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Parabolic orbifolds and the dimension of the maximal measure for rational maps

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§ 0. Introduction

Let $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map of the Riemann sphere, $\deg(f) \geqq 2$. A natural invariant measure m – the measure of maximal entropy was constructed by Ljubich [Lju] and independently by Freire, Lopes and Mañé [FLM].

The aim of this paper is to compare this measure with some Hausdorff measures. First recall the following definition. For a probability measure ν on $\bar{\mathbb{C}}$ (or, more generally, on a smooth manifold) the Hausdorff dimension of ν is defined by a formula

$$HD(\nu) = \inf_{Y:\, \nu(Y)=1} HD(Y)$$

(where $HD(Y)$ is the Hausdorff dimension of Y).

It was conjectured by Ljubich [Lju 2] that Hausdorff dimension of the measure m is strictly smaller than the Hausdorff dimension of the Julia set $J(f)$ (which is a support of m) except for some very special cases, called "critically finite with parabolic orbifold".

In the present paper we give a proof of this conjecture as well as some related results.

We shall compare the measure m with the Hausdorff measure Λ_α where $\alpha = HD(m)$.

Recall that a measure ν is said to be absolutely continuous with respect to Λ_β ($\nu \ll \Lambda_\beta$) if

$$\text{for every Borel set } E \subset \bar{\mathbb{C}} \qquad \Lambda_\beta(E) = 0 \Rightarrow \nu(E) = 0;$$

ν is said to be singular with respect to Λ_β ($\nu \perp \Lambda_\beta$) if there exists a Borel set $F \subset \bar{\mathbb{C}}$ such that

$$\nu(F) = 1 \quad \text{and} \quad \Lambda_\beta(F) = 0.$$

It is easy to see that

$$\nu \perp \Lambda_\beta \Rightarrow HD(\nu) \leqq \beta$$
$$\nu \ll \Lambda_\beta \Rightarrow HD(\nu) \geqq \beta.$$

For the measure m and $\alpha = HD(m)$ we know that

$$m \perp \Lambda_\beta \quad \text{for all} \quad \beta > \alpha$$
$$m \ll \Lambda_\beta \quad \text{for all} \quad \beta < \alpha$$

(use the remark above and ergodicity of m).

The question of the relation between m and Λ_α remains open. The answer to this question turns out to be crucial for the proof of Ljubich's conjecture.

We prove the following

Theorem 1. *Let $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map of degree $d \geqq 2$, m – the measure of maximal entropy, $\alpha = HD(m)$. Then m is singular with respect to the α-dimensional Hausdorff measure Λ_α except for the case when f is critically finite with parabolic orbifold.*

Theorem 2. *We have $HD(J(f)) > HD(m)$ iff f is not critically finite with parabolic orbifold.*

Remark 1. All maps with parabolic orbifold are classified in [DH], § 9. In § 1 we collect some useful facts on orbifolds.

§ 1. Basic notations and definitions

Orbifolds. An orbifold is a useful tool of describing the dynamics of some rational maps. The notion of orbifold was introduced by Thurston (see [T] for a general definition). We consider only orbifolds homeomorphic to the sphere $\mathbb{S}^2$. Such on orbifold can be understood to be the sphere $\mathbb{S}^2$ with a collection of "singular" points $p_1 \dots p_k \in \mathbb{S}^2$ and positive integers $\nu(p_1) \dots \nu(p_k) > 1$ ascribed to these points.

We allow some $\nu(p_i)$ to be equal ∞.

Such orbifold is denoted by $(\nu(p_1), \dots, \nu(p_k))$.

A notion of Euler characteristic of an orbifold was introduced in [T]. For our type of orbifolds it is given by the formula

$$\chi(\mathcal{O}) = 2 - \sum_{i=1}^{k} \left(1 - \frac{1}{\nu(p_i)}\right) \tag{*}$$

An orbifold $\mathcal{O}$ is called parabolic if $\chi(\mathcal{O}) = 0$. Using the formula (*) above, it is easy to write down all parabolic orbifolds homeomorphic to the sphere $\mathbb{S}^2$: (2, 2, 2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6), (2, 2, ∞), (∞, ∞).

Let f be a rational map such that the trajectories of all critical points are finite (such a map is called critically finite). There is a natural way of constructing an orbifold corresponding to f. The singular points are critical values of f (i.e. the points $f^k(c)$ for some critical point c and some $k \geqq 1$). The numbers $\nu(p_i)$ are chosen so that $\nu(f(p))$ is a multiple of $\nu(p) \cdot \deg_p f$. There is exactly one "minimal" way of such a choice. In particular, the orbifold (∞, ∞) corresponds to the map $z \to z^{\pm d}$, while $(2, 2, \infty)$ corresponds to Tchebysheff polynomials (up to sign). These are (up to a conjugacy by a Möbius transformation) the only maps with parabolic orbifold and $J(f) \neq \mathbb{C}$.

Notations. Since we are dealing with maps of the Riemann sphere, we use usually the spherical metrics. Also, all the derivatives are computed with respect to this metrics.

If B is a ball of radius r (in the usual or in the spherical metrics), then we denote by $\gamma \cdot B$ the ball with the same center and the radius $\gamma \cdot r$. The open unit disc will be denoted by D. λ denotes two-dimensional Lebesgue measure on the Riemann sphere (i.e. given by a spherical metrics).

By "critical value" we mean the image of a critical point under any iteration of f (for the first image we use rather a term "first critical value").

Very often we make use of the following Koebe Distortion Theorem:

Theorem (see [Go], Ch. 2, § 4). (1) *For very* $0<\delta<1$ *there exists* $C_\delta>0$ *such that for every univalent function* f *defined in* D

$$\log \left| \frac{f'(x)}{f'(y)} \right| \leqq C_\delta |x-y| \quad \text{for } x, y \in D_\delta$$

(*where* D_δ *is a disc of radius* δ, *centered at* 0).

In this formulation the usual derivative (rather than the spherical one) appears. It is easy to check, however, that for spherical metrics the following version is true:

(2) *For every* $\gamma>0$ *there exists a constant* K_δ *such that if* $B \subset \mathbb{C}$ *is a ball of radius* R (*with respect to the spherical metrics*), *the map* $f\colon \gamma \cdot B \to \mathbb{C}$ *is univalent and*

$$\lambda(f(\gamma \cdot B)) < \tfrac{1}{2} \lambda(\mathbb{S}^2),$$

then

$$\log \frac{|f'(x)|}{f'(y)|} \leqq K_\gamma |x-y| \quad \text{for } x, y \in B$$

(*where the distances and derivatives are computed with respect to the spherical metrics*).

§ 2. Idea of proof

We start with the well-known L.-S. Young's formula for Hausdorff dimension of an invariant ergodic measure ν.

We have (see [Y]):

$$HD(\nu) = \frac{h_\nu}{\chi_\nu} \quad \text{provided } h_\nu > 0.$$

χ_ν is the ν-Ljapunov exponent of the map f; $\chi_\nu = \int \log|f'|\,d\nu$ (notice that $h_\nu > 0$ implies $\chi_\nu > 0$, by Ruelle's inequality [R] we have

$$h_\nu(f) \leqq \int \max(0, 2\chi(f)(x))\,d\nu(x)$$

where

$$\chi(f)(x) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)|;$$

this observation was done in [P2]).

Our measure m is the measure of maximal entropy, so

$$h_m(f) = h_{\text{top}}(f) = \log d$$

and

$$\alpha = HD(m) = \frac{\log d}{\int \log|f'|\,dm}.$$

Define the function

$$\varphi = \alpha \log|f'| - \log d.$$

We have $\int \varphi\,dm = 0$.

Now, we use the results of [PUZ].

Look at the partial sums

$$S_n\varphi = \varphi + \varphi\circ f + \dots + \varphi\circ f^{n-1}.$$

Notice that

$$\exp(S_n\varphi(x)) = \frac{(|(f^n)'|(x))^\alpha}{d^n}.$$

If B is a ball around x such that $f^n|B$ is univalent and $f^n(B)$ has a big size, then $\exp S_n\varphi(x)$ equals (up to a bounded factor) $\frac{m(B)}{(\operatorname{diam} B)^\alpha}$. (Recall, that the Jacobian of m equals d, see [FLM].)

