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Weak positivity and the stability of certain Hilbert points, II

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In the first part of this paper [14] we proved that all points x of the Hilbert scheme of ν-canonically embedded compact complex manifolds are stable with respect to the usual group action and with respect to some invertible sheaf, provided that x is a smooth point of H_{red}. As explained there, the only reason that we had to restrict ourselves to smooth points of H_{red}, was our unability to give an affirmative answer to "problem 1.10" in [14]. In the second part, we will fill this gap (see 2.9) and thereby finish the proof of

Theorem 0.1. *There exists a quasi-projective coarse moduli space M_h for complex compact canonically polarized manifolds with Hilbert polynomial h.*

In fact, as in [14] our approach applies to a larger class of moduli problems:

0.2. Let $h(T)$ be a given polynomial of degree n and $\mathcal{M}'_h$ the moduli functor of complex projective normal irreducible varieties, with at most rational Gorenstein singularities and with an ample canonical sheaf ω_F satisfying $h(\nu) = \chi(F, \omega_F^\nu)$. If $n \leqq 2$ we take $\mathcal{M}_h = \mathcal{M}'_h$ and for $n = 3$ we take $\mathcal{M}_h$ to be the subfunctor of $\mathcal{M}'_h$ of families $f: X \to Y$ with $h^0(F, \omega_F) > 0$ for all singular fibers F of f. For $n > 3$ we take $\mathcal{M}_h$ to be the subfunctor of $\mathcal{M}'_h$ of smooth families. In any case $\mathcal{M}_h$ will be a separated and bounded moduli functor (see [8], §2 for those notations) and, in fact, any subfunctor $\mathcal{M}_h$ of $\mathcal{M}'_h$ having both properties can be choosen (see § 6).

Theorem 0.3. *For all the moduli functor $\mathcal{M}_h$ considered in* 0.2 *there exists a quasi-projective coarse moduli scheme M_h.*

Our result reproves D. Gieseker's theorem [4] on the existence of quasi-projective moduli spaces for surfaces of general type. However, as we point out in 6.4, the ample sheaf obtained by D. Gieseker is "better" than the one constructed here.

Narasimhan-Simha, Tankeev, Popp, Mumford-Fogarty and Kollár (see [14] for the references) have shown the existence of M_h as an analytic space. If one does not like D. Mumford's geometric invariant theory [10] one can use instead [8], 2.7, together with 5.2 (or 1.18 of [14]) in order to construct an ample sheaf on the

analytic moduli space directly. However, this only seems to work at present, if the non normal locus of M_h is compact. Otherwise this method will show only that the normalization of M_h is quasi-projective (see 5.9).

The methods used to answer "problem 1.10" of [14] in 2.9, are similar to those, already employed in §3 of [14], except one, an unpublished theorem on the extension of certain locally free sheaves to compactifications, due to Ofer Gabber (see 1.4 for the exact statement). After he told me about his result and the spirit of his proof, I was able to cook up my own, slightly different and more clumsy version (see 1.6) and its proof.

Those "Extension Theorems" together with a statement on "extensions after finite covers" in 1.7 are discussed in §1. In §2 we recall and extend the definition of weakly positive sheaves and reformulate "problem 1.10" of [14]. So we have to show, that for certain morphisms $f_0: X_0 \to Y_0$ the sheaves $f_{0*}\omega^{\nu}_{X_0/Y_0}$ are weakly positive. The usual arguments (see [9], [12] and [14]) are used in §4 to reduce this problem to the case $\nu = 1$.

"Weak positivity" is a positivity notation which uses the existence of sections (see 2.2), and the only reasonable methods to construct sections are using either ampleness criteria or vanishing theorems. Since both are only valid on compact manifolds or schemes, it is not too surprising that at some point we have to consider compactifications Y of Y_0 and extensions of $f_{0*}\omega^{\nu}_{X_0/Y_0}$ to coherent sheaves on Y. This is done in §3 using the "Extension theorem 1.7" and the "Nilpotent Orbit Theorem" of W. Schmid [11]. Unfortunately there are some complications due to the fact that we do not only want to consider moduli of non singular varieties. Those force us to include the quite technical constructions in the second half of §3.

Once an affirmative answer to "problem 1.10" is established we just have to apply the results of [14] to obtain theorem 0.3. Nevertheless we discuss in §6 the necessary assumptions a moduli functor $\mathscr{M}_h$ has to satisfy in order to get a quasi-projective moduli scheme, by using our method. Finally we recall which parts of [14] and this paper are necessary to obtain theorem 0.3 in different cases, and we make a few remarks on possible generalizations.

In § 5 we just recall the applications 2.9 has for fiber spaces. We could not resist however to formulate the ampleness criterion which, as J. Kollár pointed out, was more or less proven in § 2 and 4 of [14] but not stated there (see also [8], § 3).

This paper was written during a sabbatical term which I used to exploit the hospitality of the I.H.E.S. at Bures sur Yvette.

I thank Hélène Esnault for her help and comments during the preparation of this paper.
Special thanks I owe to Ofer Gabber. Without him, telling me about his extension theorem, this paper would not have been written.

Conventions

We try to use the usual notations of algebraic geometry (as for example in R. Hartshorne's book on this subject) and those from higher dimensional birational geometry (see [9] and [14]).

However, all varieties and manifolds are supposed to be defined over the field $\mathbb{C}$ of complex numbers, and the word "*scheme*" is used for "schemes, separated and of finite type over $\mathbb{C}$". *Locally free sheaves* on a scheme are supposed to be of finite rank.

We call $i: W_0 \to W$ a *compactification* if W is a proper scheme and W_0 an open dense subscheme. If W_0 is reduced (non singular or quasi-projective) we assume that W is reduced (non singular or projective) as well.

A morphism $\delta: W' \to W$ is called *birational* if there is an open dense subscheme W_0 of W such that $W'_0 = \delta^{-1}(W_0)$ is dense in W' and $\delta|_{W'_0}: W'_0 \to W_0$ an isomorphism. The *center* of δ is the complement of the largest W_0 with this property. An example is the blowing up of an ideal sheaf J on W, where the center is the closed subscheme where J is not invertible.

We will call δ a *desingularization*, if δ is a projective morphism and W' non singular.

A morphism $\delta: W' \to W$ will be called *finite and dominant*, if it is finite and if each component of W' is dominant over some component of W. In this case, we have an induced diagram

$$\begin{array}{ccc} \tilde{W}' & \xrightarrow{\tilde{\delta}} & \tilde{W} \\ \downarrow & & \downarrow \tau \\ W' & \xrightarrow{\delta} & W \end{array}$$

where $\tilde{W}$ and $\tilde{W}'$ are the normalizations. We have an inclusion

$$\delta_* \mathcal{O}_{W'} \to \tau_* \tilde{\delta}_* \mathcal{O}_{\tilde{W}}$$

and a trace map $\tau_* \tilde{\delta}_* \mathcal{O}_{\tilde{W}} \to \tau_* \mathcal{O}_{\tilde{W}}$. The induced map $\delta_* \mathcal{O}_{W'} \to \tau_* \mathcal{O}_{\tilde{W}}$ will be called the *trace map* of δ.

Finally, a *point* $x \in W$ is always supposed to be a closed point, if not stated otherwise.

If X is a Cohen Macauley scheme, ω_X will denote the *dualizing sheaf*. If $f: X \to Y$ is a morphism of schemes, Y Gorenstein and X Cohen Macauley, then we write $\omega_{X/Y} = \omega_X \otimes f^* \omega_Y^{-1}$.

If $f: X \to Y$ is flat and Cohen Macauley or Gorenstein, then $\omega_{X/Y}$ denotes the *relative dualizing sheaf*.

If X is a non singular scheme we call D a *normal crossing divisor* if D is a divisor on X, $X - D_{\text{red}}$ open and dense in X and if the components of D are non singular divisors intersecting transversally.

§ 1. Extensions of locally free sheaves to compactifications

1.1. Let us consider a reduced scheme W_0 and a locally free sheaf $\mathcal{F}_0$ on W_0. Assume that there is a desingularization $\delta_0: W'_0 \to W_0$, a non singular compactification $i': W'_0 \to W'$ and a locally free sheaf $\mathcal{F}'$ on W' with $i'^* \mathcal{F}' \cong \delta_0^* \mathcal{F}_0$.

We want to study conditions which imply that $\mathcal{F}'$ comes from some compactification W of W_0. More precisely we want to answer

Problem 1.2. Does there exist a compactification $i: W_0 \to W$ and a locally free sheaf $\mathscr{F}$ on W such that for all commutative diagrams of birational morphisms

$$\begin{array}{ccccc} W'' & \overset{\delta'}{\to} & W' & \overset{i'}{\leftarrow} & W_0' \\ {\scriptstyle\delta}\downarrow & & & & \downarrow{\scriptstyle\delta_0} \\ W & & \underset{i}{\leftarrow} & & W_0 \end{array}$$

with W'' non singular, one has $\delta^*\mathscr{F} = \delta'^*\mathscr{F}'$?

Here we use " = " to indicate an isomorphism of sheaves (unique if W'' is proper) which coincides over $\delta'^{-1}(W_0') \cap \delta^{-1}(W_0)$ with the one given in 1.1.

Obviously the answer is no. In fact, the following condition is necessary:

Condition 1.3. Let C be a non singular curve, C_0 an open dense subcurve and $\eta_0: C_0 \to W_0$ a morphism. Then there exists a locally free sheaf $\mathscr{G}$ on C with $\eta_0^*\mathscr{F}_0 \cong \mathscr{G}|_{C_0}$, such that for all diagrams

$$\begin{array}{ccccc} C' & \overset{\eta'}{\to} & W' & \overset{i'}{\leftarrow} & W_0' \\ {\scriptstyle\tau}\downarrow & & & & \downarrow{\scriptstyle\delta_0} \\ C & \leftarrow & C_0 & \underset{\eta_0}{\to} & W_0 \end{array}$$

with τ finite, $\eta'(\tau^{-1}C_0) \subseteq W_0'$ and $\eta_0 \circ \tau|_{\tau^{-1}(C_0)} = \delta_0 \circ \eta'|_{\tau^{-1}(C_0)}$, one has $\tau^*\mathscr{G} = \eta'^*\mathscr{F}'$.

O. Gabber studied the locally ringed space $W^\wedge$ obtained by taking the limit over all compactifications W of W_0. Then he showed that 1.3 implies that one can solve the extension problem locally for $W^\wedge$. Those solutions, however, as he can prove, must come from a locally free sheaf $\mathscr{F}$ defined on a scheme W which compactifies W_0. Thereby he obtains:

Theorem 1.4 (O. Gabber). *Problem* 1.2 *has an affirmative answer if and only if condition* 1.3 *is satisfied.*

For the purpose of this paper we will formulate (in 1.6) and prove a slightly different version of this "extension theorem". We replace 1.3 by asking for the existence of compatible extensions of $\mathscr{F}_0$ to compactifications of non singular strata of W_0. It will be quite obvious that 1.6 is weaker than O. Gabber's theorem. However, as he pointed out, it is likely that both are equivalent.

The reader should have the following type of example in mind: assume that for all desingularizations $\eta_0: Z_0 \to S_0$ of subschemes S_0 of W_0 the sheaf $\eta_0^*\mathscr{F}_0$ has a "natural" integrable connection. Then, if $Z_0 \to Z$ is a compactification of Z and $Z - Z_0$ a normal crossing divisor, $\eta_0^*\mathscr{F}_0$ has a canonical extension as a locally free sheaf $\mathscr{F}_Z$ on Z (see Deligne [2]). If all the monodromies at the boundary are unipotent, the canonical extension will be compatible with pullbacks. Under this assumption we want to extend $\mathscr{F}_0$ to some locally free sheaf $\mathscr{F}$ which gives the canonical extensions on all possible Z.

In general, the condition that the monodromies are unipotent is too stong. Hence, to include the case of quasi-unipotent monodromies, one could also consider generically finite morphisms $\eta_0: Z_0 \to S_0$ and ask for unipotent monodromies, provided the degree of η_0 is large enough. May be, with this type of example in mind, the following definition looks reasonable:

Definition 1.5. Let Y_0 be a reduced scheme.

a) A category $\mathscr{Z}$ of *compactifying tripels* for Y_0 is a category of tripels $\underline{Z} = (Z, Z_0, \eta_0)$, where Z is a non singular proper scheme, Z_0 the complement of a normal crossing divisor in Z and $\eta_0: Z_0 \to Y_0$ a proper morphism. The morphisms

$$\tau: \underline{Z} = (Z, Z_0, \eta_0) \to \underline{Z}' = (Z', Z_0', \eta_0')$$

in $\mathscr{Z}$ are supposed to be all morphisms $\tau: Z \to Z'$ with $\tau(Z_0) \subseteq Z_0'$ and $\eta_0' \circ \tau|_{Z_0} = \eta_0$.

b) We call a category of compactifying tripels *complete*, if for all $\underline{Z} = (Z, Z_0, \eta_0)$ in $\mathscr{Z}$ and for all proper morphisms $\tau: Z' \to Z$, the tripel

$$(Z', Z_0' = \tau^{-1}(Z_0), \eta_0' = \eta_0 \circ \tau|_{Z_0'})$$

belongs to $\mathscr{Z}$ if and only if Z' is non singular and $Z' - Z_0'$ a normal crossing divisor.

c) We say that $\mathscr{Z}$ *covers* Y_0, if $\mathscr{Z}$ is complete and if there are finitely many $\underline{Z}^{(i)} = (Z^{(i)}, Z_0^{(i)}, \eta_0^{(i)}) \in \mathscr{Z}$, let's say for $i = 1, \ldots, k$, such that

i) There is an open dense non singular subscheme $S^{(i)}$ of $\eta_0^{(i)}(Z_0^{(i)})$ such that $\eta_0^{(i)}|_{\eta_0^{(i)-1}(S^{(i)})}: \eta_0^{(i)-1}(S^{(i)}) \to S^{(i)}$ is finite.
ii) $Y_0 = \bigcup_{i=1}^{k} S^{(i)}$.

d) We say that $\mathscr{Z}$ *covers* Y_0 *birationally*, if we can choose $\underline{Z}^{(1)}, \ldots, \underline{Z}^{(k)}$ such that in addition to i) and ii) the morphisms $\eta_0^{(i)}|_{\eta_0^{(i)-1}(S^{(i)})}$ are isomorphisms (and hence $\eta_0^{(i)}$ is a desingularization).

e) Let $\tau_0: W_0 \to Y_0$ be a proper morphism and $i: W_0 \to W$ a compactification. We write $\mathscr{Z}_W$ for the full subcategory of $\mathscr{Z}$ consisting of all $\underline{Z} = (Z, Z_0, \eta_0) \in \mathscr{Z}$ for which $\eta_0: Z_0 \to Y_0$ factors like $Z_0 \xrightarrow{\eta_0'} W_0 \xrightarrow{\tau_0} Y_0$ and η_0' is the restriction of some morphism $\eta': Z \to W$.

f) Let $\mathscr{F}_0$ be a locally free sheaf on Y_0. An *extension* $\mathscr{G}$ *of* $\mathscr{F}_0$ *to* $\mathscr{Z}$ consists of a sheaf $\mathscr{G}_{\underline{Z}}$ on Z for all $\underline{Z} = (Z, Z_0, \eta_0) \in \mathscr{Z}$ with $\eta_0^* \mathscr{F}_0 = \mathscr{G}_{\underline{Z}}|_{Z_0}$ such that for all morphisms $\tau: \underline{Z} \to \underline{Z}'$ in $\mathscr{Z}$ one has $\tau^* \mathscr{G}_{\underline{Z}'} = \mathscr{G}_{\underline{Z}}$.

g) Using the notation from e), let $\mathscr{F}$ be a locally free sheaf on W with $i^* \mathscr{F} = \tau_0^* \mathscr{F}_0$. Let $\mathscr{G}$ be an extension of $\mathscr{F}_0$ to $\mathscr{Z}$. Then we say that $\mathscr{F}$ is *induced* by $\mathscr{G}$ if for all $\underline{Z} \in \mathscr{Z}_W$ and all prolongations $\eta': Z \to W$ of $\eta_0': Z_0 \to W_0$ one has $\eta'^* \mathscr{F} = \mathscr{G}_{\underline{Z}}$.

