathbb{C}), Y)$ *is finite in the same range as* $H_i(X, Y; \mathbb{C}) = 0$, *except in* (6.4 i) *where we need the additional assumption that* $b \leqq e/3$.

Proof. In (6.2 ii) and (6.4 iii), Y is simply connected by [F-L, Cor. 5.3, p. 47], if $b < e/2$, while in (6.4 i) it is simply connected by (6.6) (also in (6.2 i) provided $b \leqq e/3$). Now (6.2) and (6.4) imply that $H_i(X, Y; \mathbb{Z})$ is finite in the same range where $H_i(X, Y; \mathbb{C})$ is zero, and since finite groups form a Serre class (see [Sp, Chap. 9, Sec. 6]), $\pi_i(\mathbb{P}^e(\mathbb{C}), Y)$ also are finite.

If $b \geqq e/2$ in (6.2 ii) and (6.4 iii), then $i \leqq 1$, and $\pi_1(\mathbb{P}^e(\mathbb{C}), Y) = 0$ for Y connected, as can be seen from the exact sequence $\{1\} \to \pi_1(\mathbb{P}^e(\mathbb{C}), Y) \to \pi_0(Y) \xrightarrow{\cong} \pi_0(\mathbb{P}^e(\mathbb{C}))$. If $e/2 \geqq b > e/3$ in (6.2 i), then $i \leqq 1$, and $\pi_1(\mathbb{P}^e(\mathbb{C}), Y) = 0$ for the same reason, since $b \leqq e/2$ implies that Y is connected. Finally if $b > e/2$, in (6.2 i) then $i \leqq 0$ and $\pi_0(\mathbb{P}^e(\mathbb{C}), Y) = 0$ for all Y. □

It is natural to conjecture that $\pi_i(\mathbb{P}^e(\mathbb{C}), Y) = 0$ in the same range. This will be proved in an upcoming paper by the second author of this paper [Ly2].

Remark 6.8. Using (3.11) one can also prove the following result:

Let $X \subseteq \mathbb{P}^e(\mathbb{C})$ be an equidimensional closed algebraic set of dimension d, and let $Y \subseteq X$ be a closed algebraic set consisting of irreducible components of dimension $\geqq d'$. Assume that $X - Y$ is smooth, and that the singular locus of X has dimension $\leqq d - d'$. Then

i) $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq [(d + d' - e)/(d - d')]$.
ii) In addition to the above hypotheses, assume that Y is irreducible, and the singular locus of X has dimension $\leqq d - d' + 1$. Then, $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq [(d + d' - e - 1)/(d - d')] + 1$.
iii) In addition to the assumptions of ii), assume that Y is normal. Then, $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq [(d + d' - e)/(d - d' + 1)] + [(d + d' - e - 1)/(d - d' + 1)] + 1$.

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