This observation suggests that examining of the partial sums $S_n\varphi$ is a good way of comparing m and Λ_α.

This was the way chosen in [PUZ] in an analogous situation. We check that $(\varphi\circ f^n)_{n=0}^\infty$ is a sequence of weakly dependent random variables and that the Law of Iterated Logarithm holds under the essential assumption: φ is not homologous to 0 in $L^2(J, m)$. If this assumption is fulfilled, then, using the Law of Iterated Logarithm, one can prove the singularity of m with respect to Λ_α (and even a stronger singularity, see Theorem 6, § 5 in [PUZ]).

So, we have to study a situation when φ is homologous to 0, i.e. when there exists a function $u \in L^2(J, m)$ such that

(H) $$\varphi = u\circ f - u.$$

Sects. 5–8 are devoted to this problem.

First we show that u, which is a priori only a measurable function, must be actually much "better", i.e. continuous in domains not containing critical values (Lemma 2). Studying the possible singularities of u, we describe the behavior of trajectories of critical points in J (Proposition 4).

In the case $J(f)=\bar{\mathbb{C}}$ we conclude that f must be critically finite with parabolic orbifold.

The remaining case is treated in Sects. 6–8.

We already know (by Proposition 4) that $f|J$ is an expanding or (so-called) subexpanding map.

Now, in order to control the behavior of remaining critical trajectories, (as it has been done in Prop. 4 for critical trajectories in $J(f)$), we extend u beyond $J(f)$, having still the homology formula (H) fulfilled.

Now, two cases can happen (they are treated in Sects. 7 and 8). In the first case our function can be extended to the open subset Γ of $\bar{\mathbb{C}}$ containing $J(f)$ (in fact Γ is the whole $\bar{\mathbb{C}}$ minus sinks and trajectories of critical values).

The only possibility which does not lead to a contradiction is $z \to z^{\pm d}$ (in expanding case) and Tchebysheff polynomial (up to sign) in subexpanding case. These two maps correspond to the orbifolds (∞, ∞) and $(2, 2, \infty)$ respectively.

The other case is when we manage only to extend our function u to a one-dimensional real-analytic set Γ, consisting of a finite number of curves. This case is eliminated again by studying the singularities of u in Γ.

In Theorem 2 we compare Hausdorff dimension of the Julia set J with Hausdorff dimension α of the measure m. Provided φ is not homologous to zero, we find a subset $X \subset J$ invariant under some iterate f^n of f, such that $f^n|X$ is expanding and $HD(X) > \alpha$. Roughly speaking, the idea is to use only the "expanding (and rich enough) part" of the dynamics of f.

If $f|J$ is expanding, then it is easy to conclude the implication (φ not homologous to zero) $\Rightarrow HD(J) > HD(m)$ from the well-known Bowen-Manning-McCluskey picture. Consider the function $t \to P(-t \log|f'|)$ where P is the usual topological pressure. This function is decreasing and convex. Moreover

$$\left.\frac{d}{dt}\right|_{t=0} P(-t \log|f'|) = -\chi_m(f),$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} P(-t \log|f'|) = \frac{1}{\alpha^2} \cdot \sigma^2.$$

(where σ^2 is so-called asymptotic variance for the sequence $S_n \varphi$, see Proposition 3, § 4; we have $\sigma^2 = 0$ iff φ is homologous to zero).

The point of intersection of a line tangent at 0 to the graph of this function with t-axis gives us the value

$$t_0 = \frac{h(f)}{\chi_m(f)} = HD(m),$$

while the point of intersection of the graph itself with t-axis gives the value $t_1 = HD(J)$.

These two values are equal iff $\sigma^2 = 0$.

§ 3. Geometric coding tree

The geometric coding tree is a very efficient tool, which allows us to use the methods of symbolic dynamics. This construction was proposed in [J] for expanding maps, the usefulness for arbitrary maps was noticed in [P2]. The tree was the main technical tool in [PUZ]. The proof of convergence is motivated by ideas of [FLM]. Denote by Σ^d the set $\{1, \ldots, d\}^{\mathbb{Z}_+}$. Our aim is to use this space as a coding space for the dynamics of f on J. This can be done as follows (see also [PUZ], § 4).

We choose a point $z \in \bar{\mathbb{C}}$ not being a critical value and curves $\gamma_1, \ldots, \gamma_d$ joining z to all points of the set $f^{-1}(\{z\})$, such that $\gamma_i \cap \gamma_j = \{z\}$. These curves have to be chosen so that

$$\bigcup_{n=1}^{\infty} f^n(\operatorname{Crit} f) \cap \bigcup_i \gamma_i = \emptyset$$

(where Crit f is the set of critical points of f).

Now, for every sequence $\eta \in \Sigma^d$ we define a sequence $(z_n(\eta))_{n=0}^{\infty}$ by induction. First, let $z_0(\eta)$ be the endpoint of γ_{η_0} different from z. Define also the curve $\gamma_0(\eta)$ to be γ_{η_0}. Now, assume that $z_n(\eta)$ and $\gamma_n(\eta)$ are already defined. We put

$$\gamma_{n+1}(\eta) = f_{\nu(\eta)}^{-(n+1)}(\gamma_{\eta_{n+1}})$$

where $f_{\nu(\eta)}^{-(n+1)}$ is a branch of $f^{-(n+1)}$ sending z to $z_n(\eta)$. The point $z_{n+1}(\eta)$ is defined to be endpoint of $\gamma_{n+1}(\eta)$ different from $z_n(\eta)$.

Obviously, the sum $\gamma(\eta) = \bigcup_{n=0}^{\infty} \gamma_n(\eta)$ is again a curve. The whole set $\Gamma = \bigcup_{\eta} \gamma(\eta)$ forms a tree (with branches possibly intersecting).

There is a natural metrics in the space Σ^d: $d(\eta, \beta) = \frac{1}{2^i}$ where $i = \max\{j \in \mathbb{Z}_+ \, \eta_j = \beta_j\}$. Thus, a natural notion of Hausdorff dimension (with respect to this metrics) can be considered.

The following crucial lemma shows, that this tree is a good way of coding the dynamics of f.

Lemma 1. (Przytycki, [P1], compare also [P2]). *For every rational map of degree $d \geqq 2$ there exists a geometric coding tree Γ and a subset $E \subset \Sigma^d$ such that $HD(E) = 0$ and for $\eta \in \Sigma^d - E$ the branch $\gamma(\eta)$ converges exponentially fast* (*i.e.* diam $\gamma_n(\eta)$ *converge to zero exponentially*).

In this way, we obtain a coding map R: $\Sigma^d - E \to \bar{\mathbb{C}}$. (It is denoted by R to underline the similarly to boundary value of the Riemann map from the unit disc onto a simply-connected domain). Let s be the left shift on Σ^d. Then (by construction) we have

$$R \circ s = f \circ R.$$

Let μ be the measure of maximal entropy on Σ^d, $h_\mu = h_{\text{top}}(s) = \log d$.

We have $HD(\mu)>0$. By ergodicity of μ, this implies $HD(F)>0$ for every set $\mathscr{P}$ of positive μ-measure. Thus, $\mu(E)=0$ (since $HD(E)=0$).

This implies that the image $m=R_* \mu$ is well-defined. Notice that $\mathrm{supp}(m)=J$.

Proposition 1. *The measure m is the (unique) measure of maximal entropy on $J(f)$.*

Proof is contained in fact in [P2], where one gets $h_m=h=\log d$. On the other hand, $h_{\mathrm{top}}(f)=\log d$. Thus, m is the measure of maximal entropy. Proof of uniqueness is contained in [M] and [Lju]. □

§ 4. Singularity with respect to Λ_α

In this Section we collect some fact which have been proved (in a slightly different form) in [PUZ].