Again some remark about " = " is necessary: in f) and g) it means that we fix an isomorphism $\eta_0^* \mathscr{F}_0 \cong \mathscr{G}_{\underline{Z}}|_{Z_0}$ and $i^* \mathscr{F} \cong \tau_0^* \mathscr{F}_0$. The equality $\eta'^* \mathscr{F} = \mathscr{G}_{\underline{Z}}$ means that

$\eta_0'^* i^* \mathcal{F} \cong \eta_0^* \mathcal{F}_0 \cong \mathcal{G}_{\underline{Z}}|_{Z_0}$ is the restriction of $\eta'^* \mathcal{F} \cong \mathcal{G}_{\underline{Z}}$. Remark that, since Z is non singular and Z_0 dense in Z, the isomorphism $\eta'^* \mathcal{F} \cong \mathcal{G}_{\underline{Z}}$ is uniquely determined by its restriction to Z_0.

Theorem 1.6. *Let W_0 be a reduced scheme, and $\mathcal{F}_0$ be a locally free sheaf on W_0. Assume that there exists a category $\mathcal{Z}$ of compactifying triples and an extension $\mathcal{G}$ of $\mathcal{F}_0$ to $\mathcal{Z}$. If $\mathcal{Z}$ covers W_0 birationally, then there exists a compactification $i: W_0 \to W$ and a locally free sheaf $\mathcal{F}$ on W such that $i^* \mathcal{F} = \mathcal{F}_0$ and such that $\mathcal{F}$ is induced by $\mathcal{G}$.*

Again, the assumptions made in 1.6 are obviously necessary: if $\mathcal{F}$ and W are given we can take $\mathcal{Z}$ to be the category of all compactifying tripels and define $\mathcal{G}$ on $\mathcal{Z}_W$ by $\mathcal{G}_{\underline{Z}} = \eta^* \mathcal{F}$ (using the notations from 1.5.g). Before proving 1.6 let us state as consequence:

Theorem 1.7. *Let Y_0 be a reduced scheme and $\mathcal{F}_0$ a locally free sheaf on Y_0. Assume that there exists some category $\mathcal{Z}$ of compactifying tripels for Y_0 and an extension $\mathcal{G}$ of $\mathcal{F}_0$ to $\mathcal{Z}$. If $\mathcal{Z}$ covers Y_0 then we can obtain a finite dominant morphism $\tau_0: W_0 \to Y_0$ and a compactification $i: W_0 \to W$ and a locally free sheaf $\mathcal{F}$ on W such that*

a) *The trace map from $\tau_{0*} \mathcal{O}_{W_0}$ to the integral closure $\mathcal{O}_{\tilde{Y}_0}$ of $\mathcal{O}_{Y_0}$ factors over $\mathcal{O}_{Y_0}$.*
b) *$i^* \mathcal{F} = \tau_0^* \mathcal{F}_0$ and $\mathcal{F}$ is induced by $\mathcal{G}$.*

Since the assumption in 1.7 are the same as in 1.6, except of the missing word "birationally", 1.7 is a consequence of 1.6 and of the following lemma:

Lemma 1.8. *Let Y_0 be a reduced scheme and $\mathcal{Z}$ a category of compactifying tripels which covers Y_0. Then there exists a finite dominant morphism $\tau_0: W_0 \to Y_0$ such that:*

a) *The trace map from $\tau_{0*} \mathcal{O}_{W_0}$ to the integral closure $\mathcal{O}_{\tilde{Y}_0}$ of $\mathcal{O}_{Y_0}$ factors over $\mathcal{O}_{Y_0}$.*
b) *The category $\mathcal{Z}'$ of all compactifying tripels $\underline{Z} = (Z, Z_0, \eta_0)$ of W_0 with $(Z, Z_0, \tau_0 \circ \eta_0) \in \mathcal{Z}$ covers W_0 birationally.*

We will prove 1.8 at the end of this section. To prove 1.6 we need the following lemma. We remind that all schemes are of finite type over $\mathbb{C}$ and separated.

Lemma 1.9. *Let W and W' be reduced schemes, S a closed subscheme of W, S_0 an open dense subscheme of S and $\delta: W' \to W$ a desingularization with center in S such that $\delta^{-1}(S)$ is a divisor. Let T be the closure of $\delta^{-1}(S_0)$ in W' and E an effective divisor on W' with $E \cap T = \varnothing$. Then we can find a diagram of birational morphisms*

$$\begin{array}{ccc} V' & \xrightarrow{\varepsilon} & W' \\ \downarrow \rho & & \downarrow \delta \\ V & \xrightarrow{\sigma} & W \end{array} .$$

and a Cartier divisor D on V with:

i) *The centers of σ and ε lie in $S - S_0$ and $\delta^{-1}(S - S_0)$ respectively.*
ii) *If S' is the proper transform of S under σ, then ρ is a desingularization with center in S'.*
iii) *$D \cap S' = \varnothing$ and $\delta^* D = \varepsilon^* E$.*

Proof. If f is any morphism we write $f'(\ )$ instead of $f^*(\ )$/torsion. Let us denote $S - S_0$ by C. By assumption $\delta' I_s$ is invertible, where I_s is the ideal sheaf of S, and for some effective divisor Δ with $\delta(\Delta) \subset C$ we can write $\delta' I_s = \mathcal{O}_{W'}(-T-\Delta)$. We have $\delta(E) \cap S \subset C$ and, since W is non singular outside of S, $\delta(E)|_{W-C}$ is a Cartier divisor.

We choose I to be an ideal sheaf such that I coincides with $\mathcal{O}_W(-\delta(E))$ on $W - C$ and such that $\delta' I \subset \mathcal{O}_{W'}(-E-\Delta)$. For example, for $m \gg 0$ we can choose $I = \mathcal{O}_W(-\delta(E)) \cap I_C^m$. The assumptions of 1.7 are compatible with blowing up W', as long as the center lies in $\delta^{-1}(S - S_0)$. Therefore we may assume that $\delta' I = \mathcal{O}_{W'}(-\Sigma)$ for some effective divisor Σ supported in $E + \delta^{-1}(C)$. Consider the ideal sheaf $J = (I_s \cup I) \cdot \mathcal{O}_W$. We have an inclusion $I \subset J$ and $\mathrm{Im}(J \to \mathcal{O}_s) = \mathrm{Im}(I \to \mathcal{O}_s)$.

$\delta' J$ is invertible outside of $\delta^{-1}(C)$ and we can choose $\varepsilon: V' \to W'$ to be any blowing up with center in $\delta^{-1}(C)$ such that V' is non singular and $\varepsilon'\delta' J$ invertible. For some effective divisor Γ on V' supported in $\varepsilon^* E + \varepsilon^{-1}\delta^{-1}(C)$ we can write $\varepsilon'\delta' J = \mathcal{O}_{V'}(-\varepsilon^*\Sigma + \Gamma)$.

If T' denotes the proper transform of T in V' we still have

$$\mathrm{Im}(\varepsilon'\delta' J \to \mathcal{O}_{T'}) = \mathrm{Im}(\varepsilon^*\delta' I \to \mathcal{O}_{T'})$$

and hence Γ will not meet T'. Since $\varepsilon^*\delta' I_s$ is also contained in $\varepsilon'\delta' J$ we have $\varepsilon^*\Delta + \varepsilon^* T \geqq \varepsilon^*\Sigma - \Gamma$ and, by the choice of I, $\varepsilon^*\Sigma \geqq \varepsilon^* E + \varepsilon^*\Delta$. Then $\Gamma + \varepsilon^* T \geqq \varepsilon^* E$ and, since $E \cap T = \varnothing$ we find $\Gamma \geqq \varepsilon^* E$.

J and I are both invertible outside of C and we can choose $\sigma: V \to W$ to be the successive blowing up of J and I. By the universal property of blowing ups we find $\varphi: V' \to V$. Blowing up a little bit more we may assume 1.9, ii to hold true. We have an inclusion $\sigma' I \to \sigma' J$ and, since both are invertible, $\sigma' J = \mathcal{O}_V(D') \otimes \sigma' I$ for an effective Cartier divisor D'. By construction $\rho^* D' = \Gamma \geqq \varepsilon^* E$. Since $\rho^* D'$ does not meet the closure T' of $\rho^{-1}(\sigma^{-1}(S_0))$, D' can not meet the closure S' of $\sigma^{-1}(S_0)$. Therefore D' does not meet the singular locus of V and we can choose for D some subdivisor of D'.

Proof of 1.6. We proceed by induction on $\dim(W_0)$. If $\dim(W_0) = 0$ there is nothing to show. Let $\dim(W_0) = n$. Since $\mathscr{Z}$ covers W_0 birationally, it must contain some desingularization of each component of W_0. Of course, we may assume that $\mathscr{Z}$ is closed under finite disjoint unions and therefore we have:

a) There is a compactification W of W_0 and a desingularization $\delta: W' \to W$ such that $\underline{W}' = (W', W_0' = \delta^{-1}(W_0), \delta_0 = \delta|_{W_0'}) \in \mathscr{Z}$. Let us write $\mathscr{G} = \mathscr{G}_{\underline{W}'}$.

If we choose the center S of δ as small as possible, we may assume that $S_0 = S \cap W_0$ is the union of finitely many $S^{(i)}$, where we use the notation from 1.5, *c*. Therefore, if $\mathscr{Z}'$ denotes the full subcategory of $\mathscr{Z}$ consisting of all $\underline{Z} = (Z, Z_0, \eta_0) \in \mathscr{Z}$ with $\eta_0(Z_0) \subset S_0$, $\mathscr{Z}'$ will cover S_0. Moreover $\mathscr{G}$ induces an extension of $\mathscr{E}_0 = \mathscr{F}_0|_{S_0}$ to $\mathscr{Z}'$, again denoted by $\mathscr{G}$. By induction we may assume that there exists a compactification S' of S_0 and an extension $\mathscr{E}$ of $\mathscr{E}_0$ to S', which is induced by $\mathscr{G}$ restricted to $\mathscr{Z}'$. Blowing up W with centers in $S - S_0$, we may assume that S' is the closure of S_0 in W.

Let $\mathscr{B}$ be any coherent extension of $\mathscr{F}_0$ to W. Blowing up Fitting ideals we may assume that $\mathscr{B}$ is locally free. Since moreover, after blowing up, $W - W_0$ will become the exact support of an effective Cartier divisor, we can choose $\mathscr{B}$ as large as we want. Hence we can obtain:

b) There exists a locally free sheaf $\mathscr{B}$ on W with $\mathscr{B}|_{W_0} = \mathscr{F}_0$ such that $\mathscr{G} \subset \delta^*\mathscr{B}$ and $\mathscr{E} \subset \mathscr{B}|_{S'}$, where, of course, all inclusions coincide with the given isomorphisms on W_0.

Both, *a* and *b* are compatible with further blowing ups and we may assume in addition:

c) $S = S'$ is the closure of S_0 in W and $\delta^{-1}(S) \cup \delta^{-1}(W - W_0)$ is a normal crossing divisor. If T denotes the closure of $\delta^{-1}(S_0)_{\mathrm{red}}$ in W', then

$$\underline{T} = (T, T_0 = W'_0 \cap T, \eta_0 = \delta_0|_{T_0}) \in \mathscr{Z}'.$$

If we write $\eta = \delta|_T$, then $\mathscr{G}_{\underline{T}} = \eta^*\mathscr{E}$. Consider the restriction maps $\varphi: \mathscr{B} \to \mathscr{B}|_S$ and $\varphi': \delta^*\mathscr{B} \to \delta^*\mathscr{B}|_T$. We define $\mathscr{B}' = \varphi^{-1}(\mathscr{E})$ and $\mathscr{K} = \varphi'^{-1}(\mathscr{G}_{\underline{T}})$. If we write again $\delta'(\mathscr{B}')$ instead of $\delta^*(\mathscr{B}')$/torsion, we have an inclusion $\delta'(\mathscr{B}') \to \mathscr{K}$. Since $\mathscr{G}|_T = \mathscr{G}_{\underline{T}}$, we have a second inclusion $\mathscr{G} \to \mathscr{K}$, and both are isomorphisms over W'_0. Replacing W' by some blowing up, we may assume that $\delta'(\mathscr{B}')$ is invertible and (by abuse of notations) $\mathscr{K}$ as well. Moreover, by construction

$$\delta'(\mathscr{B}')|_T = \mathscr{K}|_T = \mathscr{G}|_T = \eta^*\mathscr{E}.$$

Therefore we can find some effective divisor E on W', not meeting T and contained in $\delta^{-1}(S - S_0)$ such that

$$\mathscr{G} \to \mathscr{K} \to \delta'(\mathscr{B}') \otimes \mathscr{O}_{W'}(E)\,.$$

Let us apply Lemma 1.9. Using the notations introduced there, we may assume that $\delta^{-1}\sigma^{-1}(W - W_0)$ is a normal crossing divisor on V' and that $\sigma'(\mathscr{B}')$ is locally free. Then V' comes from some $\underline{V}' \in \mathscr{Z}$ and

$$\mathscr{G}_{\underline{V}'} = \varepsilon^*\mathscr{G} \subset \varepsilon^*\delta'\mathscr{B}' \otimes \rho^*\mathscr{O}_V(D) = \rho^*(\tau'\mathscr{B}' \otimes \mathscr{O}_V(D))\,.$$

After renaming we may replace b by the assumption
b') $\mathscr{G}$ is contained in $\delta^*\mathscr{B}$, $\mathscr{E} = \mathscr{B}|_S$ and $\mathscr{G}|_T = \delta^*\mathscr{B}|_T$.

Now let E be the support of $\delta^*\mathscr{B}/\mathscr{G}$. Again E is a divisor not meeting T. By 1.9, again, we can blow up W and W' such that E will become the pullback of some Cartier divisor D on W which is not meeting S. In other words, we can assume that for some neighbourhood U of S, the inclusion $\mathscr{G} \to \delta^*\mathscr{B}$ is an isomorphism over $\delta^{-1}(U)$. Glueing together $\mathscr{G}|_{W' - \delta^{-1}(S)}$ and $\mathscr{B}|_U$ along $\delta^{-1}(U) - \delta^{-1}(S) \simeq U - S$, we obtain an extension $\mathscr{F}$ of $\mathscr{F}_0$ to W. By construction $\delta^*\mathscr{F} = \mathscr{G}$ and, since all $\mathscr{G}_{\underline{Z}}$ are compatible, $\eta^*\mathscr{F} = \mathscr{G}_{\underline{Z}}$ for all $\underline{Z} \in \mathscr{Z}_W$.

This ends the proof of 1.6. For 1.7 we still have to prove 1.8. Before doing so we need some method to enforce the condition *a* of 1.8:

Lemma 1.10. *Let $\tau: W'_0 \to Y_0$ be a dominant finite morphism of reduced schemes and W'_0 normal. Then there exists a birational morphism $\eta: W'_0 \to W_0$ and a finite*

morphism $\tau_0: W_0 \to Y_0$ *such that* $\tau = \tau_0 \circ \eta$ *and:*

a) *The trace map from* $\tau_{0*}\mathcal{O}_{W_0}$ *to the integral closure* $\mathcal{O}_{\tilde{Y}_0}$ *of* $\mathcal{O}_{Y_0}$ *factors over* $\mathcal{O}_{Y_0}$.
b) *If* Y_1 *is an open subscheme of* Y_0, $W_1' = \tau^{-1}(Y_1)$, $\tau_1: W_1' \to Y_1$ *a morphism satisfying a) and if* $\tau|_{W_1'}$ *factors like*

$$W_1' \to W_1 \xrightarrow{\tau_1} Y_1$$

then the restriction of τ_0 *to* $\tau_0^{-1}(Y_1)$ *factors like* $\tau_0^{-1}(Y_1) \to W_1 \xrightarrow{\tau_1} Y_1$.

Proof. Let us choose $\mathcal{N}$ to be the sheaf of all local sections s of $\tau_*\mathcal{O}_{W_0'}$ such that the traces of all local sections of $s \cdot \tau_*\mathcal{O}_{W_0'}$ lie in $\mathcal{O}_{Y_0}$. Then $\mathcal{N}$ contains $\mathcal{O}_{Y_0}$ and is obviously an $\mathcal{O}_{Y_0}$ algebra. $\mathcal{N}$ contains $J \cdot \tau_*\mathcal{O}_{W_0'}$ for any ideal sheaf J of $\mathcal{O}_{Y_0}$ with $J \cdot \mathcal{O}_{\tilde{Y}_0} \subset \mathcal{O}_{Y_0}$. Therefore the integral closure of $\mathcal{N}$ is $\tau_*\mathcal{O}_{W_0'}$. So we found on $\mathcal{N}$ such that $W_0 = \mathrm{Spec}_{Y_0}(\mathcal{N})$ satisfies a. We can replace $\mathcal{N}$ by any larger subalgebra of $\tau_*\mathcal{O}_{W_0'}$, as long as the trace of all local sections s of $\mathcal{N}$ lies in $\mathcal{O}_{Y_0}$. Then, if $\tau_1: W_1 \to Y_1$ is as in b, we can assume that $\tau_*\mathcal{O}_{W_1}$ is contained in $\mathcal{N}|_{Y_1}$.

Proof of 1.8. Let $\tau_0: W_0 \to Y_0$ be any finite morphism and $\mathcal{Z}'$ as in 1.8, b.

a) If $\mathcal{Z}$ covers Y_0 then $\mathcal{Z}'$ covers W_0. Moreover, if $\mathcal{Z}$ covers Y_0 birationally, then the same holds for $\mathcal{Z}'$.

In fact, we just have to choose the generators $\underline{Z}^{(i)}$ of $\mathcal{Z}$ (as defined in 1.5c) such that $\tau_0^{-1}(S^{(i)})$ is non singular.

If $j: U_0 \to Y_0$ is an open subscheme we will write $j^*\mathcal{Z}$ for the category of all compactifying triples $\underline{Z}' = (Z', Z_0', \eta_0')$ of U_0 for which there exists some $\underline{Z} \in \mathcal{Z}$ with $Z = Z'$, $Z_0' = \eta_0^{-1}(U_0)$ and $\eta_0' = \eta_0|_{Z_0'}$. Choosing the generators $\underline{Z}^{(i)}$ of $\mathcal{Z}$ such that $U_0 = \bigcup_{i=\tau'}^{k} S^{(i)}$ for some τ' we obtain:

b) If $\mathcal{Z}$ covers Y_0 then $j^*\mathcal{Z}$ covers U_0.

Now let $\underline{Z}^{(i)}$ be generators of $\mathcal{Z}$ and let S_0 be the union of the closures of all $S^{(i)}$ for which $\eta_0^{(i)}$ is not birational. We may assume that all $Z^{(i)}$ are varieties, that

$$\dim(Z^{(1)}) = \dim(Z^{(2)}) = \ldots = \dim(Z^{(\tau)})$$

and that S_0 is the closure of $\bigcup_{i=1}^{\tau} S^{(i)}$. Moreover we may assume that $\mathbb{C}(Z^{(i)})$ is a Galois extension of $\mathbb{C}(S^{(i)})$. We can find an open affine subscheme Y_1 of Y_0 containing all general points of components of Y_0 and of the $S^{(i)}$ for $i = 1, \ldots, \tau$.

We can choose primitive polynomials $G_i(t)$ for $\mathbb{C}(Z^{(i)})$ whose coefficients are regular functions on $S^{(i)} \cap Y_1$. If $Y_1 \subset \mathbb{A}^m$ is an embedding, those $G_i(t)$ lift to some $F_i(t) \in \mathbb{C}[x_1 \ldots x_m, t]$. Let $\gamma: B \to \mathbb{A}^m$ be a finite Galois cover such that all $F_i(t)$ for $i = 1, \ldots, \tau$ split over $\mathbb{C}(B)$ in linear factors.

Define τ_1 to be the restriction of γ to $W_1 = \gamma^{-1}(Y_1)$. Then, by construction, the trace map from $\tau_{1*}\mathcal{O}_{W_1}$ maps to $\mathcal{O}_{Y_1}$. By Lemma 1.10 we choose $\tau_0: W_0 \to Y_0$, where W_0' is the union of the normalizations of the components of Y_0 in the function fields of the components of W_1. Let S' be the closure of an irreducible component of $\tau_0^{-1}(S^{(i)})$ in W_0 for $i \in \{1, \ldots, \tau\}$. By construction $G_i(t)$ will have a root in $\mathbb{C}(S')$ and

$\mathbb{C}(S')$ will contain $\mathbb{C}(Z^{(i)})$.Therefore we can find some $\underline{Z} \in \mathscr{Z}$, such that $\eta_0 : Z_0 \to Y_0$ factors over a birational map $\eta_0' : Z_0 \to S'$. We can add those to a generator system $\underline{Z}^{(i)'}$ of $\mathscr{Z}'$. As we have seen $j'^*\mathscr{Z}'$ covers $\tau_0^{-1}(U_0)$ birationally and since we may choose a system of generators of $\mathscr{Z}'$ containing those of $j'^*\mathscr{Z}'$ we have for $\mathscr{Z}'$ a generator system $\underline{Z}^{(i)'}$ such that $\dim(Z^{(i)'}) < \dim(Y_0 - U_0)$, whenever $\eta_0^{(i)'}$ is not birational.

§ 2. Weak positivity, again

2.1. Let Y_0 be a reduced quasi-projective scheme, $\mathscr{H}_0$ an ample invertible sheaf on Y_0, U an open subscheme of Y_0 and $\mathscr{F}_0$ a coherent torsion free sheaf on Y_0.

If $i : V \to Y_0$ is the largest open subscheme such that $i^*\mathscr{F}_0$ is locally free, we define $S^\mu(\mathscr{F}_0) = i_* S^\mu(i^*\mathscr{F}_0)$ and $\Lambda^\mu(\mathscr{F}_0) = i_* \Lambda^\mu(i^*\mathscr{F}_0)$. Recall

Definition. a) $\mathscr{F}_0$ is called *globally generated by A over U* if A is a subspace of $H^0(Y_0, \mathscr{F}_0)$ and $A \otimes_{\mathbb{C}} \mathscr{O}_U \to \mathscr{F}_0|_U$ surjective.
b) $\mathscr{F}_0$ is called *globally generated over U* if $\mathscr{F}_0$ is globally generated by $H^0(Y_0, \mathscr{F}_0)$ over U.
c) $\mathscr{F}_0$ is called *weakly positive over U* if $\mathscr{F}_0|_U$ is locally free and if for all $a > 0$ we can find some $b > 0$ such that $S^{a \cdot b}(\mathscr{F}_0) \otimes \mathscr{H}_0^b$ is globally generated over U.

The definition of weakly positive is independent of the ample sheaf $\mathscr{H}_0$ choosen. If $\mathscr{F}_0$ is locally free on Y_0 and Y_0 projective, then "$\mathscr{F}_0$ weakly positive over Y_0" is the same as "$\mathscr{F}_0$ semi-positive" in the sense of Kawamata [5].

For technical reasons we want to "bound the poles" of the sections needed in 2.1, c in order to generate the sheaf over U. May be, one can get along without, but in any case it seems to be of interest to keep control, where those sections are coming from. Also it seems that this boundedness makes it easier to glue local informations together (see 2.11). N. Nakayama gave an example of an invertible sheaf, where the boundedness does not follow from the weak positivity. Hence, we add it to the definition:

Definition 2.2. Keeping the notations from 2.1, we will call $\mathscr{F}_0$ *weakly positive over U with respect to* $(Y', \mathscr{F}')$ if the following conditions hold:

i) Y' is a projective non singular scheme, $\mathscr{F}'$ a coherent torsion free sheaf on Y' and $\mathscr{F}_0|_U$ is locally free.
ii) Y' is obtained as a desingularization $\delta : Y' \to Y$ of a projective compactification $j : Y_0 \to Y$.
iii) For all $\eta > 0$, one has an inclusion

$$\varphi_\eta : S^\eta(\mathscr{F}_0) \to j^* \delta_* S^\eta(\mathscr{F}') .$$

iv) Given an ample invertible sheaf $\mathscr{H}$ on Y and $a > 0$ we can find some $b > 0$ such that $S^{a \cdot b}(\mathscr{F}_0) \otimes j^*\mathscr{H}^b$ is globally generated over U by

$$A = H^0(Y, \delta_* S^{a \cdot b}(\mathscr{F}') \otimes \mathscr{H}^b) \cap H^0(Y_0, S^{a \cdot b}(\mathscr{F}_0) \otimes j^*\mathscr{H}^b)$$

(Of course, the intersection takes place in $H^0(Y_0, j^*\delta_* S^{a\cdot b}(\mathscr{F}') \otimes j^*\mathscr{H}^b)$ using $\varphi_{a\cdot b}$).

2.3. Some properties. If $j: Y_0 \to Y$ is a compactification and $n: \tilde{Y} \to Y$ the normalization, we write $\mathscr{A}_Y = j_*\mathscr{O}_{Y_0} \cap n_*\mathscr{O}_{\tilde{Y}}$. $\mathscr{A}_Y$ is an $\mathscr{O}_Y$-algebra, quasi-coherent, as the intersection of quasi-coherent sheaves, and coherent, as subsheaf of the coherent sheaf $n_*\mathscr{O}_{\tilde{Y}}$.

Let us write $(\ )^{**}$ instead of $S^1(\ )$ for the reflexive hull on Y'. Let $Y' \xrightarrow{\delta} Y \xleftarrow{j} Y_0$ be as in 2.2 i, ii and let $\sigma: Y'' \to Y'$ be a birational morphism with Y'' non singular and projective.

a) We can, in 2.2, assume that $\mathscr{A}_Y = \mathscr{O}_Y$.

b) If $\mathscr{A}_Y = \mathscr{O}_Y$ and if $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$, then $\mathscr{F} = j_*\mathscr{F}_0 \cap \delta_*(\mathscr{F}'^{**})$ is a coherent and torsion free sheaf, weakly positive over U.

c) If $\mathscr{F}_0$ is weakly positive over U with respect to $(Y'', \mathscr{F}'')$, then the same holds with respect to $(Y', \sigma_*\mathscr{F}'')$.

d) If $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$, then we can find some $\mathscr{F}''$ with $\sigma_* S^\eta(\mathscr{F}'') = S^\eta(\mathscr{F}')$. Especially $\mathscr{F}_0$ will be weakly positive over U with respect to $(Y'', \mathscr{F}'')$.

Proof. For a we just have to replace Y by $\mathrm{Spec}_Y(\mathscr{A}_Y)$.

In b it is obvious that $\mathscr{F}$ is coherent and torsion free. For simplicity we can write $\mathscr{F}'$ instead of $\mathscr{F}'^{**}$. For A as in 2.2., iv, we have to show that $A \otimes \mathscr{O}_Y$ maps to $S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b$. By abuse of notations we may assume that $\mathscr{F}$ is locally free. For $\eta > 0$ we have, for $\delta': Y' \to \tilde{Y}$ with $\delta = n \circ \delta'$,

$$S^\eta(n^*\mathscr{F}) = S^\eta(n^* j_*\mathscr{F}_0 \cap \delta'_*\mathscr{F}') = n^* S^\eta(j_*\mathscr{F}_0) \cap S^\eta(\delta'_*\mathscr{F}') .$$

Outside of the center of δ' both, $\delta'_* S^\eta(\mathscr{F}')$ and $S^\eta(\delta'_*\mathscr{F}')$ coincide and we get a map from the first sheaf to the second. Then $A \otimes \mathscr{O}_Y$ maps to

$$n_* n^* S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b = S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b \otimes n_*\mathscr{O}_Y$$

and to

$$j_* S^{a\cdot b}(\mathscr{F}_0) \otimes \mathscr{H}^b = S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b \otimes j_*\mathscr{O}_{Y_0}.$$

Since $\mathscr{A}_Y = \mathscr{O}_Y$ we are done.

c) follows since we have again a natural map $\sigma_* S^\eta(\mathscr{F}'') \to S^\eta(\sigma_*\mathscr{F}'')$.

For d we choose some effective divisor E in the exceptional locus of σ such that $\mathscr{O}_{Y''}(-E)$ is relatively ample for σ. For some $\mu \gg 0$ and $\mathscr{F}'' = \sigma^*\mathscr{F}'/\text{torsion}$ the sheaf $\mathscr{F}''(\mu \cdot E)|_E$ will be contained in $\oplus\, \mathscr{O}_E$. Then $S^\eta(\mathscr{F}''(\mu \cdot E))$ will have the same property. Therefore $\sigma_*(S^\eta(\mathscr{F}''(\mu \cdot E)) \otimes \mathscr{O}(\nu \cdot E))$ is for all $\nu > 0$ the same as $\sigma_*(S^\eta(\mathscr{F}''(\mu \cdot E)))$. Replacing $\mathscr{F}''$ by $\mathscr{F}''(\mu \cdot E)$ we found $\mathscr{F}''$.

In §3 of [14] and §5 of [9] the reader can find several properties of weakly positive sheaves. They carry over, in an obvious way, to "weakly positive with

respect to". In any case, the only non trivial properties we will need are:

2.4. Let us fix some notations:

$$\begin{array}{ccccc} Z_0 & \overset{i}{\to} & Z & \overset{\sigma}{\leftarrow} & Z' \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ Y_0 & \underset{j}{\to} & Y & \underset{\delta}{\leftarrow} & Y' \end{array} \qquad (*)$$

is supposed to be a commutative diagram with: i and j are projective compactifications, σ and δ desingularizations and τ_0 is a projective morphism of reduced schemes.

Functorial properties. Assume that $\mathscr{F}_0$ is locally free in some neighbourhood of the non normal locus of Y_0.

a) Assume that the morphism τ_0 in $(*)$ satisfies: $V = \tau_0^{-1}(U) \to U$ is finite, dominant and the trace map from $\tau_{0*}\mathcal{O}_V$ to the integral closure of $\mathcal{O}_U$ factors over $\mathcal{O}_U$. Then $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$, if $\tau_0^*\mathscr{F}_0$ is weakly positive over V with respect to $(Z', \tau'^*\mathscr{F}')$.

b) $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$ if for some $\mu > 0$ one has: For all diagrams $(*)$ with τ_0 finite and for all ample invertible sheaves $\mathscr{A}$ on Z, the sheaf $\tau_0^*\mathscr{F}_0 \otimes i^*\mathscr{A}^\mu$ is weakly positive over $\tau_0^{-1}(U)$ with respect to $(Z', \tau^*\mathscr{F}' \otimes \sigma^*\mathscr{A}^\mu)$.

c) Assume that Y is projective and $\mathscr{F}$ locally free on Y, and let $\delta: Y' \to Y$ be a desingularization. Then $\delta^*\mathscr{F}$ is weakly positive over Y', if and only if $\mathscr{F}$ is weakly positive over Y with respect to $(Y', \delta^*\mathscr{F})$.

Proof. c) The "with respect to $(Y', \mathscr{F}')$", is automatically satisfied if $\mathscr{F}$ is known to be weakly positive over Y. Hence c is a special case of [14], 3.4 *e*. However, a simple proof is as follows: by [14], 3.4. *d*, one may assume $\mathscr{F}$ to be an invertible sheaf. Seshadri's numerical criterion for ampleness together with [14], 3.2, show that it is enough to use the obvious fact, that $\mathscr{F}$ is numerically effective if and only if the same holds for $\delta^*\mathscr{F}$.

a) and b) are nearly the same as [14], 3.4, b and c. The proof uses the same arguments. One just has to add, that the trace maps on Y and Y' are compatible. For example, assume that $\tau_0^*\mathscr{F}_0$ is weakly positive over V with respect to $(Z', \tau'^*\mathscr{F}')$. We may assume by 2.3, a that $\mathscr{A}_Y = \mathcal{O}_Y$ and moreover that $\mathscr{F}_0$ is locally free. As in 2.3, the sheaf $\sigma_*\tau'^*\mathscr{F}' \cap i_*\tau_0^*\mathscr{F}_0$ is weakly positive over V.