Proposition 2 (see [PUZ], § 5, Lemma 4, 5, 6).

(a) *The function $\Psi=\log|f'|\circ R$ is in the class $L^p(\mu)$ for every $0<p<\infty$.*

(b) *For every $p>0$ there exist $K>0$, $\beta\in(0, 1)$ such that for every $n\geqq 0$*

$$\int |\Psi - E_\mu(\Psi|\mathscr{A}_n)|^p \, d\mu < K\beta^n \tag{1}$$

(where $\mathscr{A}^n$ is a partition into cylinders of length n, $E(\Psi|\mathscr{A}_n)$ is the conditional expectation),

$$\int |(\Psi - \int \Psi \, d\mu)\cdot(\Psi - \int \Psi \, d\mu)\circ s^n| \leqq K\beta^n. \quad \triangle \tag{2}$$

Let $\varphi=\alpha\cdot\Psi-\log d$.

Using the assertion of Proposition 2 and the mixing properly of μ, we conclude (compare [Ph-St]. Th. 7.1, and [PUZ], Lemma 6, § 5).

Proposition 3. *The limit (called: asymptotic variance)*

$$\sigma^2 = \lim_{n\to\infty} \frac{\int (S_n \varphi)^2 \, d\mu}{n} \quad \textit{exists.}$$

Moreover, if $\sigma^2 \neq 0$, then the sequence $(\varphi\circ s^n)_{n=1}^{\infty}$ satisfies the Law of Iterated Logarithm. If $\sigma^2=0$, then the sequence $n\to\int (S_n \varphi)^2 \, d\mu$ is bounded.

Corollary. *If $\sigma^2 \neq 0$, then $m\perp\Lambda_\alpha$ and even a stronger singularity (due to the Law of Iterated Logarithm) occurs:*

$$m\perp\Lambda_{\eta_c} \quad \textit{for } c>c_0$$

where

$$c_0=\frac{2\sigma^2}{\int \log|f'| \, dm}$$

and Λ_{η_c} is the Hausdorff measure corresponding to the function

$$\eta_c(t)=t^\alpha \exp\left(c \log\frac{1}{t} \log\log\log\frac{1}{t}\right)^{\frac{1}{2}}. \quad \triangle$$

Proof can be derived from [PUZ], Theorem 6. □

Now, we come back to the original space $L^2(J(f), m)$. Denote $\phi = \alpha \log|f'| - \log d$; we have $\varphi = \phi \circ R$. Assume $\sigma^2 = 0$. Then by Proposition 3 we know that the integrals $\int (S_n \phi)^2 \, dm$ are bounded. Then, by standard consideration (quoted, for example, in [PUZ], Lemma 1, § 1) we conclude that ϕ is homologous to 0 in $L^2(J(f), m)$, i.e. there exists a function $u \in L^2(J(f), m)$ such that

$$\phi = u \circ f - u. \tag{H}$$

§ 5. Properties of the function *u*

In this section we show that our function u, which was a priori only an element of L^2, must be actually better. The following lemma is crucial for understanding when (H) can happen.

Lemma 2. *Assume $\phi = u \circ f - u$ for some $u \in L^2(J(f), m)$. If p is not a critical value (i.e. $f^n c \neq p$ for all $n \geq 1$ and all critical points c) then there exists a neighbourhood U of p and a continuous function $w: U \to \mathbb{R}$ such that $u = w$ m-almost everywhere in U.*

Proof. In the case when $f|J$ is expanding, one can use the ideas coming from [Li]. (The proof of a similar fact in the expanding situation was given in [PUZ], Lemma 1, § 1).

We try to use an analogous way of reasoning in non-expanding case.

We know that u is m-measurable, thus by Luzin theorem there exists a set F of measure m bigger than $\frac{3}{4}$ such that $u|F$ is uniformly continuous.

We claim that

$$(*) \qquad \text{there exists } \delta > 0 \text{ such that}$$

if B is a disc (small enough) centered at p, then there exists a subset $E \subset B$ of full measure such that if $x, y \in E$, then one can find a sequence $m_i \to \infty$ and a holomorphic branch $f_\nu^{-n_i}$ defined on $2 \cdot B$ for which

$$\operatorname{diam}(f_\nu^{-n_i}(B) \leqq K \exp(-n_i \delta),$$
$$f_\nu^{-n_i}(x) \in F, \quad f_\nu^{-n_i}(y) \in F.$$

(K is some constant independent of i).

Assume (*) is true. Then we have

$$u(x) - u(y) = \log \left| \frac{(f^{n_i})'(f_\nu^{-n_i} x)}{(f^{n_i})'(f_\nu^{-n_i} y)} \right| + u(f_\nu^{-n_i} x) - (f_\nu^{-n_i} y).$$

The first summand can be estimated by $c \cdot |x - y|$, where c is some constant, by Distortion Theorem. The second summand tends to zero as $i \to \infty$, since $\operatorname{dist}(f_\nu^{-n_i} x, f_\nu^{-n_i} y) \to 0$ and $u|F$ is uniformly continuous. Thus, it is enough to prove that (*) is true.

Proof of (*): We have to pass to the natural extension $(\tilde{J}, \tilde{m}, \tilde{f})$. Let $\pi: \tilde{J} \to J$ be the projection onto 0-th coordinate. We fix a ball B centered at p such that there are no critical values up to order M in $2 \cdot B$ (M is a positive integer to be specified later on). Fix also a positive number K.

Let f_ν^{-n} be a branch of f^{-n} defined in a neighbourhood of p. We say that this branch is good if

(1) $$f_\nu^{-n} \text{ is well-defined in } 2 \cdot B$$

(2) $$\operatorname{diam}(f_\nu^{-n}(B)) < K \exp(-n\delta).$$

We say that $(f_\nu^{-n})_{n=1}^{\infty}$ is a sequence of branches if

$$f \circ f_\nu^{-(n+1)} = f_\nu^{-n}.$$

The following lemma, motivated by the paper [FLM] was proved in [PUZ] (Lemma 8, § 5). Here, we formulate it in a more convenient form.

Basic lemma. *For every $\varepsilon > 0$ there exist constants $M > 0$ (fixing the size of B), $\delta > 0$ and a subset $\tilde{K} \subset \pi^{-1}(B)$ such that $\dfrac{\tilde{m}(\tilde{K})}{m(B)} > 1 - \varepsilon$ and if $(\ldots x_{-k}, x_{-k+1}, \ldots, x_0, x_1, \ldots)$ is an element of $\tilde{K}$, then $x_{-k} = f_\nu^{-k}(x_0)$ for some good branch of f^{-k}.*

Proof. We sketch the proof here, since we shall need the explicit construction of $\tilde{K}$ later on.

The idea is to remove consecutively "bad" branches f_ν^{-n}.

We start with d^M branches of f^{-M} defined on $2 \cdot B$. We remove those branches for which $f_\nu^{-M}(2 \cdot B)$ contains a first critical value. Thus, we remove at most $2d - 2$ branches.

Assume that the good branches $f_\nu^{-(n-1)}$ have been already chosen and the images $f_\nu^{-(n-1)}(2 \cdot B)$ do not contain critical values. We consider all branches $f_\eta^{-1} \circ f_\nu^{-(n-1)}$ (i.e. good branches $f_\nu^{-(n-1)}$ are composed with d possible branches f_η^{-1} defined on $f_\nu^{-(n-1)}(2 \cdot B)$).

Among them the branches to be removed are those branches f_ν^{-n} for which

$$\lambda(f_\nu^{-n}(B)) > \exp(-2n\delta)$$

or $f_\nu^{-n}(2 \cdot B)$ contains a first critical value.

We procede by induction.