If one chooses $\mathscr{H}$ and E and $b \gg 0$ as in [14], 3.4, b one finds a map, surjective over U,

$$\oplus\, \tau_*\mathcal{O}_Z \otimes \mathscr{H}^b \to S^{2\cdot a\cdot b}(\sigma_*\tau'^*\mathscr{F}' \cap i_*\tau_0^*\mathscr{F}_0) \otimes \mathscr{H}^{2b} \otimes \tau_*\mathcal{O}_Z(-b\cdot E)\,.$$

The trace maps induce a map, surjective over U, from the right hand side to $S^{2\cdot a\cdot b}(j_*\mathscr{F}_0) \otimes \mathscr{H}^{2\cdot b}$ and another map, compatible with the first one, to

$$S^{2\cdot a\cdot b}(\delta_*\mathscr{F}') \otimes \mathscr{H}^{2\cdot b}\,.$$

The same argument proves b.

Later, the reference sheaves $\mathscr{F}'$ on Y' will come from direct images of powers of dualizing sheaves. The following "base change lemma", similar to [12], 3.2, will allow to compare different compactifications and desingularizations.

Lemma 2.5. *Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{\delta} & Z = Y'' \times_{Y'} X' & \xrightarrow{p_2} & X' \\ {\scriptstyle f''}\searrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f'} \\ & & Y'' & \underset{\rho}{\to} & Y' \end{array}$$

be a commutative diagram of proper morphisms between reduced schemes, where δ is birational X'', Y'', X' and Y' are non singular, and $\rho' = p_2 \circ \delta$. Let $\mathscr{L}'$ be an invertible sheaf on X' and $\mathscr{L}'' = \rho'^*\mathscr{L}'$. Then for $v \geqq 1$ we have:

a) *If ρ is birational, then*

$$f'_*(\omega^v_{X'/Y'} \otimes \mathscr{L}') \quad \textit{contains} \quad \rho_* f''_*(\omega^v_{X''/Y''} \otimes \mathscr{L}'') .$$

b) *If ρ is finite then*

$$\rho^*(f'_*(\omega^v_{X'/Y'} \otimes \mathscr{L}'))^{**} \quad \textit{contains} \quad (f''_*(\omega^v_{X''/Y''} \otimes \mathscr{L}''))^{**} .$$

Proof. a) is trivial since one has an inclusion $\mathcal{O}_{Y''} \to \omega^v_{Y''/Y'}$ and

$$\rho'_*(\omega^v_{X''/Y'} \otimes \mathscr{L}'') = \omega^v_{X'/Y'} \otimes \mathscr{L}' .$$

b) Let Z' be the normalization of Z and let $X'' \xrightarrow{\delta''} Z' \xrightarrow{\delta'} Z$ be the factorization of δ. Since ρ is flat, the pullback of a reflexive sheaf under ρ remains reflexive and, leaving out codimension two subschemes of Y', we may assume that f' is flat and Gorenstein. Then Z will be Gorenstein and $\omega_{Z/Y''} = p_2^*\omega_{X'/Y'}$. If $\omega_{Z'}$ denotes the reflexive hull of the canonical sheaf on the smooth locus, one has

$$\delta'_*\omega_{Z'} = \mathscr{H}\mathrm{om}(\delta'_*\mathcal{O}_{Z'}, \omega_Z) \to \omega_Z$$

and a surjection $\delta'^*\delta'_*\omega_{Z'} \to \omega_{Z'}$. Since ω_Z is invertible we obtain $\omega_{Z'} \to \delta'^*\omega_Z$ and, for some exceptional divisor E of δ'' an inclusion $\omega_{X''}^{v-1} \to \delta^*\omega_Z^{v-1}(E)$. Altogether

$$\begin{aligned} \delta_*(\omega^v_{X''} \otimes \mathscr{L}'') \to \delta'_*(\delta''_*\omega_{X''}(E) \otimes \delta'^*(\omega_Z^{v-1} \otimes p_2^*\mathscr{L}')) \to \\ \to \delta'_*\omega_{Z'} \otimes \omega_Z^{v-1} \otimes p_2^*\mathscr{L}' \to \omega_Z^v \otimes p_2^*\mathscr{L}' . \end{aligned}$$

By flat base change we get

$$f''_*(\omega^v_{X''/Y''} \otimes \mathscr{L}'') \to p_{1*}p_2^*(\omega^v_{X'/Y'} \otimes \mathscr{L}) = \rho^*(f'_*(\omega^v_{X'/Y'} \otimes \mathscr{L})) .$$

Assumptions 2.6. Let

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{\sigma} & X' \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y_0 & \xrightarrow{j} & Y & \xleftarrow{\delta} & Y' \end{array}$$

be a commutative diagram of morphisms of reduced quasi-projective schemes such that:

a) i and j are compactifications and σ and δ desingularizations.
b) f and f' are surjective and $X_0 = f^{-1}(Y_0)$.
c) f_0 is flat and all fibers of f_0 are reduced normal varieties of dimension n with at most rational Gorenstein singularities.

Theorem 2.7. *We keep the notations and assumptions from* 2.6. *Let* $v \geqq 1$ *be a natural number. Assume that one of the following conditions hold:*

i) $v = 1$.
ii) f_0 *is smooth and for some* $N > 0$ *the map* $f_0^* f_{0*}\omega^N_{X_0/Y_C} \to \omega^N_{X_0/Y_0}$ *is surjective.*
iii) $f_{0*}\omega^{v\cdot\mu}_{X_0/Y_0}$ *is locally free and commutes with arbitrary change for* $\mu > 0$. *Moreover* $f_0^* f_{0*}\omega^N_{X_0/Y_0} \to \omega^N_{X_0/Y_0}$ *is surjective for some* $N > 0$. *Then* $f_{0*}\omega^v_{X_0/Y_0}$ *is weakly positive over* Y_0 *with respect to* $(Y', f'_*\omega^v_{X'/Y'})$.

Remarks 2.8. 1. Since the total space of a deformation of rational Gorenstein singularities over a smooth base has again rational Gorenstein singularities, the assumption 2.6 implies:
c') The general fiber of f_0 is normal, f_0 is Gorenstein and, if $Y'_0 \to Y_0$ is a desingularization and $W'_0 \to Y'_0$ a non singular finite cover, then $X_0 \times_{Y_0} W'_0$ is normal with at most rational Gorenstein singularities. Moreover, the singular locus of $X_0 \times_{Y_0} W'_0$ will not contain a whole fiber of pr_2.

2. It follows from c' that for $Y'_0 = \delta^{-1}(Y_0)$ one has

$$f'_*\omega^v_{X'/Y'}|_{Y'_0} = \mathrm{pr}_2^*\omega^v_{X_0 \times_{Y_0} Y'_0/Y'_0} = \delta^* f_{0*}\omega^v_{X_0/Y_0}\,.$$

Especially the condition iii of 2.2 holds.

3. It should be possible to replace in theorem 2.7 the condition c of 2.6 by the weaker condition c'.

4. The sheaf $f_{0*}\omega_{X_0/Y_0}$ is locally free and compatible with arbitrary base change. This result, due to Deligne [2], 5.5 in the smooth case, follows from an argument, which J. Kollár told me: By "Cohomology and Base Change" we only have to show that the sheaves $R^i f_{0*}\omega_{X_0/Y_0}$ are all locally free for all $i \geqq 0$. As well known one can find a complex $E^\bullet$ of locally free sheaves on Y_0 such that for all closed subschemes C of Y_0 and $W = X_0 \times_{Y_0} C$ the sheaves $R^i f_{0*}\omega_{X_0/Y_0} \otimes \mathcal{O}_W$ are given by the cohomology of $E^\bullet \otimes \mathcal{O}_C$. Therefore we may assume Y_0 to be a curve and, using flat base change, we can assume as well that Y_0 is smooth. Without changing $R^i f_{0*}\omega_{X_0/Y_0}$ we can replace X_0 by its desingularization and the local freeness follows from [6].

5. Using 4 (in fact only in the smooth case) we will show in 4.3 that ii) implies iii). Therefore, in the proof of 2.7 we will use base change whenever it is convenient.

Corollary 2.9. *Under the assumptions made in* 2.6 *assume that, for all fibers* F *of* f_0, ω_F *is semi-ample and of maximal Kodaira dimension. Then* $f_{0*}\omega^v_{X_0/Y_0}$ *is weakly positive over* Y_0 *for* $v \geqq 1$.

Proof. If ω_F is semi-ample then one obtains from the Grauert-Riemenschneider vanishing theorem (see also [3], § 2) that $H^i(F, \omega_F^\nu) = 0$ for $i > 0$ and $\nu > 1$. Then $f_{0*}\omega^\nu_{X_0/Y_0}$ is compatible with base change. Since f_0 is flat, the Euler-Poincaré characteristic $\chi(\omega_F^\nu)$ is locally constant. As in 2.8,4 this follows since $H^\cdot(F, \omega_F^\nu)$ is given by a complex of locally free sheaves on Y_0, restricted to $f_0(F)$. Therefore $f_{0*}\omega^\nu_{X_0/Y_0}$ is locally free and compatible with base change for all $\nu \geqq 1$. Since ω_F is semi-ample for all fibres the assumption iii) of 2.7 holds true.

As an intermediate step we will obtain in §4 as well.

Theorem 2.10. *Under the assumptions made in* 2.6 *let* $\mathscr{L}'$ *and* $\mathscr{L}_0$ *be invertible sheaves on* X' *and* X_0 *respectively. Assume that* $\mathscr{L}_0 \subset i^*\sigma_*\mathscr{L}'$ *and that for some* $N > 0$ *the sheaf* $\mathscr{L}_0^N$ *is globally generated over* X_0 *by* $H^0(X', \mathscr{L}'^N) \cap H^0(X_0, \mathscr{L}_0^N)$. *Assume moreover that* $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ *is locally free and compatible with base change. Then* $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ *is weakly positive over* Y_0 *with respect to* $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

Lemma 2.11. a) *In order to prove* 2.7 *or* 2.10 *we may blow up* Y' *and* X'.
b) *In order to prove* 2.10 *it is enough to show that each point* $y \in Y_0$ *has an open neighbourhood* U *such that for the restriction* h *of* f_0 *to* $V = f_0^{-1}(U)$ *one has:* $h_*(\omega_{V/U} \otimes \mathscr{L}_0|_V)$ *is weakly positive over* U *with respect to* $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

Proof. a) Follows directly from 2.5, a.

b) Obviously it is enough to show that $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is weakly positive over U with respect to $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$. To this aim let us write $Y'_0 = \delta^{-1}(Y_0)$, $X'_0 = \sigma^{-1}(X_0)$ and

$$\begin{array}{ccccc} V & \xrightarrow{\beta} & X_0 & \xleftarrow{\sigma_0} & X'_0 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'_0} \\ U & \xrightarrow[\alpha]{} & Y_0 & \xleftarrow[\delta_0]{} & Y'_0 \end{array}$$

for the induced morphisms. Consider as in 2.3 $\mathscr{A}_Y = j_*\alpha_*\mathscr{O}_U \cap \delta_*\mathscr{O}_{Y'}$.

$\mathscr{A}_Y$ is a coherent $\mathscr{O}_Y$-algebra, isomorphic to $\mathscr{O}_{Y_0}$ over U. By 2.4,a and since $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is locally free and commuting with base change we are allowed to blow up Y as long as the center stays in $Y - U$. Therefore, replacing Y by $\mathrm{Spec}_Y(\mathscr{A}_Y)$ we can assume that $\mathscr{A}_Y = \mathscr{O}_Y$.

We may also assume that $\mathscr{L}'|_{X'_0} = \sigma_0^*\mathscr{L}_0$. In fact, if not we replace $\mathscr{L}'$ by the largest subsheaf with this property and blow X' up to make it invertible.

Since we assume that 2.10 holds for h we know from 2.3, b that

$$\mathscr{F} = \delta_* f'_*(\omega_{X'/Y'} \otimes \mathscr{L}') \cap (\alpha \circ j)_* h_*(\omega_{V/U} \otimes \mathscr{L}_0|_V)$$

is weakly positive over U. Moreover, $\mathscr{F}$ coincides with $h_*(\omega_{V/U} \otimes \mathscr{L}_0|_V)$ on U. Therefore it is enough to show that $j^*\mathscr{F}$ is contained in $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$. Obviously $j^*\mathscr{F}$ lies in $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0) \otimes \alpha_*\mathscr{O}_U$.

On the other hand, we have a natural map

$$f'_{0*}(\omega_{X'_0/Y'_0} \otimes \sigma_0^*\mathscr{L}_0) \to \mathrm{pr}_{2*}(\omega_{X_0 \times_{Y_0} Y'_0/Y'_0} \otimes \mathrm{pr}_1^*\mathscr{L}_0)$$

and, by assumption, the right hand side is $\delta_0^* f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$. Therefore we have

$$j^*\mathscr{F} \to \delta_{0*} f'_*(\omega_{X'_0/Y'_0} \otimes \sigma_0^* \mathscr{L}_0) \to f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0) \otimes \delta_{0*} \mathscr{O}_{Y'_0}$$

and, since $\mathscr{A}_Y = \mathscr{O}_Y$, we are done.

Remarks 2.12. a) The argument used in 2.11, b shows, how to use "with respect to $(Y', f'_* \omega^{\nu}_{X'/Y'})$" in 2.7 to extend "weak positivity" to sheaves coming from compactifications. For example, one gets easily, that under the assumptions made in 2.6, one also has:

Let Y_1 be an open subscheme of Y, containing Y_0 and f_1 the restriction of f to $X_1 = f^{-1}(Y_1)$. Assume that one of the following conditions holds true:
i) *f_1 is flat, Cohen Macauley and $f_{1*}\omega_{X_1/Y_1}$ locally free and compatible with base change. We have $\nu = 1$ and $\sigma_* \omega_{X'/Y'}|_{X_1}$ is larger than ω_{X_1/Y_1}.*
ii) *f_1 is flat, Gorenstein and $f_{1*}\omega^{\nu \cdot \mu}_{X_1/Y_1}$ is locally free and compatible with base change, for $\mu > 0$. Moreover, $\sigma_* \omega^{\nu \cdot \mu}_{X'/Y'}|_{X_1}$ is larger than $\omega^{\nu \cdot \mu}_{X_1/Y_1}$ and, for some $N > 0$, $f_1^* f_{1*} \omega^N_{X_1/Y_1} \to \omega^N_{X_1/Y_1}$ is surjective over X_0.*

Then $f_{1*}\omega^{\nu}_{X_1/Y_1}$ is weakly positive over Y_0 with respect to $(Y', f'_* \omega^{\nu}_{X'/Y'})$.

b) The assumption that, for $\nu > 1$, $f_0^* f_{0*} \omega^N_{X_0/Y_0}$ maps surjectively to $\omega^N_{X_0/Y_0}$ in 2.7, or the assumption "ω_F semi ample" in 2.9 can be replaced by some weaker condition in case that f_0 is smooth:

Assume that there exists an effective divisor D_0 on X_0, which has normal crossings relative to f_0, such that for some $N > 0$ the following two conditions are satisfied:

i) *For all $\mu > 0$ one has*

$$f_{0*}(\omega^{N \cdot \mu}_{X_0/Y_0} \otimes \mathscr{O}_{X_0}(-\mu \cdot D_0)) = f_{0*} \omega^{\mu \cdot N}_{X_0/Y_0}$$

ii)

$$f_0^* f_{0*}(\omega^N_{X_0/Y_0} \otimes \mathscr{O}_{X_0}(-D_0)) \to \omega^N_{X_0/Y_0} \otimes \mathscr{O}_{X_0}(-D_0)$$

is surjective (or: for all fibres F of f_0 $\omega_F^N \otimes \mathscr{O}_F(-D_0|_F)$ is semi ample and of maximal Iitaka dimension).

In other words, we ask for some very nice "Zariski-decomposition". The proof remains more or less the same. Just, in 2.10 and 3.16–24 one has to replace "$\mathscr{L}_0^N$ globally generated over X''_0 by $\mathscr{L}_0^N(-D_0)$ globally generated over X_0, and to work with $\mathscr{L}_0\left(-\left[\frac{D_0}{N}\right]\right)$ instead of $\mathscr{L}_0$. We leave it to the reader to check all details.

§ 3. Direct images of dualizing sheaves

We want to apply 1.6 and 1.7 to construct extensions of direct images of dualizing sheaves to certain compactifications. The starting point is:

Theorem 3.1 (W. Schmid, [11]). *Let $g'_0 : V'_0 \to W'_0$ be a proper surjective smooth morphism of non singular schemes such that all fibres are irreducible and of dimension n. Let $W'_0 \to W'$ be a non singular compactification such that $W' - W'_0$ is a normal crossing divisor. Assume that the monodromies of $R^n g'_{0*} \mathbb{C}_{V'_0}$ around the components*

of $W' - W'_0$ *are unipotent. Let* $\mathscr{H}'$ *be the canonical extension of* $\mathscr{H}'_0 = (R^n g'_{0*}\mathbb{C}_{V'_0}) \otimes_{\mathbb{C}} \mathscr{O}_{W'_0}$ *to* W' *(in the sense of Deligne,* [2], *II,* 5.2). *Then the subbundle* $\mathscr{F}'_0 = g'_{0*}\omega_{V'_0/W'_0}$ *of* $\mathscr{H}'_0$ *extends to a subbundle* $\mathscr{F}'$ *of* $\mathscr{H}'$.

Definition 3.2. For $g'_0: V'_0 \to W'_0$ as in 3.1, we will call $\mathscr{F}'$ the *Schmid extension* of $g'_{0*}\omega_{V'_0/W'_0}$ to W'. Hence, if we say that there *exists a Schmid extension* this means especially that the monodromy condition of 3.1 is satisfied.

Theorem 3.3 (Y. Kawamata, [5]). *Keeping the assumptions and notations from* 3.1, *let* $V'_0 \to V'$ *be a non singular compactification and* $g': V' \to W'$ *a morphism with* $g'|_{V'_0} = g'_0$. *Then*

a) $\mathscr{F}' = g'_*\omega_{V'/W'}$ *is the Schmid extension of* $g'_{0*}\omega_{V'_0/W'_0}$
b) $\mathscr{F}'$ *is weakly positive over* W'.

Kawamata's proof of b in [5] is quite complicated and uses the theory of variations of Hodge Structures. The Tankeev-Kollár vanishing theorem, applied to products of g'_0 with itself, gives a simpler proof of 3.3, b (see [6] or [13] for similar arguments).

3.4. If $f_0: X_0 \to Y_0$ is a smooth morphism of reduced schemes, all of whose fibers are compact manifolds of dimension n, we can construct a category of compactifying tripels $\mathscr{Z}$, which covers Y_0, and an extension $\mathscr{G}$ of $f_{0*}\omega_{X_0/Y_0}$ in the following way (see 1.5 for the notations):

3.5. Let S_0 be a closed subscheme of Y_0 and $\sigma_0: S'_0 \to S_0$ a desingularization. If $h'_0: T'_0 \to S'_0$ is the pullback of f_0 and $S'_0 \to S'$ a compactification such that $S' - S'_0$ is a normal crossing divisor, then the monodromy theorem ([2], III or [11]) says that the monodromy of $R^n h'_{0*}\mathbb{C}_{T'_0}$ around the components of $S' - S'_0$ are quasi-unipotent. Using for example [5], 18, or [9], 4.5, we find a non singular covering $\tau: W' \to S'$ such that $W'_0 = \tau^{-1}(S'_0)$ is the complement of a normal crossing divisor and the monodromies of $\tau^* R^n h'_{0*}\mathbb{C}_{T'_0}$ are unipotent.

Choose $\mathscr{Z}$ to be the smallest complete category of compactifying tripels which contains all the tripels $\underline{W}' = (W', W'_0, \sigma_0 \circ \tau|_{W'_0})$ obtained in this way.

3.6. In order to construct an extension $\mathscr{G}$ of $f_{0*}\omega_{X_0/Y_0}$ to $\mathscr{Z}$, it is enough to define $\mathscr{G}_{\underline{W}'}$ for the generators $\underline{W}'$ from 3.5. For those, if $g'_0: V'_0 \to W'_0$ is the pullback of f_0, $g'_{0*}\omega_{V'_0/W'_0}$ has a Schmid-extension $\mathscr{F}'$ and we define $\mathscr{G}_{\underline{W}'} = \mathscr{F}'$. In order to extend $\mathscr{G}$ to $\mathscr{Z}$ by pullback we need that the Schmid extensions are compatible.

Lemma 3.7. *Under the notations and assumptions of* 3.1, *let* $\eta: Z \to W'$ *be a proper morphism of non singular schemes and* $\eta^{-1}(W' - W'_0)$ *a normal crossing divisor. For* $Z_0 = \eta^{-1}(W'_0)$ *the morphism* $\mathrm{pr}_2: V'_0 \times_{W'_0} Z_0 \to Z_0$ satisfies again the assumptions of 3.1 and $\eta^* \mathscr{F}'$ *is the Schmid extension of* $\mathrm{pr}_{2*}\omega_{V'_0 \times_{W'_0} Z_0/Z_0}$.

Proof. This is just saying that the unipotency of the monodromy and the canonical extension $\mathscr{H}'$ of $\mathscr{H}'_0$ are compatible with pullbacks.

By 3.7 we can extend $\mathscr{G}$ to $\mathscr{Z}$ and found the extension asked for. By 1.7 we get:

Theorem 3.8. *Let $f_0\colon X_0 \to Y_0$ be a smooth morphism of reduced schemes, all of whose fibers are compact manifolds of dimension n. Then there exists a dominant finite covering $\tau_0\colon W_0 \to Y_0$, a compactification $i\colon W_0 \to W$ and a locally free sheaf $\mathcal{F}$ on W with:*

a) *The trace map from $\tau_{0*}\mathcal{O}_{W_0}$ to the integral closure $O_{\tilde{Y}_0}$ of $\mathcal{O}_{Y_0}$ factors over $\mathcal{O}_{Y_0}$.*

b) $$\tau_0^* f_{0*}\omega_{X_0/Y_0} = i^*\mathcal{F}$$

c) *If $\delta\colon W' \to W$ is a desingularization and $W'_0 = \delta^{-1}(W_0)$ the complement of a normal crossing divisor, then $\delta^*\mathcal{F}$ is the Schmid extension, of $\mathrm{pr}_{2*}\omega_{X_0 \times_{Y_0} W'_0/W'_0}$.*

Corollary 3.9. *If f_0 is smooth and $v = 1$, then $f_{0*}\omega_{X_0/Y_0}$ is weakly positive over Y_0 and, moreover, theorem 2.7 holds true.*

In fact, 3.9 is a direct consequence of 3.8 and 2.4, a. To keep control of the reference sheaf on Y' one has to use 3.3, b and 2.5. Since the exact formulation is repeated in the proof of 3.24, we do not give the details.

In §4 we will show, that 2.11 and 2.7 can be deduced quite easily from 3.9 if f_0 is smooth. Hence, *the reader just interested in moduli of manifolds can skip the rest of this paragraph and go to* 4.2.

For the non smooth case we consider $f_{0*}(\omega_{V_0/W_0} \otimes \mathcal{L}_0)$ directly, allowing $\mathcal{L}_0$ to be trivial. To this aim we have to have a closer look to the Schmid extension first:

3.10. We want to show, that the Schmid extension of $g'_{0*}\omega_{V'_0/W'_0}$ is compatible with the Schmid extension of certain components of the boundary. To this aim, let g'_0 and $g'\colon V' \to W'$ be as in 3.1 and 3.3. We may assume that $\Sigma = g'^*(W' - W'_0)$ is a normal crossing divisor. Let Z be an irreducible component of $W' - W'_0$ and $E = (W' - W'_0) - Z$. Let Γ be a component of Σ of multiplicity one in Σ and dominant over Z. Write $W'_1 = W' - E$, $V_1 = g'^{-1}(W'_1)$, $\Gamma_0 = \Gamma \cap g'^{-1}(Z_0)$ for Z_0 open in Z, and assume that $Z - Z_0$ is a normal crossing divisor.

Write

$$\begin{array}{ccccccc} V'_1 & \to & V' & \leftarrow & \Gamma & \leftarrow & \Gamma_0 \\ \downarrow g'_1 & & \downarrow g' & & \downarrow \alpha & & \downarrow \alpha_0 \\ W'_1 & \to & W' & \leftarrow & Z & \leftarrow & Z_0 \end{array}$$

for the corresponding morphisms.

If $R = g'^*(Z) - \Gamma$, we have natural adjunction maps

$$\begin{array}{ccc} g'_*\omega_{V'/W'}(-R) & \overset{\Phi'}{\to} & \alpha_*\omega_{\Gamma/Z} \\ \downarrow & & \downarrow \eta \\ g'_*\omega_{V'/W'} & \underset{\Phi}{\to} & \alpha_*\omega_{\Gamma/Z}(R \cap \Gamma) \end{array}$$

Proposition 3.11. *Assume that $\alpha_0\colon \Gamma_0 \to Z_0$ is smooth and that the monodromy of $R^{n-1}\alpha_{0*}\mathbb{C}_{\Gamma_0}$ around the components of $Z - Z_0$ is unipotent, and that $\eta|_{Z_0}$ is an isomorphism. Then the image of Φ is $\alpha_*\omega_{\Gamma/Z}$.*

Remark 3.12. It is always true that Φ' is surjective,

Proof of 3.12. The question is local and we may assume therefore that W' is a projective manifold. For an ample invertible sheaf $\mathscr{H}$ on W', the Kollár-Tankeev vanishing theorem ([6], see also [3], §3) tells us that

$$H^0(V', \omega_{V'}(\Gamma) \otimes g'^* \mathscr{H}) \to H^0(\Gamma, \omega_\Gamma \otimes \alpha^* \mathscr{H}/_Z)$$

is surjective. Since this adjunction map is induced by Φ', and since we can choose $\mathscr{H}$ as ample as we want, Φ' must be surjective.

Proof of 3.11. Let us write $R = R_1 + R_2$, where R_1 is the sum of the components of R which are dominant over Z and R_2 the others.

In order to prove 3.11 we can blow up W' and replace it by finite coverings. In fact, since both, $g'^* \omega_{V'/W'}$ and $\alpha_* \omega_{\Gamma/Z}$ are Schmid extensions, they are compatible with pullbacks as well as the surjection, induced by Φ, from $g'_* \omega_{V'/W'}|_{W'-(Z-Z_0)}$ to $\alpha_{0*} \omega_{\Gamma_0/Z_0}$.

By 4.6 of [9] or 6.1 of [12] we can make semi-stable reduction in codimension one and, replacing Z_0 by some smaller open set, assume that R_1 is a reduced normal crossing divisor. Blowing up we may as well assume that $Z_0 = Z - (Z \cap E)$ and that g'_1 is flat. Making semi-stable reduction a second time we can finally assume that all components of Σ which are dominant over components of $W' - W'_0$ are reduced in Σ.

For some $b \gg 0$, Φ will induce a map

$$g'_* \omega_{V'/W'}(-b \cdot R_2) \to \alpha_* \omega_{\Gamma/Z}$$

and using Φ', we see that $b = 1$ is enough. Hence 3.11 follows from

Claim 3.13. Under the assumptions made above $g'_* \omega_{V'/W'}(-R_2) \to g'_* \omega_{V'/W'}$ is an isomorphism.

Proof. Write $R_2 = \sum_{\nu=1}^{r} \mu_\nu \cdot \Sigma_\nu$, for $\mu_\nu > 0$, and $\Sigma = \sum_{\nu=1}^{k} \eta_\nu \cdot \Sigma_\nu$. Since R_2 is supported in $g'^{-1}(E)$ we have $\mu_\nu < \eta_\nu$ for $\nu = 1, \ldots, r$. Let M be a number, larger than η_ν for $\nu = 1, \ldots, r$.

Since our question is local in W' we can write $\mathscr{O}_{W'}(E + Z) = \mathscr{H}^M$ for some invertible sheaf $\mathscr{H}$. Let S be the normalization of the covering, obtained by taking the M-th root out of $E + Z$, and T the normalization of that, obtained by taking the M-th root out of Σ, where for $\mathscr{L} = g'_* \mathscr{H}$ we write $\mathscr{L}^M = \mathscr{O}_{V'}(\Sigma)$ (see 3.14 and [9], §4 or [3], 2.7 for this well known construction).

Let

$$\begin{array}{ccc} T & \xrightarrow[\tau]{} & V' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g'} \\ S & \xrightarrow[\rho]{} & W' \end{array}$$

denote the induced morphisms. We have $\omega_S = \rho^* \omega_{W'} \otimes \mathscr{H}^{M-1}$ and

$$\tau_* \omega_T = \bigoplus_{i=0}^{M-1} \omega_{V'} \otimes \mathscr{L}^i\left(-\left[\frac{i \cdot \Sigma}{M}\right]\right).$$

Therefore $g'_* \tau_* \omega_{T/S}$ contains $g'_* \omega_{V'/W'} \otimes \mathcal{O}_{V'}\left(-\left[\frac{M-1}{M} \Sigma \right] \right)$ as a direct summand. This remains true if we replace S by some higher cover and T by the normalization of the pullback. Since S has at most quotient singularities we may, by abuse of notations, assume S to be non singular and $\delta^{-1}(E + Z)$ to be a normal crossing divisor. Since T has at most rational singularities $h_* \omega_{T/S}$ will again be a Schmid extension and hence locally free. Therefore

$$g'_*\left(\omega_{V'/W'} \otimes \mathcal{O}_{V'}\left(-\left[\frac{M-1}{M} \Sigma \right] \right) \right)$$

is locally free. This sheaf is contained in $g'_* \omega_{V'/W'}$ and, since all components of Σ, dominant over components of $W' - W'_0$, are of multiplicity one, both sheaves must be isomorphic. On the other hand,

$$\frac{M-1}{M} \cdot \eta_\nu \geqq \eta_\nu - 1 \geqq \mu_\nu .$$

Therefore $\left[\frac{M-1}{M} \Sigma \right] \geqq R_2$.

3.14. In the proof of 3.13 we used the coverings obtained by taking roots out of divisors. Recall:

Let Z_0 be a scheme, $\mathcal{L}$ an invertible sheaf, D an effective Cartier divisor and $N > 0$. If $\mathcal{L}^N = \mathcal{O}_{Z_0}(D)$ and if s is the corresponding section, then we call the flat finite morphism

$$\tau_0 \colon Z_1 = \operatorname{Spec}_{Z_0}\left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/s^{-1} \right) \to Z_0$$

the *covering obtained by taking the N-th root out of D*

Properties 3.15. Assume that Z_0 is flat and Gorenstein over some scheme S_0.

a) Z_1 is a reduced scheme, flat and Gorenstein over S_0, and

$$\tau_{0*} \omega_{Z_1/S_0} = \bigoplus_{i=0}^{N-1} \mathcal{L}^i \otimes \omega_{Z_0/S_0}.$$

b) If $S_0 = \operatorname{Spec} \mathbb{C}$ and if Z_0 is normal with rational singularities, let us assume that $\mathcal{L}^N$ is generated by some finite dimensional vector space A of global sections and that $s \in A$ is a general section. Then Z_1 is again normal with at most rational Gorenstein singularities.