A straightforward computation (compare [PUZ], § 5, Lemma 8) shows, that the remaining set $\tilde{K}$ (consisting of sequences $(\ldots x_{-k}, x_{-k+1}, \ldots, x_0, x_1, \ldots)$ such that $x_0 \in B$ and $x_{-k} = f^{-k}(x_0)$ for some branch f^{-k} which has not been removed) has measure m as close to $\tilde{m}(B)$ as we want (if δ is small and M large enough). Notice, that every branch chosen in this way is good (use the Koebe Distortion Theorem). □

Notice that the set $\tilde{K}$ has a natural product structure. There is a bijection $\varphi_{x,y}$ between the fibres $\pi^{-1}(\{x\})\cap\tilde{K}$ and $\pi^{-1}(\{y\})\cap\tilde{K}$, namely:

$$\varphi_{x,y}((\dots x_{-k}, x_{-k+1}, \dots, x_0, x_1, \dots)) = (\dots y_{-k}, y_{-k+1}, \dots, y_0, y_1, \dots)$$

if x_{-k} and y_{-k} are obtained by use of the same branch of f^{-k} defined in B.

Moreover,

$$(\varphi_{x,y})_* \tilde{m}_x = \tilde{m}_y$$

where $\tilde{m}_x, \tilde{m}_y$ are conditional measures on fibres of the partition into sets

$$\pi^{-1}(\{x\})\cap\tilde{K} \qquad (x\in B).$$

Now, from the ergodicity of $\tilde{m}$ it follows that there exists a subset $\tilde{E}\subset\tilde{K}$ of full measure such that for $\tilde{x}\in\tilde{E}$

$$\tilde{f}^{-n}(\tilde{x})\in\tilde{F}=\pi^{-1}(F)$$

happens with frequency $m(F)$ (bigger than $\frac{3}{4}$).

Since $\tilde{m}(\tilde{E})=\tilde{m}(\tilde{K})$, for m-almost all $x\in B$

$$\tilde{m}_x(\tilde{E}\cap\pi^{-1}(\{x\})\cap\tilde{K})=\tilde{m}_x(\pi^{-1}(\{x\})\cap\tilde{K}).$$

Thus, for almost all $y\in B\cap\pi(\tilde{E})$

$$\tilde{m}_y(\varphi_{x,y}(\tilde{E}\cap\pi^{-1}(\{x\})\cap\tilde{K}))=\tilde{m}_y(\pi^{-1}(\{y\})\cap\tilde{K})$$

i.e. $\varphi_{x,y}(\tilde{E}\cap\pi^{-1}(\{x\})\cap\tilde{K})$ has a full measure in the fibre $\pi^{-1}(\{y\})\cap\tilde{K}$ (due to the product structure of $\tilde{K}$).

It follows, that for these x, y there exists a common sequence of branches f_ν^{-n} such that both $f_\nu^{-n}(x)$ and $f_\nu^{-n}(y)$ fall into F with a frequency bigger than $\frac{3}{4}$. Thus, one can find a sequence n_i such that

$$f_\nu^{-n_i}(x)\in F \quad\text{and}\quad f_\nu^{-n_i}(y)\in F.$$

In this way, (*) is proved, completing the proof of Lemma 2. □

As a corollary, we get an important

Proposition 4.

(1) *If $f^k(d_1)=f^l(d_2)$ for some $d_1, d_2\in J$, d_1, d_2 not being critical values, then*

$$\deg_{d_1}(f^k)=\deg_{d_2}(f^l).$$

(2) *If c is a critical point in J and $f^k(c)=f(x)$ for some x, then x is either a critical point or a critical value.*

(3) *The trajectories of all critical points in J are finite.*

Proof. (1) It follows from Lemma 2, that u is bounded a.e. in some neighbourhood of d_1. Using the homology formula (H) we conclude that u has a singularity

$$\frac{s-1}{s}\log|y-f^k(d_1)| \qquad (s=\deg_{d_1} f^k)$$

in the neighbourhood of $f^k(d_1)$, i.e. $u(y)-\frac{s-1}{s}\log|y-f^k(d_1)|$ is bounded for y close to $f^k(d_1)$. By the same reasoning we get a singularity

$$\frac{t-1}{t}\log|y-f^l(d_2)| \qquad \text{where } t=\deg_{d_2} f^l$$

in the neighbourhood of $f^l(d_2)=f^k(d_1)$. Comparing these two results we get $t=s$.

Now, (3) follows easily from (2) while (2) is a consequence of (1). □

Remark 2. Denote $\mathscr{P}_f=\bigcup_{n\geq 1} f^n$ (critical points in J). Assume ϕ is homologous to zero. Then to every point $b\in\mathscr{P}_f$ we can ascribe a positive integer $v(b)$ such that

$$v(f(b))=\deg_b f\cdot v(b) \tag{1}$$

and

$$u(x)\approx\alpha\left(1-\frac{1}{v(b)}\right)\log|x-b| \tag{2}$$

in the neighbourhood of b (the sign $\approx$ means here: the difference is bounded).

This is possible by Proposition 4.

We get an important

Corollary. *If ϕ is homologous to zero and $J(f)$ is the whole $\mathbb{C}$, then f is critically finite with parabolic orbifold.*

Proof. The numbers $v(b)$ are precisely the numbers ascribed to critical values in the definition of an orbifold. Moreover Proposition 4 together with the property (1) of Remark 2 above show that $f\colon \mathcal{O}_f\to\mathcal{O}_f$ is a covering map of orbifolds (see [T] for the definition; for our orbifolds it means just, that $v(f(b))=\deg_b f\cdot v(b)$). Thus (as it is shown in [T])

$$\chi(\mathcal{O}_f)=d\cdot\chi(\mathcal{O}_f), \qquad \text{hence } \chi(\mathcal{O}_f)=0. \quad \square$$

Next, we assume that ϕ is homologous to 0 and $J(f)\neq\mathbb{C}$.

First, notice that there are neither Siegel discs nor Herman rings in the complement of $J(f)$ (since the boundary of such domain is contained in the closure of trajectories of critical points in $J(f)$ and here the trajectories of critical points in $J(f)$ are finite.

Thus, there are only basins of sinks and their preimages in the complement of $J(f)$. Moreover, parabolic basins are also excluded, since we have

Lemma 3. (a) *If $p \in J$ is a periodic point of period n and there exists a "good way back" from p (i.e. a sequence $(x_i)_{i=1}^{\infty}$ of points such that $f(x_1)=p$, $f(x_{i+1})=x_i$ for $i \geqq 1$ and none of x_i is a critical point), then*

$$|(f^n)'(p)|^\alpha = d^n \qquad (\textit{thus, } p \textit{ is a source}).$$

(b) *If $p \in J$ is a periodic point of period n and $p=f^k(c)$, where c is a critical point not being a critical value, then*

$$|(f^n)'(p)|^\alpha = d^{ns} \qquad \textit{where } s=\deg_c f^k$$

(thus, p is also a source).

Proof relies on a straightforward computation and will be omitted. □

§ 6. Case $J(f) \neq \mathbb{C}$

Here, we want to describe the trajectories of critical points outside $J(f)$.

Our first step will be to extend u beyond $J(f)$ as far as possible; we require the extended function to satisfy the homology formula:

$$u(f(x)) - u(x) = \alpha \log|f'(x)| - \log d$$

whenever $u(f(x))$, $u(x)$ are defined.

We have two (slightly different) cases: either there are no critical points in J (expanding case) or critical points in J satisfy the statement of Proposition 4, in particular their trajectories are finite (subexpanding case).

For a subexpanding map it is convenient to introduce a new, "adapted" metrics (compare [DH 2]) defined by the function

$$v(x) = \sum_{b \in \mathscr{P}_f} \frac{1}{|x-b|^{\left(1-\frac{1}{v(b)}\right)}}.$$

The derivative in this new metrics is $Df = |f'| \dfrac{v \circ f}{v}$ and $\log Df$ is homologous to $\log d$, $\log Df = \log d + w \circ f - w$ where $w = \log v + u$.

The function w is bounded (see Prop. 4 for a discussion of singularities of u). In particular, we have $Df^n > 1$ for some n (since $Df^n = n \log d + w \circ f^n - w$).

Both cases (expanding and subexpanding ones) will be treated in the same way.