Proof. τ_0 is flat and Z_1 reduced. Moreover ω_{Z_1/S_0} is just the sheaf on Z_1 corresponding to the $\bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}$ module $\mathcal{H}om(\bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}, \omega_{Z_0/S_0})$. To prove b we consider a desingularization

$$\eta_0 \colon Z'_0 \to Z_0 \quad \text{and} \quad D' = \eta_0^* D .$$

since D was in general position D' will be non singular. Let Z'_1 be the covering

obtained by taking the N-th root out of D' and

$$\begin{array}{ccc} Z_1' & \xrightarrow[\tau_0']{} & Z_0' \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_0} \\ Z_1 & \xrightarrow[\tau_0]{} & Z_0 \end{array}$$

the induced morphisms. Then Z_1' is non singular and

$$\tau_{0*}\eta_{1*}\mathcal{O}_{Z_1'} = \eta_{0*}\tau_{0*}'\mathcal{O}_{Z_1'} = \eta_{0*}\bigoplus_{i=0}^{N-1}\eta_0^*\mathcal{L}^{-i} = \tau_{0*}\mathcal{O}_{Z_1}$$

gives the normality of Z_1. Z_1 can only have rational singularities since

$$\tau_{0*}\eta_{1*}\omega_{Z_1'/S_0} = \eta_{0*}\tau_{0*}'\omega_{Z_1'/S_0} = \eta_{0*}\bigoplus_{i=0}^{N-1}\omega_{Z_0'/S_0}\otimes\eta_0^*\mathcal{L}^{-i} = \tau_{0*}\omega_{Z_1/S_0}\,.$$

Assumptions 3.16. $f_0\colon X_0 \to Y_0$ is supposed to be a flat Gorenstein morphism of reduced quasi-projective schemes, $\mathcal{L}_0$ an invertible sheaf on X_0, D_0 an effective Cartier divisor and $N \geqq 1$. Assume that $\mathcal{L}^N = \mathcal{O}_{X_0}(D_0)$ where: If $N = 1$, then D_0 should be empty. If $D_0 = \varnothing$ then N should be as small as possible. Moreover, D_0 should not contain any fiber of f_0. Assume moreover:

i) $f_{0*}(\omega_{X_0/Y_0}\otimes\mathcal{L}_0)$ is locally free and commutes with arbitrary base change
ii) Let $\tau_0\colon X_1 \to X_0$ be the covering obtained by taking the N-th root out of D_0 and $f_1 = f_0\circ\tau_0\colon X_1 \to Y_0$. Then all fibers of f_1 are reduced normal varieties with at most rational (Gorenstein) singularities.

Remarks 3.17. a) If f_0 is smooth, we can replace ii) by:
ii') D_0 is a normal crossing divisor relative to f_0 and all components of D_0 have a multiplicity strictly smaller than N. Let $\tau_0\colon X_1 \to X_0$ be the normalization of the covering obtained by taking the N-th root out of D_0. Then all fibers of $f_1 = f_0\circ\tau_0$ should be reduced normal varieties.
b) As in 2.6 it should be enough to ask in ii) or ii') that f_1 satisfies the weaker assumption c' from 2.8, 1.

We have to define some kind of Schmid extension of $f_{0*}(\omega_{X_0/Y_0}\otimes\mathcal{L}_0)$. To this aim (and not just to increase confusion) let us formulate:

Statement 3.18. There exists a diagram of fiber products

$$\begin{array}{ccccc} V_1' & \to & V_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow{\scriptstyle \beta_0'} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \tau_0} \\ V_0' & \xrightarrow{\delta'} & V_0 & \xrightarrow{\pi'} & X_0 \\ \downarrow{\scriptstyle g_0'} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f_0} \\ W_0' & \xrightarrow{\delta_0} & W_0 & \xrightarrow{\pi_0} & Y_0 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} & & \\ W' & \xrightarrow{\delta} & W & & \end{array}$$

with $f_1 = f_0 \circ \tau_0$, $g_1 = g_0 \circ \beta_0$, $g'_1 = g'_0 \circ \beta'_0$, $\mathscr{M}_0 = \pi'^* \mathscr{L}_0$, $\mathscr{M}'_0 = \delta'^* \mathscr{M}_0$, $B_0 = \pi'^* D_0$ and $B'_0 = \delta'^* B_0$, such that:

a) π_0 is finite, dominant and the trace map from $\pi_{0*} \mathscr{O}_{W_0}$ to the integral closure of $\mathscr{O}_{Y_0}$ factors over $\mathscr{O}_{Y_0}$.
b) i and i' are projective compactifications and δ: $W' \to W$ is a desingularization such that $\delta^{-1}(W - W_0)$ is a normal crossing divisor.
c) There exists an open subscheme W'_s of W'_0 such that $W' - W'_s$ is a normal crossing divisor and a desingularization σ_s: $T'_s \to V'_s = g'^{-1}_1(W'_s)$ which is smooth over W'_s.

3.19. Assume that 3.16 holds true and that we have a diagram as in 3.1. Assume moreover that the smooth morphism $h_s = g'_1|_{V'_s} \circ \sigma_s$: $T'_s \to W'_s$ satisfies the assumptions made in 3.1 on the monodromies. Let $\mathscr{G}'$ be the Schmid extension of $h'_{s*} \omega_{T'_s/W'_s}$ to W'. By assumption the fibers of g'_1 have rational Gorenstein singularities. Therefore the same holds true for the total space V'_1. If σ: $T'_1 \to V'_1$ is any desingularization we will have by 3.3, a, and 3.15, a, that $\mathscr{F}'_0 = g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}'_0)$ is a direct summand of

$$g'_{1*} \omega_{V'_1/W'_0} = (g'_1 \circ \sigma)_* \omega_{T'_1/W'_0} = \mathscr{G}'|_{W'_0} .$$

Therefore we can define:

Definition 3.20. We say that $\mathscr{F}'_0 = g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}'_0)$ *has a Schmid extension*, if the assumptions of 3.18, b and c, and of 3.19 hold true and we call in that case $\mathscr{F}' = \mathscr{G}' \cap i'_* \mathscr{F}'_0$ *the Schmid extension* of $\mathscr{F}'_0$ (with respect to B'_0)

Properties 3.21.
i) $\mathscr{F}'$ is locally free and weakly positive over W'.
ii) $\mathscr{F}'$ is compatible with pullback under dominant morphisms.
iii) Let V' be a compactification of V'_0 and g': $V' \to W'$ an extension of g'_0 to V'. Let σ: $V'' \to V'$ be a desingularization, $g'' = g' \circ \sigma$ and σ_0 the restriction of σ to $V''_0 = \sigma^{-1}(V'_0)$. Assume that there exist an effective normal crossing divisor B'' and an invertible sheaf $\mathscr{M}''$ on V'' with

$$\mathscr{O}_{V''}(B'') = \mathscr{M}''^N, \; B''|_{V''_0} = \sigma_0^* B'_0 \quad \text{and} \quad \mathscr{M}''|_{V''_0} = \sigma_0^* \mathscr{M}'_0 .$$

Then
$$\mathscr{G}' = \bigoplus_{i=0}^{N-1} g''_* \left(\omega_{V''/W'} \otimes \mathscr{M}''^i \left(- \left[\frac{i \cdot B''}{N} \right] \right) \right)$$

and
$$\mathscr{F}' = g''_* \left(\omega_{V''/W'} \otimes \mathscr{M}'' \left(- \left[\frac{B''}{N} \right] \right) \right) \subseteq g''_* (\omega_{V''/W'} \otimes \mathscr{M}'') .$$

Proof. If T'_1 from 3.19 is choosen large enough, we can take a non singular compactification T' of T'_1 which is dominating V''. Then, for h': $T' \to W'$ the direct sum in iii) is nothing but $h'_* \omega_{T'/W'}$ (see for example [9], §4 or [3], 2.7), and $\mathscr{F}'$ must be the summand for $i = 1$. Since $\mathscr{F}'$ is a direct summand of $\mathscr{G}'$ we obtain i) from 3.3, b, and ii) from 3.7.

Corollary 3.22. *Let Z be a smooth irreducible divisor in W' such that $Z_0 = W'_0 \cap Z$ is the complement of a normal crossing divisor and let φ_0 be the restriction of g'_0 to*

$\Delta'_0 = g_0'^{-1}(Z_0)$. *If* $\varphi_{0*}(\omega_{\Delta_0/Z_0} \otimes \mathscr{M}'_0)$ *has a Schmid extension* $\mathscr{F}'_Z$ *to* Z, *with respect to* $B'_0|_{\Delta'_0}$, *then* $\mathscr{F}'|_Z = \mathscr{F}'_Z$.

Proof. Returning to the notations from 3.21, let us consider

$$\begin{array}{ccccccc}
\Gamma & \to & T' & & & & \\
\downarrow \beta & & \downarrow & & & & \\
\Delta & \to & V'' & \overset{\sigma}{\to} & V' & \leftarrow & \Delta' \\
\downarrow \rho & & \downarrow g'' & & \downarrow g' & & \downarrow \varphi \\
Z & \to & W' & = & W' & \leftarrow & Z
\end{array}$$

where Δ' is the closure of Δ_0 in V', Δ the proper transform of Δ' in V'', Γ the proper transform of Δ in T' and $\alpha = \rho \circ \beta$. We may assume that $\Delta + B''$ is a normal crossing divisor and that $\beta: \Gamma \to \Delta$ is a desingularization of the covering obtained by taking the N-th root out of $B''|_\Delta$. Then Γ will be a desingularization of a compactification of $\Delta'_1 = g_1'^{-1}(Z_0)$. Write $\varphi'_1: \Delta'_1 \to Z_0$ for the induced map. $\mathscr{F}'_Z$ will be a direct summand of $\alpha_* \omega_{\Gamma/Z}$. We have a natural map

$$\phi_0: \mathscr{G}'|_{W'_0} = g'_{1*}\omega_{V'_1/W'_0} \to \varphi'_{1*}\omega_{\Delta'_1/Z_0} = \alpha_* \omega_{\Gamma/Z}|_{Z_0}$$

which induces an isomorphism $\mathscr{G}'|_{Z_0} \to \alpha_* \omega_{\Gamma/Z}|_{Z_0}$. If $Z_0 \cap W_s \neq \varnothing$ we are done. If not, this implies that the assumptions of 3.11 are satisfied and Φ_0 extends to a surjection $\mathscr{G}' \to \alpha_* \omega_{\Gamma/Z}$, or an isomorphism $\mathscr{G}'|_Z \to \alpha_* \omega_{\Gamma/Z}$.

Theorem 3.23. *Under the assumptions made in* 3.16 *we can construct a diagram satisfying* 3.18 *and a locally free sheaf* $\mathscr{F}$ *on* W *such that* $g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}_0)$ *has a Schmid extension* $\mathscr{F}'$ *with respect to* B'_0 *and* $\mathscr{F}' = \delta^* \mathscr{F}$.

Proof. Of course we can find such a diagram, for example for $\pi_0 = \mathrm{id}$, such that all the properties asked for in 3.18 hold true. However, the morphism $h_s = g'_1|_{V'_s} \circ \sigma_s$ will not satisfy the assumption made in 3.1 on the monodromies. Using [5], 18, (see also [9], 4.5) we can find some non singular covering of W' with sufficiently high ramification along $W' - W'_s$ such that the monodromy condition holds true over this covering. Renaming and using 1.10 to get back W_0, we found a diagram 3.18, such that $g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}'_0)$ has a Schmid extension.

We may repeat this construction for all closed subschemes of Y_0. Let $\mathscr{Z}$ be the smallest complete category of compactifying tripels, as defined in 1.5, which contains all those tripels $\underline{W}' = (W', W'_0, \tau_0 \circ \delta_0)$ thereby obtained. Obviously $\mathscr{Z}$ covers Y_0.

By construction we have a Schmid extensions $\mathscr{F}'$ of $g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}'_0)$ to W' (we keep the notation from 3.18, even if we start with a subscheme of Y_0). Let us denote it by $\mathscr{G}_{\underline{W}'}$. If $\tau: \underline{Z} = (Z, Z_0, \eta_0) \to \underline{W}'$ is a morphism in $\mathscr{Z}$ and $\tau: Z \to W'$ dominant, then $\tau^* \mathscr{G}_{\underline{W}'}$ is the Schmid extension on Z by 3.21. If $\tau(Z)$ is a smooth irreducible divisor in W', then the same holds true by 3.22, and, since we can blow up W' to desingularize $\tau(Z)$, a combination of both steps gives the same result, whenever $\tau(Z)$ is a divisor. By induction on the dimension of $\tau(Z)$ we find that $\tau^* \mathscr{G}_{\underline{W}'}$ always is the Schmid extension. Therefore, if we define for $\underline{Z} \in \mathscr{Z}$, $\mathscr{G}_{\underline{Z}}$ by

pullback, this is well defined and compatible with morphisms in $\mathscr{Z}$. Hence we found an extension $\mathscr{G}$ of $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ to $\mathscr{Z}$, as defined in 1.5, f.

By 1.7 we find a new covering W_0 of Y_0 satisfying 3.18, a, a compactification W of W_0 and a locally free sheaf $\mathscr{F}$ on W which is induced by $\mathscr{G}$. Finally, desingularizing W we will get back the whole diagram asked for in 3.23.

Corollary 3.24. *Under the assumptions made in* 3.16, *let* $i\colon X_0 \to X$ *and* $j\colon Y_0 \to Y$ *be compactifications,* $\delta\colon Y' \to Y$ *and* $\sigma\colon X' \to X$ *desingularizations and* $f'\colon X' \to Y'$ *a morphism induced by* f_0. *Let* $\mathscr{L}'$ *be an invertible sheaf of* X' *and* D' *an effective divisor with* $\mathscr{L}'^N = \mathscr{O}_{X'}(D')$. *Assume that* $\mathscr{L}'|_{\delta^{-1}(Y_0)} = \delta^*(\mathscr{L}_0)$ *and* $D'|_{\delta^{-1}(Y_0)} = \delta^*(D_0)$. *Then* $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ *is weakly positive over* Y_0 *with respect to* $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

Proof. We may use the diagram 3.18 existing by 3.23, and the compactification V' from 3.21, iii. Moreover we may assume that π_0 and π' induce morphisms $\tau\colon W' \to Y'$ and $\gamma\colon V' \to X'$. By 2.11, a, (or better, using the same simple argument) we are allowed to blow up Y' and hence we may assume that the ramification locus of τ in Y' is a normal crossing divisor. The normalization $\tilde{Y}$ of Y' in the function ring of W' need not be non singular, but by [5], 19, (see also [9], 4.7) there is a finite non singular covering dominating $\tilde{Y}$. Replacing W', V' etc, by the induced covering and using 1.10 to enforce condition 3.18 a, we may assume that W' is a desingularization of a non singular finite cover of Y'.

Since $\delta^*\mathscr{F} = \mathscr{F}'$ is weakly positive over W' we find by 2.4, c, that $\mathscr{F}$ is weakly positive over W with respect to $g''_*(\omega_{V''/W'} \otimes \mathscr{M}'')$, where, in the notations of 3.21, iii, $\mathscr{M}'' = \sigma^*\gamma^*\mathscr{L}'$. By 2.5, a and b, and 2.4, a we find $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ to be weakly positive over Y_0 with respect to $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

§4. Direct images of powers of dualizing sheaves

We want to prove theorem 2.7. The methods employed are similar to those used in [12] (see also [9], §5 and [14] §3).

4.1. *Proof of 2.10.* We will use the notations of 2.6 and 2.10.

Let $y \in Y_0$ be a given point. Since $\mathscr{L}_0^N$ is generated over X_0 by $A = H^0(X', \mathscr{L}'^N) \cap H^0(X_0, \mathscr{L}_0^N)$, we can take a general section of the image of A in $H_0(f_0^{-1}(y), \mathscr{L}_0^N|_{f_0^{-1}(y)})$ and lift it to $s \in A$. Let D_0 be the zero divisor of s on X_0 and X_1 the covering obtained by taking the N-th root out of D_0 (see 3.14). By 3.15, b, the fiber of $f_1\colon X_1 \to X_0 \xrightarrow{f_0} Y_0$ over y' will be a normal variety with rational Gorenstein singularities for all y' in a neighbourhood of y. By 2.11, b we are allowed to replace Y_0 by this neighbourhood. Hence we may assume that the assumptions made in 3.16 are all satisfied. By construction, the pullback of D_0 extends to some divisor D' on X' with $\mathscr{O}_{X'}(D') = \mathscr{L}'^N$. Then 3.24 implies the positivity, asked for in 2.10.

4.2. For those who did not want to read the constructions 3.10–3.24, we sketch a

slightly different proof of 2.10 in the smooth case:

Let $y \in Y_0$ be given and, again, $f_1\colon X_1 \to Y_0$ the morphism obtained by taking the N-th root out of a general section

$$s \in H^0(X', \mathscr{L}'^N) \cap H^0(X_0, \mathscr{L}_0^N) .$$

If D_0 is the zero divisor of S, D_0 will be non singular and will meet all fibers over some neighbourhood of y transversally. By 2.11,b we can replace Y_0 by this neighbourhood.

Then however, $f_1\colon X_1 \to Y_0$ is smooth. Moreover, $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ will be direct summand of $f_{1*}\omega_{X_1/Y_0}$. Hence 3.9 implies that $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is weakly positive. Since s extends to some section of $\mathscr{L}'^N$, we can extend the covering X_1 of X_0 to some compactification and obtain 2.10 as stated.

Corollary 4.3. *If $f_0\colon X_0 \to Y_0$ is a smooth morphism and $\mathscr{L}_0$ a sheaf such that $f_0^* f_{0*} \mathscr{L}_0^N \to \mathscr{L}_0^N$ is surjective, for some $N > 0$, then $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is locally free and commutes with arbitrary base change.*

Proof. Since the question is local, we can assume that $\mathscr{L}_0^N$ is generated by global sections. In the notations from 4.2 $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is a direct summand of $f_{1*}\omega_{X_1/Y_0}$. This sheaf however is locally free and commutes with base change, as it was shown in [1], 5.5, and in 28,4.

4.4. *The proof of* 2.7. If $\nu = 1$ and if f_0 is smooth, 2.7 has been shown in 3.9. In general, for $\nu = 1$, 2.7 is a special case of 3.24 (for $N = 1$ and $\mathscr{L} = \mathscr{O}_{X_0}$). Therefore, let us assume that $\nu > 1$. Using the notations from 2.6, we choose $\mathscr{H}$ to be an ample invertible sheaf on Y, $\mathscr{H}' = \delta^* \mathscr{H}$ and $\mathscr{H}_0 = j^* \mathscr{H}$.

By assumption $f_0^* f_{0*} \omega_{X_0/Y_0}^N \to \omega_{X_0/Y_0}^N$ is surjective. Replacing N by some multiple we may assume moreover that $S^b(f_{0*}\omega_{X_0/Y_0}^N) \to f_{0*}\omega_{X_0/Y_0}^{b \cdot N}$ is surjective for all $b > 0$, and that ν divides N.

Let us define

$$r(\mu) = \operatorname{Min}\{s > 0; f_{0*}\omega_{X_0/Y_0}^{\mu} \otimes \mathscr{H}_0^{s \cdot \mu - 1}$$

$$\text{weakly positive over } Y_0 \text{ with respect to}$$

$$(Y', f'_* \omega_{X'/Y'}^{\mu} \otimes \mathscr{H}'^{s \cdot \mu - 1})\} .$$

$r(\mu)$ is finite. In fact, we just have to use 2.8, 2, and choose some coherent extension $\mathscr{F}$ of $f_{0*}\omega_{X_0/Y_0}^{\mu}$ which is contained in $\delta_* f'_* \omega_{X'/Y'}^{\mu}$. Then $\mathscr{F} \otimes \mathscr{H}^n$ will be globally generated for $n \gg 0$. We write $r = r(N)$.

Choose $\mathscr{L}_0 = \omega_{X_0/Y_0}^{\mu-1} \otimes f_0^* \mathscr{H}_0^{r(\mu-1)}$ and, for some large divisor E on X' with $\operatorname{codim}_{Y'}(f'(E)) \geqq 2$, $\mathscr{L}' = \omega_{X'/Y'}^{\mu-1} \otimes f'^* \mathscr{H}'^{r \cdot (\mu-1)} \otimes \mathscr{O}_{X'}(E)$. For some b sufficiently large,

$$S^{(\mu-1)\cdot b}(f_{0*}\omega_{X_0/Y_0}^N \otimes \mathscr{H}_0^{r \cdot N - 1}) \otimes \mathscr{H}_0^{(\mu-1)\cdot b}$$

will globally generated over Y_0 by sections lying in

$$H^0(Y', S^{(\mu-1)\cdot b}(f'_* \omega_{X'/Y'}^N \otimes \mathscr{H}'^{r \cdot N - 1}) \otimes \mathscr{H}'^{(\mu-1)\cdot b}) .$$

Therefore the quotient sheaf

$$f_{0*}\omega_{X_0/Y_0}^{(\mu-1)\cdot b\cdot N}\otimes \mathscr{H}_0^{(\mu-1)\cdot b\cdot r\cdot N}=f_{0*}\mathscr{L}_0^{b\cdot N}$$

is generated over Y_0 by

$$H^0(Y',((f'_*\omega_{X'/Y'}^{(\mu-1)\cdot b\cdot N})\otimes \mathscr{H}'^{(\mu-1)\cdot b\cdot r\cdot N})^{**}).$$

If we choose E large enough this vector space is $H^0(X',\mathscr{L}'^{b\cdot N})$. Since

$$f_{0*}\mathscr{L}_0\otimes\omega_{X_0/Y_0}=f_{0*}\omega_{X_0/Y_0}^{\mu}\otimes\mathscr{H}_0^{r\cdot(\mu-1)}$$

we find, if ν divides μ, that all assumptions of 2.10 are satisfied for $\mathscr{L}_0$ and $\mathscr{L}'$. Therefore we obtain:

Claim 4.5. $f_{0*}(\omega_{X_0/Y_0}\otimes\mathscr{L}_0)=f_{0*}\omega_{X_0/Y_0}^{\mu}\otimes\mathscr{H}_0^{r\cdot(\mu-1)}$ is weakly positive over Y_0 with respect to $(Y',f'_*\omega_{X'/Y'}^{\mu}\otimes\mathscr{H}'^{r\cdot(\mu-1)})$.

Claim 4.6. If ν divides μ, then $(r(\mu)-1)\cdot\mu-1\leqq r(N)\ (\mu-1)$. Moreover $r(N)\leqq N$.

Proof. By the definition of $r(\mu)$ as a minimum, the sheaf

$$f_{0*}\omega_{X_0/Y_0}^{\mu}\otimes\mathscr{H}_0^{(r(\mu)-1)\cdot\mu-1}$$

will not be weakly positive over Y_0 with respect to

$$(Y',f'_*\omega_{X'/Y'}^{\mu}\otimes\mathscr{H}'^{(r(\mu)-1)\cdot\mu-1}).$$

Then however $r\cdot(\mu-1)$ must be larger than $(r(\mu)-1)\cdot\mu-1$. Since we assumed that ν divides N, we obtain $r\leqq N$.

Claim 4.7. $f_{0*}\omega_{X_0/Y_0}^{\nu}\otimes\mathscr{H}_0^{N(\nu-1)}$ is weakly positive over Y_0 with respect to

$$(Y',f'_*\omega_{X'/Y'}^{\nu}\otimes\mathscr{H}'^{N(\nu-1)}).$$

Proof. This for $r=r(N)$ and $\mu=\nu$ just following from 4.5 and 4.6.

The proof of 2.7 ends with the usual argument: 4.7 holds true for all ample sheaves $\mathscr{H}$ and, since N remains the same for all pullback families, we can apply 2.4, b, using 2.5, b, to obtain 2.7 from 4.7.

§5. Applications to fibre spaces

In [14] we considered two methods to show the quasi-projectivity of moduli spaces: one, using "Geometric Invariant Theory" and a second one, which needs however the existence of a universal family, and hence applies only to fine moduli spaces. Let us recall the second one, in a slightly more general set up.

Assumptions 5.1. Let

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{\sigma} & X' \\ \downarrow f_0 & & \downarrow f & & \downarrow f' \\ Y_0 & \xrightarrow{j} & Y & \xleftarrow{\delta} & Y' \end{array}$$

be a commutative diagram of morphisms of reduced irreducible separated analytic spaces (we assume, for simplicity, that they are irreducible, even if the same arguments work if they are of finite type). Let us assume that:

a) i and j are open embeddings, X and Y are compact, and σ and δ are desingularizations. $X - X_0$ and $Y - Y_0$ are supposed to be closed proper analytic subspaces.

b) f and f' are surjective and $X_0 = f^{-1}(Y_0)$.

c) f_0 is flat and projective and all fibres of f_0 are reduced normal varieties of dimension n with at most rational Gorenstein singularities.

d) For all fibres F of f_0, the sheaf ω_F is semi-ample and of maximal Kodaira dimension.

e) For all $y \in Y_0$, the set $\{y' \in Y_0, f^{-1}(y') \text{ birational to } f^{-1}(y)\}$ is finite.

Theorem 5.2. *Under the assumptions made in* 5.1 *write*

$$\mathscr{L}_0 = \det(f_{0*}\omega_{X_0/Y_0}^{\nu \cdot \mu})^a \otimes \det(f_{0*}\omega_{X_0/Y_0}^{\nu})^b$$

and

$$\mathscr{L}' = \det(f'_*\omega_{X'/Y'}^{\nu \cdot \mu})^a \otimes \det(f'_*\omega_{X'/Y'}^{\nu})^b .$$

Then, for $a, b, \mu, \nu \gg 0$ *the sheaf* $\mathscr{L}_0$ *is globally generated over* Y_0 *by*

$$A = H^0(Y_0, \mathscr{L}_0) \cap H^0(Y', \mathscr{L}')$$

and the induced morphism $Y_0 \to \mathbb{P}(A)$ *is an embedding. (Definition* 2.1, *a, makes perfectly sense for analytic spaces).*

Remarks 5.3. a) The assumption 5.1, d can be replaced by the weaker assumption discussed in 2.12, b, provided f_0 is smooth.

b) A closer look to the proof of 5.2 should reveal that, even without the assumption 5.1, e, for some values of a and b, one still finds enough sections to generate $\mathscr{L}_0$ over Y_0, provided that Y_0 is normal and ω_{X_0/Y_0} relatively ample.

c) Since 5.1, d implies that $f_{0*}\omega_{X_0/Y_0}^{\eta}$ is locally free and compatible with base change (see 2.9), one can choose the numbers ν and μ in 5.2 in the following way: for all fibres F of f_0 one wants $\varphi_\nu\colon F \to \mathbb{P}(H^0(F, \omega_F^\nu))$ to be birational and the ideal of $\varphi_\nu(F)$ to be generated by homogeneous polynomials of degree μ. Moreover, the multiplication map

$$S^\mu(H^0(F, \omega_F^\nu)) \to H^0(F, \omega_F^{\nu \cdot \mu})$$

should be surjective.

In fact, one can find such ν, μ for a given F, and by base change, the same ν, μ will work for all fibres which are close by.

Before starting to sketch the proof of 5.2 it is convenient to have the following generalization of our Definition 2.2.

Definition 5.4. Let Y a reduced irreducible analytic space, $i\colon Y_0 \to Y$ a Zariski open dense subspace and $\delta\colon Y' \to Y$ a desingularization. Let $\mathscr{F}_0$ be a locally free sheaf on Y_0 and $\mathscr{F}'$ a coherent reflexive sheaf on Y'. We call $\mathscr{F}_0$ *weakly positive over* Y_0 *with*

respect to $(Y', \mathscr{F}')$ if one has:

i) $\mathscr{F}_0 \subset i^* \delta_* \mathscr{F}'$

ii) Let

$$\begin{array}{ccccc} Z_0 & \xrightarrow{j} & Z & \xleftarrow{\sigma} & Z' \\ {\scriptstyle \alpha_0}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha'}\downarrow \\ Y_0 & \xrightarrow{i} & Y & \xleftarrow{\delta} & Y' \end{array}$$

be a commutative diagram, where $Z_0 = \alpha_0^{-1}(Y_0)$, Z is projective and σ a desingularization. Then $\alpha_0^* \mathscr{F}_0$ is weakly positive over Z_0 with respect to $(Z', \alpha'^* \mathscr{F}'/\text{torsion})$.

Remark 5.5. It is likely, that this definition is not the "right" one, if Y is not Moišezon. However, in all cases where we use it, it will turn out later, that Y was Moišezon.

Proof of 5.2. Since by 2.9 the sheaves $f_{0*} \omega^{\eta}_{X_0/Y_0}$ are locally free and compatible with base change, for $\eta > 1$, they are weakly positive over Y_0 with respect to $(Y', f'_* \omega^{\eta}_{X'/Y'})$ by 2.7. Then, if we choose ν and μ as in 5.3, c, the multiplication map $S^{\mu}(f_{0*} \omega^{\nu}_{X_0/Y_0}) \to f_{0*} \omega^{\nu \cdot \mu}_{X_0/Y_0}$ is surjective and its kernel in $y \in Y_0$ determines the birational equivalence class of $f_0^{-1}(y)$. Moreover, by 2.9, the sheaf $f_{0*} \omega^{\nu}_{X_0/Y_0}$ is weakly positive over Y_0 with respect to $(Y', f'_* \omega^{\nu}_{X'/Y'})$. One can finish the proof as we did for 1.18 in [14]. However, we want to reformulate the principle behind the arguments used there.

In fact, as J. Kollár pointed out, there is hidden in [14] some quite general ampleness criterion. It follows from arguments similar to those used in [14], §2 and §4, however it was not stated there. We use the pretext of 5.2 to formulate it:

5.6. For Y_0, Y and Y' as in 5.4 and $i = 1, \ldots, s$, let $\mathscr{F}_0^{(i)}$ be locally free sheaves on Y_0, $\mathscr{F}'^{(i)}$ coherent reflexive sheaves on Y', such that $\mathscr{F}_0^{(i)}$ is weakly positive over Y_0 with respect to $(Y', \mathscr{F}'^{(i)})$. Let $r_i = \operatorname{rank}(\mathscr{F}_0^{(i)}) = \operatorname{rank}(\mathscr{F}'^{(i)})$ and let T_i: $Gl(r_i, \mathbb{C}) \to V_i$ be finite dimensional positive representations ([14] §2 or [8] §3). As we did in §2 of [14] we can consider the induced tensor bundles $T_i(\mathscr{F}_0^{(i)})$ and "taking reflexive hulls" $T_i(\mathscr{F}'^{(i)})$. Let Q_0 be a locally free sheaf on Y_0, Q' a coherent reflexive sheaf on Y' with $Q_0 \subset i^* \delta_* Q'$. Moreover, assume we have a surjective map

$$m_0 \colon \bigotimes_{i=1}^{s} T_i(\mathscr{F}_0^{(i)}) \to Q_0$$

and a map

$$m' \colon \bigotimes_{i=1}^{s} T_i(\mathscr{F}'^{(i)}) \to Q'$$

which are compatible with each other under $i^* \delta_*$. For $y \in Y_0$ the kernel of m_0 in y is a $k = \operatorname{rank}(\bigotimes_{i=1}^{s} T_i(\mathscr{F}_0^{(i)})) - \operatorname{rank}(Q_0)$ dimensional subspace of $\bigotimes_{i=1}^{s} T_i(\mathbb{C}^{r_i})$. For $G = \times_{i=1}^{s} \mathbb{P} Gl(r_i, \mathbb{C})$ we obtain thereby a G orbit $G \cdot y$ in the Grassmannian $\operatorname{Gr}(k, \bigotimes_{i=1}^{s} T_i(\mathbb{C}^{r_i}))$.