First, remind that a sequence of branches is a sequence such that

$$f(f_v^{-(n+1)}(x)) = f_v^{-n}(x).$$

Define a set $\Gamma = \{x \in \bar{\mathbb{C}}$: x is neither a critical point nor a sink and for $x_0 \in J$ not being a critical value and an arbitrary curve joining x to x_0 and not passing through critical values and sinks the formula

$$(\#) \qquad \bar{u}(x) = u(x_0) + \alpha \sum_{i=1}^{\infty} (\log|f'(f_{v,\gamma}^{-i} x)| - \log|f'(f_{v,\gamma}^{-i} x_0)|)$$

gives the same result (i.e. independent of the choice of x_0, γ and a sequence of branches $f_{v,\gamma}^{-i}$ along γ.

In the following lemmas we list some properties of the set Γ.

Lemma 4. (1) *If* $x \in J - \{$*critical values in* $J\}$ *then* $x \in \Gamma$ *and* $\bar{u}(x) = u(x)$.

(2) $f^{-1}(\Gamma) \subset \Gamma$ *and the function* $\bar{u}$ *satisfies the homology formula* (H) *whenever* $x, f(x) \in \Gamma$.

Proof. Let $x \in J$. Choose $x_0 \in J$ and a curve γ as in definition of Γ. We have

$$f_{v,\gamma}^{-n}(\gamma) \rightrightarrows J$$

Indeed, remind that there are only basins of sinks and their preimages in the complement of J. Hence, for n large $f_{v,\gamma}^{-n}(\gamma)$ is contained in a neighbourhood V of J in which f is expanding with respect to the usual metrics or to the adapted one (in the subexpanding case).

In both cases we have

$$\operatorname{diam}(f_{v,\gamma}^{-n}(B)) \xrightarrow[n\to\infty]{} 0.$$

It follows that

$$u(f_{v,\gamma}^{-n}(x)) - u(f_{v,\gamma}^{-n}(x_0)) \xrightarrow[n\to\infty]{} 0.$$

Thus,

$$\begin{aligned}
&u(x_0) + \alpha \sum_{i=1}^{\infty} (\log|f'(f_{v,\gamma}^{-i}(x))| - \log|f'(f_{v,\gamma}^{-i}(x_0))|) \\
&= \lim_{n\to\infty} \left(\left(u(x_0) + \alpha \sum_{i=1}^{n} \log|f'(f_{v,\gamma}^{-i}(x))| - n \log d \right) \right. \\
&\quad \left. - \left(\alpha \sum_{i=1}^{n} \log|f'(f_{v,\gamma}^{-i}(x_0))| - n \log d \right) \right) \\
&= \lim_{n\to\infty} (u(x_0) + u(x) - u(f_{v,\gamma}^{-n}(x)) - u(x_0) + u(f_{v,\gamma}^{-n}(x_0))) = u(x)
\end{aligned}$$

and obviously the result does not depend on the way we have chosen.

Proof of (2) is based on a straightforward computation and will be omitted. □

Lemma 5. *Let y_0 be neither a critical value nor a sink. Take a disc B around y_0 containing no critical values. Then $\Gamma \cap B$ is*

(1) *the whole B or*

(2) *an empty set or*

(3) *the sum of a finite number of real-analytic curves and isolated points.*

Proof. Fix a point $x_0 \in J$ and a curve γ joining x_0 and y_0 as in the definition of Γ.

Notice that the formula (#) defines a harmonic function on B (where $f_{v,\gamma}^{-n}(y)$ is understood to be that branch of f^{-n} on B which maps y_0 to $f_{v,\gamma}^{-n}(y_0)$.

Consider two such functions $\bar{u}_1, \bar{u}_2$ (obtained by a procedure above). Then the set

$$V_{\bar{u}_1, \bar{u}_2} = \{z : \bar{u}_1(z) = \bar{u}_2(z)\}$$

is the set of zeros of a harmonic function, thus either the whole B or the sum of a finite number of real-analytic curves.

Now, take another pair $V_{\bar{u}_3, \bar{u}_4}$ and consider the set $V_{\bar{u}_1, \bar{u}_2} \cap V_{\bar{u}_3, \bar{u}_4} \cdot V_{\bar{u}_3, \bar{u}_4}$ is again a sum of a finite number of analytic curves $t_1 \dots t_k$ (or the whole B). Moreover, if $s_i \cap t_j$ has a condensation point then $s_i = t_j$. Indeed, t_j is described by the $\mathbb{R}$-analytic parametrization $\phi = (\phi_1, \phi_2)$. Hence, the function $(\bar{u}_1 - \bar{u}_2) \circ \phi$ is $\mathbb{R}$-analytic and equals zero on a set having a condensation point. Thus, it equals zero everywhere and $t_j \subset V_{\bar{u}_1, \bar{u}_2}$. It follows that $V_{\bar{u}_1, \bar{u}_2} \cap V_{\bar{u}_3, \bar{u}_4}$ is a sum of a finite number of real-analytic curves and isolated points (or the whole B). It is easy to see that the same is true for the full intersection $\Gamma = \bigcap_{(\bar{u}_1, \bar{u}_2)} V_{\bar{u}_1, \bar{u}_2}$ (the intersection is taken over all possible pairs as above). □

Now, we have two cases which will be treated in §§ 7, 8. In the first case the set Γ consists of a finite number of real-analytic curves. In the second case Γ is an open subset of $\bar{\mathbb{C}}$.

§ 7. One-dimensional set Γ

In this section we assume that

$$\operatorname{int} \Gamma \cap J = \emptyset.$$

Under this assumption we have

Lemma 6. (a) *Take a point $y_0 \in J$ not being a critical value. If a ball $B(y_0, \rho)$ is small enough and does not contain critical values, then $\Gamma \cap B(y_0, \rho)$ is a real-analytic curve.*

(b) *If $y \in \bar{\mathbb{C}}$ and the ball $B(y, \rho)$ does not contain critical values, then $B(y, \rho) \cap \Gamma$ contains at most one analytic curve or is a set (possibly empty) of isolated points.*

Proof. (a) Since $y_0 \in J$ and $J \subset \Gamma$, then y_0 is not an isolated point in Γ. Thus, one can assume that there are no isolated points of Γ in B. We have to check that Γ cannot contain two analytic curves intersecting at y. Assume that two such curves exist. But one can find an infinite number of branches $f_v^{-n_i}$ defined in B such that $f_v^{-n_i}(B) \subset B$ and the set of points $(f_v^{-n_i}(y))_{i=1}^{\infty}$ is infinite. All these points are in Γ (by Lemma 4) and all of them are points of intersection of curves contained in Γ. This contradicts Lemma 5.

(b) We fix a point $y_0 \in J$ and a ball B as in (a). There exists a branch f_v^{-n} such that $f_v^{-n}(B(y,\rho)) \subset B$. Since $B \cap \Gamma$ is an analytic curve, then $\Gamma \cap B(y,\rho)$ is contained in the analytic curve $f^n(\Gamma \cap B)$. □

Now, we describe connected components of Γ.

Lemma 7. *Assume $y_0 \in J$ is not a critical value. If s is a connected component of y_0 in Γ, then $\bar{s}$ is either an analytic Jordan curve, or an embedded closed interval with endpoints being critical points or sinks.*

Proof. y_0 is not an isolated point in Γ. Thus, by Lemma 6 we conclude that s is locally a real-analytic curve. Notice that the curve s may have only two (perhaps coinciding) condensation points not belonging to s. (For, by Lemma 6 (b) a condensation point must be either a critical value or a sink). Thus, $\bar{s}$ is an embedded closed interval (if such points exist) or an analytic Jordan curve (if $\bar{s} - s = \phi$). □

Denote by S the set of all connected components of Γ intersecting J. S is a finite set (since in the neighbourhood of every point $y_0 \in J$ we have only a finite number of curves in Γ (even if y_0 is a critical value).

Lemma 8. *Let p be an endpoint of the curve $s \in S$. Then* $\lim_{\substack{x \to p \\ x \in S}} \bar{u}(x) = -\infty$.