Ampleness Criterion 5.7. *Under the assumptions made in* 5.6, *assume that for all* $y \in Y_0$ *the set* $\{y' \in Y_0;\ G \cdot y = G \cdot y'\}$ *is finite and moreover* $\dim(G \cdot y) = \dim(G)$. *Then, for* $a, b_1, \ldots, b_s \gg 0$, *the sheaf* $\mathscr{L}_0 = \det(Q_0)^a \otimes \bigotimes_{i=1}^s \det(\mathscr{F}_0^{(i)})^{b_i}$ *is globally generated over* Y_0 *by*

$$A = H^0(Y_0, \mathscr{L}_0) \cap H^0(Y', \det(Q')^a \otimes \bigotimes_{i=1}^s \det(\mathscr{F}'^{(i)})^{b_i})$$

and the induced morphism $Y_0 \to \mathbb{P}(A)$ *is an embedding.*

Remark 5.8. a) Obviously the multiplication map considered in the proof of 5.2 satisfies for $s = 1$, all the assumptions made in 5.6 and 5.7.
b) J. Kollár replaced in [8], §3, the descend argument from [14], 2.7 by some kind of Nakai-Moišezon criterion. One advantage is that he obtains $b_i = 0$ in 5.7, the disadvantage is that Y_0 must be compact in his case.
c) 5.7 holds true without "with respect to $(Y', \ldots)$", in the set up of "weak positivity over Y_0". The proof remains the same.

Sketch of the proof of 5.7. As in 2.3 we may replace Y by $\mathrm{Spec}_Y(\mathscr{A}_Y)$ and assume that $\mathscr{A}_Y = \mathscr{O}_Y$. Let us write $\mathscr{F}^{(i)} = \delta_* \mathscr{F}'^{(i)} \cap i_* \mathscr{F}_0^{(i)}$ and $Q = \delta_* Q' \cap i_* Q_0$. Blowing up Y we can assume that $\mathscr{F}^{(i)}$ and Q are locally free and that the induced map $m: \bigotimes_{i=1}^s T_i(\mathscr{F}^{(i)}) \to Q$ is surjective. For $\mathscr{E}_i = \bigoplus^{r_i} \mathscr{F}^{(i)}$ we consider

$$\mathbb{P}_i = \mathbb{P}(\mathscr{E}_i^\vee) \xrightarrow{\pi_i} Y$$

$$\mathbb{P} = \mathbb{P}_1 \times_Y \ldots \times_Y \mathbb{P}_s \xrightarrow{\pi} Y$$

and $\mathbb{P}_0 = \pi^{-1}(Y_0)$. The maps $\pi_i^* \mathscr{E}_i^\vee \to \mathscr{O}_{\mathbb{P}_i}(1)$ induce "universal bases"

$$s_i: \bigoplus^{r_i} \mathrm{pr}_i^* \mathscr{O}_{\mathbb{P}_i}(-1) \to \pi^* F^{(i)}$$

and

$$s = \bigoplus_{i=1}^s s_i: \bigoplus_{i=1}^s (\bigoplus^{r_i} \mathrm{pr}_i^* \mathscr{O}_{\mathbb{P}_i}(-1)) \to \bigoplus_{i=1}^s \pi^* \mathscr{F}^{(i)}\,.$$

For $\eta \in \mathbb{Z}$, let us denote

$$\mathscr{O}_{\mathbb{P}}(\eta \cdot \underline{r}) = \left(\bigotimes_{i=1}^s \mathrm{pr}_i^* \mathscr{O}_{\mathbb{P}_i}(r_i) \right)^\eta \quad \text{and} \quad \mathscr{F} = \bigoplus_{i=1}^s \mathscr{F}^{(i)}\,.$$

If D is the degeneration divisor of s, then $\mathscr{O}_{\mathbb{P}}(D) = \pi^* \det(\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}}(\underline{r})$. As in 2.6 of [14] one obtains:

The natural inclusion

$$\rho^\eta: \mathscr{O}_Y \to \pi^* \mathscr{O}_{\mathbb{P}}(\eta \cdot D) = \det(\mathscr{F})^\eta \otimes \mathscr{O}_{\mathbb{P}}(\eta \cdot \underline{r})$$

splits.

In fact ρ is induced by the natural map

$$\det(\mathscr{F})^{-1} = \bigotimes_{i=1}^s \det(\mathscr{F}^{(i)})^{-1} \to \bigotimes_{i=1}^s (\bigotimes^{r_i} \mathscr{F}^{(i)\vee}) \to \bigotimes_{i=1}^s S^{r_i}(\mathscr{E}_i^\vee) = \pi_* \mathscr{O}_{\mathbb{P}}(\underline{r})\,.$$

On $\mathbb{P}$ we have the induced map

$$M: \bigotimes_{i=1}^s \mathrm{pr}_i^* T_i(\bigoplus^{r_i} \mathscr{O}_{\mathbb{P}}(-1)) \to \bigotimes_{i=1}^s T_i(\pi^* \mathscr{F}^{(i)}) \to \pi^* Q\,.$$

and M is surjective over $\mathbb{P}_0 - D$. Let

$$h\colon \mathbb{P}_0 - D \to \mathrm{Gr}\left(k, \bigotimes_{i=1}^{s} T_i(\mathbb{C}^{r_i})\right) \subseteq \mathbb{P}^M$$

be the induced map where the right hand inclusion is given by the Plücker coordinates. By assumption h is quasi-finite.

Let $\tau\colon \mathbb{P}' \to \mathbb{P}$ be a blowing up with centers outside $\mathbb{P}_0 - D$, such that $h' = h_0 \circ \tau\colon \mathbb{P}' \to \mathbb{P}^M$ is a morphism, and such that for some $a > 0$ $h'^* \mathcal{O}_{\mathbb{P}^M}$ (a) contains an ample sheaf $\mathcal{H}$, isomorphic to $h'^* \mathcal{O}_{\mathbb{P}^M}$ (a) over $\mathbb{P}_0 - D$. Write $\pi' = \pi \circ \tau\colon \mathbb{P}' \to Y$.

Since $h^* \mathcal{O}_{\mathbb{P}^M}(1) = \pi^*(\det(Q) \otimes \det(\bigotimes_{i=1}^{s} T_i(\mathcal{F}^{(i)})^{-1})$ we can find some divisor D', supported in $\tau^{-1}(D)$, an effective divisor E, supported in $\tau^{-1}(\mathbb{P} - \mathbb{P}_0)$, and some $b_i' \in \mathbb{Z}$ such that

$$\pi'^*(\det(Q)^a \otimes \bigotimes_{i=1}^{s} \det(F^{(i)})^{b_i'}) \otimes \mathcal{O}_{\mathbb{P}'}(D' - E)$$

is ample.

The assumptions made on $\mathcal{F}'^{(i)}$ and $\mathcal{F}_0^{(i)}$, the choice of Y and 2.3, b imply that $\pi'^* \mathcal{F}$ is weakly positive over $\pi'^{-1}(\mathbb{P}_0)$. Then as in 4.7 of [14] one finds $\mathcal{O}_{\mathbb{P}'}(\tau^* D) \otimes \bigotimes_{i=1}^{s} \pi'^* \det(\mathcal{F}^{(i)})^{r_i^{-1}}$ to have the same property. Tensorizing with a high power of this sheaf, we can, for $b_i \gg 0$, force D' to become effective. Then, 2.7 of [14] allows to "descend sections to Y" and to prove 5.7.

Remark 5.9. Let $\mathcal{M}_h$ be one of the moduli functors considered in 0.2. Then there exists an analytic coarse moduli space M_h^{an} for $\mathcal{M}_h$. One can use 5.2 to show that the normalization $M_h^{\sim}$ of M_h^{an} is quasi-projective:

In fact, by [8] §2, we know inbetween that over some finite cover $\tau_0\colon Y_0 \to M_h^{an}$ there exists some "universal family" $f\colon X_0 \to Y_0$. By 5.2 we find an ample sheaf on Y_0, which is the pullback of some invertible sheaf on M_h^{an}.

However, as long as M_h^{an} is not normal, the property of ampleness is not compatible with finite coverings. Therefore, in this way we only obtain:

a) $M_h^{\sim}$ is quasi-projective

b) If the non normal locus of M_h^{an} is compact, then M_h^{an} is quasi-projective.

However, the only obstruction to obtain 0.3 from 5.2 seems to be, that we do not know whether we can choose $\tau_0\colon Y_0 \to M_h^{an}$ in such a way that the trace map from $\tau_{0*} \mathcal{O}_{Y_0}$ has its image in $\mathcal{O}_{M_h^{an}}$. May be, a closer look to the arguments used in [8] §2 would settle this problem. Since [8] only considers compact moduli spaces (for more general moduli problems), the assumption made in b holds there.

§6. Applications to moduli problems

Fortunately we formulated 1.7, b in [14] as well as its prove in such a way, that 2.9 implies theorem 0.3. Let us nevertheless make some comments on 0.3, [14] 1.7, b, and their proves.

The assumptions on the moduli functors $\mathscr{M}_h$ in 0.2 are needed in order to have "nice Hilbert schemes". Let us state what we need exactly:

Assumptions 6.1. Let $h(T)$ be a polynomial of degree n and $\mathscr{M}'_h$ the moduli functor of complex projective normal irreducible varieties F of dimension n with at most rational Gorenstein singularities, such that ω_F is semi-ample, of Kodaira dimension n and such that for $v > 0$ one has $h(v) = \chi(F, \omega_F^v)$.

Let $\mathscr{M}_h$ be a subfunctor of $\mathscr{M}'_h$ such that for some number $v > 1$ one has:

i) For all $F \in \mathscr{M}_h(\mathbb{C})$ the sheaf ω_F^v is generated by its global sections and for the induced map $\varphi_v\colon F \to \mathbb{P}(H^0(F, \omega_F^v))$, the image $\varphi_v(F)$ is normal and birational to F.

ii) There exists some separated scheme H of finite type over $\mathbb{C}$ and a universal family $h\colon \mathscr{X} \to H \in \mathscr{M}_h(H)$ together with an isomorphism

$$\Phi_v\colon \mathbb{P}(h_* \omega^v{}_{\mathscr{X}/H}) \cong \mathbb{P}^{r-1} \times H$$

over H such that (h, Φ_v) is universal, i.e. for all $f\colon X \to Y \in \mathscr{M}_h(Y)$ and $\rho_v\colon \mathbb{P}(f_* \omega_{X/Y}^v) = \mathbb{P}^{r-1} \times Y$ there is a unique morphism $Y \to H$ such that (f, ρ_v) is obtained as pullback of (h, Φ_v).

iii) The action of $G = \mathbb{P}Gl(r, \mathbb{C})$ on H obtained by "change of coordinates in $\mathbb{P}^{r-1}$" is proper.

Let us write $\lambda_\eta = \det(h_* \omega^\eta{}_{\mathscr{X}/H})$ and $r(\eta) = \operatorname{rank}(h_* \omega^\eta{}_{\mathscr{X}/H})$. Obviously for all η the sheaf λ_η is G-linearized (see [10] Def. 1.6). Recall that we denote by $H(\mathscr{L})^s$ the stable points of H with respect to $\mathscr{L}$ and under the G-action (see [10], Def. 1.7 and [14], 5.2).

Then the main result of [14] and of this paper is

Theorem 6.2. *Under the assumptions made in* 6.1 *one has*

a) *For* $a, b, \mu \gg 0$ *one has* $H = H(\lambda_{v \cdot \mu}^a \otimes \lambda_v^b)^s$.
b) *There exists a coarse quasi-projective moduli scheme* M_h *for* $\mathscr{M}_h$
c) $\lambda_{v \cdot \mu}^a \otimes \lambda_v^b$ *descends for* $a, b, \mu \gg 0$ *to an ample invertible sheaf on* M_h.

Remark 6.3. It seems that, using the assumptions mentioned in 2.12, b, one can as well consider $\mathscr{M}'_h$ to be the moduli functor of complex projective manifolds F such that for some fixed N one finds an invertible semi-ample subsheaf $\mathscr{P}$ of ω_F^N of maximal Iitaka dimension and

$$\dim H^0(F, \mathscr{P}^v) = \dim H^0(F, \omega_F^{v \cdot N}) = h(v \cdot N)\,.$$

However, one would need that $\mathscr{P}$ behaves well on $\mathscr{X} \to H$.

Remark 6.4. If $\mathscr{M}_h$ is the moduli functor of curves or surfaces of general type, then D. Mumford [10] and D. Gieseker [4] proved 6.2. However, they were able to take $b = -\dfrac{r(v \cdot \mu) \cdot \mu}{r(v)} \cdot a$. Since the sheaf corresponding to λ_v is weakly positive over M_h their result implies ours for $n \leqq 2$, but not vice versa.

Some remarks on the proof of 2.9 *and on the proof of* "2.9 $\Rightarrow$ 6.2" *in* [14], §5:

6.5. The starting point is 2.9. If $\mathcal{M}_h$ is the moduli functor of canonically polarized manifolds with Hilbert polynomial h, then 2.9 is just needed for smooth morphisms $f_0\colon X_0 \to Y_0$. In this case the part "3.10–4.1" of this paper is unnecessary.

If $\mathcal{M}_h$ is the moduli functor of surfaces of general type, then again it would be sufficient to know 2.9 for smooth morphisms, provided that for all families $f_0\colon X_0 \to Y_0 \in \mathcal{M}_h(Y_0)$ of normal surfaces with rational double points, one has "simultaneous resolution". That means that one can find $\tau_0\colon X_0' \to X_0$ such that all fibres of $f_0' = f_0 \circ \tau_0$ are minimal desingularizations of the corresponding fibres of f_0. However, I do not know whether this can be done.

6.6. The second part, "how to use 2.9 in order to get 6.2" is completely contained in §5 of [14]. The proof given there uses only in 5.9 the special structure of the universal family and the prove of 5.9 in 5.14 of [14] works as well and without any change under the assumptions of 6.2. The sections §2–§4 of [14] are no longer necessary in the proof of 6.2. The properties, recalled in 2.4, and 2.9 are enough.

Comments and open problems

6.7. It might be possible to extend the results of this paper to moduli problems of multicanonically polarized varieties with canonically singularities, at least if one has a reasonable Hilbert scheme and if one knows that any deformation of those objects over a non singular base has canonical singularities in the total space. We leave it to the reader to estimate the technical difficulties one could run into, trying to generalize §3 to this case.

6.8. In principle the methods from [14] together with 2.10 should enable us to construct quasi-projective moduli spaces for arbitrary polarized compact complex manifolds, let us say with trivial Pic^0. However one has to find a natural polarization $\mathcal{L}$ on the total space X for all $f\colon X \to Y \in \mathcal{M}_h(Y)$, such that $f_*\mathcal{L}$ is weakly positive over Y. Since $\mathcal{L}$ is only determined up to $f^*(\mathrm{Pic}(Y))$ we can always force $f_*\mathcal{L}$ to be weakly positive, however in order to apply the methods from §5 of [14] we have to do it in a functorial way. It seems that this can be done and we hope to be able to use this construction to prove in a third part of this paper the existence of quasi-projective moduli schemes for bounded and separated moduli functors of polarized normal varieties with rational Gorenstein singularities and irregularity zero, whose canonical sheaves are numerically effective.

6.9. And what about moduli of vector bundles? Again the problem seems to be to find a good universal bundle to start with. The only case I know, where an approach similar to [14] was applied successfully, is J. Kollár's proof of the projectivity of the compactified Picard scheme in [8], §6. May be, his construction of a rigidification of the Poincaré bundle and his method to apply the ampleness criterion [8], 3.9 (see also 5.7) could give some hint, how to attack moduli of vector bundles.

6.10. As the reader has seen, we needed at some point some extension of the sheaves considered to a compactification, because 2.4, c, was the only way to descend "weak positivity" to singular varieties. It would have been more natural to this aim, to look for "good compactifications of morphisms" instead of "Schmid extensions" of direct images of dualizing sheaves. May be, one can use methods from §1 to construct those.

6.11. Of course, the existence of good compactifications of morphisms would also follow from compactifications of the moduli functors $\mathcal{M}_h$. Since in Theorem 0.3 and 6.2 we even know that the sections of $\lambda_{\nu\cdot\mu}^a \otimes \lambda_\nu^b$ which embedd M_h in a projective space are coming from some compactification of a desingularization H, it seems possible that some of the methods used in this paper could be of some help, if one wants to construct compactifications $\bar{\mathcal{M}}_h$ of $\mathcal{M}_h$.

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