Proof. Use the homology formula

$$\bar{u}(x) = \bar{u}(f_v^{-n}(x)) + \alpha \log |(f^n)'(f_v^{-n}(x))| - n \log d$$

for an appropriate branch of f^{-n}. □

Now, using singularities of $\bar{u}$ we describe the dynamics of f on the curves belonging to S.

Proposition 5. *$f|S$ is a permutation of curves, i.e. for every $s \in S$ $\bar{s}$ is mapped onto $\bar{s}'$ (for some $s' \in S$). Moreover, a curve of a given type (i.e. a closed one or homeomorphic to the interval) is mapped onto a curve of the same type.*

Proof. Let $s \in S$. Obviously, there exists a curve $s' \in S$ such that $f(s) \cap s' \neq \emptyset$.

First we assume that s' is homeomorphic to the interval. If there exists a point $y \in s$ such that $f(y)$ is an endpoint of s' then y must be a critical point (because $\bar{u}(x)$ tends to $-\infty$ as $x \to f(y)$ and by the homology formula). Thus, $f^{-1}(s')$ contains an arc passing through the point y, hence a neighbourhood of y in Γ.

This implies that

$$f(\bar{s})\subset\bar{s}'.$$

Actually, we have $f(\bar{s})=\bar{s}'$, since otherwise some endpoint of $f(s)$ (being a critical value or a sink) would lie in s'. But there are neither critical values nor sinks in s'.

If s' is a Jordan curve, then s must be Jordan curve, too (since the endpoints of s are critical values or sinks and there are no such points in s'). As before, one can check that $f(s)\subset s'$. Thus, there are no critical points in s (since there are no critical values in s') and $f|s$ is locally one-to-one.

Hence, $f(s)=s'$.

Obviously, a curve s homeomorphic to the interval is mapped onto a curve of the same type. Thus, a Jordan curve $t\in S$ must be mapped onto a Jordan curve (because each curve is the image of some other curve). □

Corollary. *There exists a component $t\in S$ periodic for f (i.e. $f^k(t)=t$ and $f^{-k}(t)=t$ for some k).* △

First, assume that t is a Jordan curve.

Proposition 6. *If there is a Jordan curve $t\in S$ periodic under f, then f equals $z\to z^{\pm d}$ up to a Möbius transformation.*

Proof. Passing to some iterate of f, one can assume that $f(t)=f^{-1}(t)=t$. Then the Julia set is just our curve t. To see that, take a point $x\in J\cap t$. Then $J=\mathrm{cl}(\bigcup_n\{y: f^n y=t\})\subset t$ since $f^{-n}(t)\subset t$. On the other hand, if $y\in t-J$ then $f^n(y)$ tends to a sink and belongs to t. This is impossible, since t is separated from sinks.

Thus, the situation must be as follows: J is an analytic Jordan curve dissecting $\mathbb{S}^2$ into two simply-connected domains D_1, D_2. One can assume (taking $f\circ f$) that $f(D_1)=D_1, f(D_2)=D_2$. Then f is conjugate by Möbius transformation to the Blaschke product and J is a circle (by the argument due to Sullivan [Su]).

Now, among these map there is only one (up to a conjugacy by a Möbius transformation) for which $\log|f'|$ is homologous to $\log d$, this is $z\mapsto z^d$.

Since we have passed to some iterate f^k, we know up to now that f^k equals (up to a Möbius transformation) $z\to z^{d^k}$. But then f itself equals $z\mapsto z^d$ or $z\mapsto z^{-d}$ (up to a Möbius transformation). □

Now, we assume that there exists a curve $s\in S$ periodic under f and homeomorphic to the interval.

Suppose $f^k(s)=s$. Obviously, the endpoints of s are mapped by f^k to the endpoints. Thus, there exists an endpoint p of s periodic for f; one can assume that f is a fixed point.

First, notice that p cannot be a superattractive fixed point. To see this, take a small annulus $\mathscr{P}$ around p. Since p is the endpoint of s and Γ is invariant under f^{-1}, there exists a dense subset of $\mathscr{P}$ contained in Γ. This is a contradiction (we already know, that $\Gamma\cap\mathscr{P}$ consists of analytic curves).

Thus, $\lambda=|f'(p)|\neq 0$. We already know, that

$$\lim_{\substack{x\to p\\ x\in S}} \bar{u}(x)=-\infty.$$

It is easy to compute that in the neighbourhood of p

$$\bar{u}(x)\approx\log|x-p|\left(\alpha-\frac{\log d}{\log\lambda}\right).$$

On the other hand, p has a preimage q different from p and not being a critical value.

If $f^n(q)=p$ and $\deg_q f^n=t$, then $\bar{u}\approx\alpha\left(1-\frac{1}{t}\right)\log|x-p|$ in the neighbourhood of p (see Corollary after Lemma 2).

This gives $t=\alpha\dfrac{\log\lambda}{\log d}$. Thus, $\lambda>1$ and p is a source. As in the proof of Proposition 6 we check that the curve s and the Julia set J coincide.

Obviously, we can assume that the endpoints of s are $-1, 1$ and that $\infty\notin s$. Consider a two-sheet cover of $\mathbb{S}^2$ ramified over $1, -1$, given by the map π: $\bar{\mathbb{C}}\to\bar{\mathbb{C}}$,

$$\pi(z)=\frac{z+z^{-1}}{2}.$$

The preimage of s under π is the piecewise smooth Jordan curve t dissecting $\bar{\mathbb{C}}$ into two topological discs D_1, D_2, each of them being mapped by π onto the complement of s.

The map $\tilde{f}$ defined by $\pi^{-1}\circ f\circ\pi$ on D_1 and D_2 extends to a continuous (and thus also analytic and rational) map on $\bar{\mathbb{C}}$ with $J(\tilde{f})=t$. Moreover, $\log|\tilde{f}'|$ is homologous to $\log d$ and we conclude from Proposition 6 that t is a geometric circle and $\tilde{f}$ is conjugate to $z\to z^d$ by some homography $\tilde{h}$. Since $\tilde{f}\left(\frac{1}{z}\right)=\frac{1}{\tilde{f}(z)}$, t must be the unit circle S^1 and h may be taken as $\tilde{h}(z)=\dfrac{z-a}{1-\bar{a}z}$; $a\in D$ is the superattractive fixed point of f. The other fixed point is $\frac{1}{a}$ $\left(\text{since } \tilde{f}\left(\frac{1}{z}\right)=\frac{1}{\tilde{f}(z)}\right)$; it must be equal to $\tilde{h}^{-1}(\infty)=\frac{1}{\bar{a}}$, hence $a\in\mathbb{R}$. Then $\tilde{h}\left(\frac{1}{z}\right)=\frac{1}{\tilde{h}(z)}$ and there exists a homography h such that $\pi\circ\tilde{h}=h\circ\pi$. This homography gives the conjugacy between f and the Tchebysheff polynomial.

In fact, we have only checked that some iterate of f is conjugate to Tchebysheff polynomial. But now we already know that $J(f)$ is an interval (since $J(f)=J(f^k)$) and repeating the reasoning above we conclude that f or $-f$ is the Tchebysheff polynomial.

Thus, we have proved

Proposition 7. *If there exists a curve $s \in S$ homeomorphic to the interval and periodic under f, then f is conjugate by Möbius transformation to the Tchebysheff polynomial (up to sign).*

Actually, it is easy to check that in both cases described in Propositions 6 and 7 the set Γ is two-dimensional (i.e. $\operatorname{int} \Gamma \cap J \neq \emptyset$). Thus we have

Corollary. *If $\alpha \log|f'|$ is homologous to $\log d$ then $\operatorname{int} \Gamma \cap J \neq \emptyset$.*

So, it remains to consider the case $\operatorname{int} \Gamma \cap J \neq \emptyset$. This will be done in the next section.

§ 8. Two-dimensional set Γ

Throughout this section we assume that $\operatorname{int} \Gamma \cap J \neq \emptyset$.

Lemma 8. *If $\operatorname{int} \Gamma \cap J \neq \emptyset'$, then Γ is an open connected set and every point in $\partial \Gamma$ is either a critical value or a sink.*

Proof. There exists $y \in J \cap \operatorname{int} \Gamma$. Let s be (as before) the connected component of y in Γ. We claim that s is an open set. Indeed, the set $\{x \in s : x \in \operatorname{int} \Gamma\}$ is open and closed in Γ. (If $x_n \in \operatorname{int} \Gamma$ and $x_n \to x \in \Gamma$, then x must be in $\operatorname{int} \Gamma$. Otherwise (by Lemma 5) in the neighbourhood of $x \Gamma$ consists of a finite number of analytic curves, thus $x_n \notin \operatorname{int} \Gamma$ for large n).

Now, let z belong to ∂s. We claim that z is a critical value or a sink. Otherwise, as $z \in \partial s \subset \bar{s}$, then $z \in \Gamma$ (by definition of Γ). Thus, in a small ball B around z the set Γ is a sum of a finite number of curves and isolated points (but then there are no points of $s = \operatorname{int}(s)$ in B) or the whole B (but then $z \in \partial s$).

It follows that $s \cup \partial s = \bar{\mathbb{C}}$ (since ∂s is at most countable). This ends the proof. □

The next lemma is in fact a repetition of Proposition 4 of Sect. 5 and therefore the proof will be only sketched.

Lemma 9. *If $c \notin J$ is a critical point then $c \in \partial s$ and c is periodic.*

Proof. Since the function u can be extended to the whole set $s = \bar{\mathbb{C}} - \partial s$, we can use the same method (studying of singularities of $\bar{u}$) as in the proof of Proposition 4, § 5. If $c \notin \partial s$, then the function $\bar{u}$ is bounded in the neighbourhood of c and we conclude (as in Proposition 4 and Lemma 3b) that some image of c is a source. But there are no sources outside of J. Thus, $c \in \partial s$ and c is a critical value by Lemma 8 above. Since there are only finitely may critical points, it follows that c is periodic.

Take an arbitrary critical point $c_0 \notin J$. We can assume that $f(c_0) = c_0$ (replacing f by some iterate of f). We claim that $\deg_{c_0} f = d$. Otherwise, there exists a point $x \neq c_0$ such that $f(x) = c_0$. The point x must be a critical value (again by a reasoning like in the proof of Prop. 4). Thus, there exists a critical point $c_1 \neq c_0$ such that $f^k(c_1) = c_0$. Obviously, one can require that c_1 is not a critical value. But this contradicts Lemma 9 above. □

Now, we have again two cases. The first possibility is that there are no critical points in J. Then f must have two superattractive points with maximal degree. Then the Julia set is a circle and f is conjugate by a Möbius transformation to $z \mapsto z^d$ (compare the proof of Proposition 6). Since we have replaced f by some iterate f^k, actually we know that f^k is conjugate to $z \mapsto z^{d^k}$. This implies that f itself is conjugate to $z \mapsto z^d$ or $z \mapsto z^{-d}$. Notice that the corresponding orbifold is $\mathcal{O} = (\infty, \infty)$ and $\chi(\mathcal{O}) = 0$, thus $\mathcal{O}$ is parabolic.

The second possibility is that there are critical points in J. Then there is only one critical superattractive point of maximal degree in $\partial\Gamma$ and (sending this point to ∞ by a rotation) we can assume that f is a polynomial.

Moreover, the map $f\colon \mathcal{O}_f \to \mathcal{O}_f$ is a covering map of orbifolds. It follows (as in the corollary after Proposition 4) that $\mathcal{O}_f$ is parabolic. The only parabolic orbifold corresponding to the polynomial with critical points in the Julia set is $(2, 2, \infty)$. It corresponds to the Tchebysheff polynomial (up to sign). (Compare [DH], § 9).

We summarize the results of this section in

Proposition 8. *If* $\alpha \log|f'|$ *is homologous to* $\log d$ *and* $\operatorname{int} \Gamma \cap J \neq \phi$, *then* f *is conjugate by a Möbius transformation to one of the following maps*:

$$z \mapsto z^d \quad or$$

$$z \mapsto z^{-d} \quad or$$

$\pm$ *Tchebysheff polynomial.*

In this way, the proof of Theorem 1 has been completed.

§ 9. Hausdorff dimension of the Julia set

In this section we shall prove

Theorem 2. *Hausdorff dimensions of the Julia set* J *and of the measure* m *are equal iff* f *is critically finite with parabolic orbifold (i.e.* $\alpha \log|f'| - \log d$ *is homologous to zero).*

Proof. We shall work in the natural extension $(\tilde{J}, \tilde{m}, \tilde{f})$.

Let B be a ball in $\bar{\mathbb{C}}$. Recall that in § 5 we introduced a notion of good branches of f^{-n} defined on B; a branch f_v^{-n} is good if

$$f_v^{-n} \text{ is well-defined in } 2 \cdot B$$

and

$$\operatorname{diam} f_v^{-n}(B) < K \exp(-n\delta).$$

In the Basic Lemma (§ 5) we proved the following: there exists $\delta > 0$ such that for every $\tilde{\varepsilon} > 0$ there is $M \in \mathbb{Z}_+$ so that if there are no critical values up to order M in B then one can find a subset $\tilde{K}_B \subset \tilde{B} = \pi^{-1}(B)$ of $\tilde{m}$-measure bigger than $(1 - \tilde{\varepsilon})\, m(B)$ and consisting of "good" trajectories. (The trajectories $(\dots x_{-k}, x_{-k+1}, \dots, x_0, x_1, \dots)$ is good if x_{-k} is an image of x_0 under some "good" branch of f^{-k} defined on B.)

Let $p_1, \dots, p_s$ be critical values up to order M. Take $r > 0$ small and $\varepsilon > 0$.

Let $B_1, \ldots, B_s$ be balls centered at p_i's with radius r. Let $\mathscr{B}$ be a cover of the remaining set $\bar{\mathbb{C}} \cup B_i$ with balls of radius $\frac{r}{4}$. If r is small enough then

$$\tilde{m}(\bigcup_{B \in \mathscr{B}} \tilde{K}_B) > 1 - \varepsilon.$$

Fix a ball $B \subset \bar{\mathbb{C}}$.

Let $\mathscr{F}_n$ be the set of branches f_ν^{-n} defined in B such that f_ν^{-n} is well-defined in $2 \cdot B$, diam $f_\nu^{-n}(B) \leqq \exp\left(-n\frac{\delta}{2}\right)$ and $f_\nu^{-n}(B) \subset \frac{1}{2} B$.

For $t \in \mathbb{R}$ we define

$$S_n^t(B) = \sum_{\nu \in \mathscr{F}_n} \sup_{x \in B} |(f_\nu^{-n})'(x)|^t.$$

Assume that $\alpha \log|f'| - \log d$ is not homologous to zero. Then the asymptotic variance σ^2 (see Prop. 3, § 4) is non-zero.

Proposition 9. *If $\sigma^2 \neq 0$, then there exists a ball B such that the sequence $S_n^\alpha(B)$ is unbounded.*

Proof. We know that the sequence $\phi, \phi \circ f, \ldots, \phi \circ f^n, \ldots$ satisfies the Central Limit Theorem, since $\sigma^2 \neq 0$ (compare Prop. 3, § 4), (recall that $\phi = \alpha \log|f'| - \log d$). It follows that

$$\tilde{m}(\{\tilde{x} \in \tilde{J} : S_n \phi(\tilde{x}) < -A\sigma\sqrt{n}\}) \to \Psi(-A)$$

where Ψ is the distributant of the normal distribution.

Obviously (if ε is small) there exists a ball B of our cover $\mathscr{B}$ such that the inequality

$$\tilde{m}(\{\tilde{x} \in \tilde{J} : S_n \phi(\tilde{x}) < -A\sigma\sqrt{n} \quad \text{and} \quad \tilde{f}^n(\tilde{x}) \in \tilde{K}_B\}) > \beta > 0$$

holds for some β and infinitely many n.

By the topological exactness of f we know that for some $l \in \mathbb{Z}_+$ $f^l(\frac{1}{4}B) \supset J$.

Let $q_1 \ldots q_m$ be critical values up to order l.

Let $D_i (i = 1, \ldots, m)$ be a ball of radius ρ around q_i.

Choose $\rho > 0$ small enough to have

$$(*) \quad \tilde{m}(\{\tilde{x} \in \tilde{J} : \pi(\tilde{x}) \notin \bigcup_{i=1}^m 2 \cdot D_i, S_{n-l} \phi(\tilde{x}) < -A\sigma\sqrt{n-l}, \tilde{f}^{n-l}(\tilde{x}) \in \tilde{K}_B\}) > \beta' \text{ for some}$$

$\beta' > 0$ and infinitely many n.

Denote by $\mathscr{D}_n$ the set of points satisfying the condition above. For every $\tilde{x} \in \mathscr{D}_n$ we choose a preimage of $x_0 = \pi(\tilde{x})$ under f^l lying in $\frac{1}{4}B$ (this can be done since $f^l(\frac{1}{4}B) \supset J$. This preimage will be denoted by x^l.

The point x^l corresponds to some branch f_ν^{-n} defined on $2 \cdot B$; this is a composition of a branch $f_\tau^{-(n-l)}$ sending $x_{n-l} = \pi(f^{n-l}(\tilde{x}))$ to $x_0 = \pi(\tilde{x})$ and a branch f_η^{-l} sending x_0 to x^l. This branch is well-defined on the image $f_\tau^{-(n-l)}(B)$

for n large, because $f_\tau^{-(n-l)}$ is a good branch, i.e. $\operatorname{diam}(f_\tau^{-(n-l)}(B))<K \exp(-(n-l)\,\delta)$. Since x_0 lies outside $2\cdot D_i$, the whole image $f_\tau^{-(n-l)}(B)$ does not intersect D_i. Moreover,

$$\operatorname{diam}(f_\nu^{-n}(B)) \leqq \sup_{z\notin \cup D_i} |(f_\eta^{-l})'(z)|\cdot \operatorname{diam}(f_\tau^{-(n-l)}(B))$$

$$\leqq \sup_{z\notin \cup D_i} |(f_\eta^{-l})'(z)|\cdot K\exp(-\delta(n-l)) \leqq \exp\left(-\frac{\delta}{2}n\right)$$

if n is large. Also, $f_\nu^{-n}(B)\subset \frac{1}{2}B$ for large n, since $x^l\in\frac{1}{4}B$ and $\operatorname{diam}(f_\nu^{-n}(B))\subset \exp\left(-\frac{\delta}{2}n\right)$.

Thus, f_ν^{-n} is in $\mathscr{F}_n$. Denote the set of branches obtained in this way by $\mathscr{G}_n$. We have

$$\sup_{y\in B} |(f_\nu^{-n})'(y)|^\alpha \geqq |(f_\nu^{-n})'(f^n(x^l))|^\alpha = \exp(-\alpha \log|(f^n)'(x^l)|$$

$$+n\log d - n\log d) = \frac{1}{d^n}\exp(-S_n\,\phi(x^l)) \geqq \frac{1}{d^n}\exp(A'\sigma\sqrt{n})$$

$$= m(f_\nu^{-n}(B))\cdot \exp(A'\sigma\sqrt{n})$$

(the constant $A'<A$ was introduced here to neglect the derivative of f^l).

Moreover,

$$m\Big(\bigcup_{\nu\in\mathscr{G}_n} f_\nu^{-n}(B)\Big) = \frac{1}{d^l}\, m\Big(\bigcup_{\nu\in\mathscr{G}_n} f_\nu^{-(n-l)}(B)\Big) \geqq \frac{1}{d^l}\, m(\pi(\mathscr{D}_n)) = \frac{1}{d^l}\cdot \beta'.$$

Thus,

$$\sum_{\nu\in\mathscr{F}_n} \sup_{x\in B} |(f_\nu^{-n})'(x)|^\alpha \geqq \sum_{\nu\in\mathscr{G}_n} \sup_{x\in B} |(f_\nu^{-n})'(x)|^\alpha \geqq \exp(A'\sigma\sqrt{n})\frac{1}{d^l}\beta'.$$

and the sequence $S_n^\alpha(B)$ is unbounded. □

Remark. In fact, $S_n^\alpha(B)$ grows exponentially with n. △

We fix this ball B. (Actually, the statement of Proposition 9 is true for every small ball B). Keeping the assumption $\sigma^2 \neq 0$ we have

Proposition 10. *There exists a subset $X\subset J$ invariant under f^n (for some $n\in\mathbb{Z}_+$) such that $f^n|X$ is expanding and $HD(X)>\alpha$.*

Proof. Fix n large (to be precised later on).

Define a set

$$X_1 = \bigcup_{\nu\in\mathscr{F}_n} f_\nu^{-n}(B).$$

Now, we define X:

$$X = \{x\in B\colon \forall k\geqq 1\; f^{nk}(x)\in X_1\},$$

i.e. X is an intersection of a descending sequence of sets X_k; every X_k is a sum of topological discs and

$$X_{k+1}=\bigcup_{v\in\mathscr{F}_n} f_v^{-n}(X_k).$$

The set X is invariant under forward iterations of f^n and $f^n|X$ is expanding (by the definition of $\mathscr{F}_n$).

We estimate the usual topological pressure $P_X(-\alpha\log|(f^n)'|)$ for the map $f^n|X$ and the function $-\alpha\log|(f^n)'|$ (which is Lipschitz continuous on X).

In the following computation D is a component of X_k; C is a component of X_1.

$$\begin{aligned}
P_X(-\alpha\log|(f^n)'|)&=\lim_{k\to\infty}\frac{1}{k}\log\left(\sum_D \inf_{x\in D}\frac{1}{|(f^{nk})'|^\alpha(x)}\right)\\
&\geqq\frac{1}{k}\log\left(\sum_C \inf_{x\in C}\frac{1}{|(f^n)'(x)|^\alpha}\right)^k\\
&=\log(\sum_{v\in\mathscr{F}_n}\inf_{y\in B}|(f_v^{-n})'(y)|^\alpha\geqq -\log L\\
&\quad+\log(\sum_{v\in\mathscr{F}_n}\sup_{y\in B}|(f_v^{-n})'(y)|^\alpha)\geqq -\log L+\log S_n^\alpha(B)
\end{aligned}$$

where L is an estimate of a distortion of f^{-n} in B (common for all branches, by the Distortion Theorem).

We fix n so that

$$\log S_n^\alpha(B)-\log L>0$$

(this is possible since, by the previous Proposition, the sequence S_n is unbounded).

By the variational principle we know that

$$P_X(-\alpha\log|(f^n)'|)=\sup_\kappa(h_\kappa-\alpha\int\log|(f^n)'|\,d\kappa)$$

where supremum is taken over all measures κ f^n-invariant and ergodic. Thus, there exists a measure κ f^n-invariant ergodic with supp $\kappa\subset X$ such that

$$h_\kappa(f^n)-\alpha\int\log|(f^n)'|\,d\kappa>0.$$

Hence, $HD(X)\geqq HD(\kappa)=\alpha\dfrac{h_\kappa}{\log|(f^n)'|\,d\kappa}$.

This completes the proof of Proposition 10. □

To finish the proof of Theorem 2, it remains to check that for maps with parabolic orbifold we have $HD(J)=HD(m)$.

For the map $z\mapsto z^{\pm d}$ m is just the Lebesgue measure on the circle. For Tchebysheff polynomials m is equivalent to the Lebesgue measure on the interval. Thus, we have $\alpha=1=HD(J)=HD(m)$.

If f has a parabolic orbifold and $J(f)=\mathbb{C}$, then $\alpha=HD(m)=2=HD(J)$. It is so, because every parabolic orbifold can be obtained as a quotient space of action of a subgroup of Aut($\mathbb{C}$) on $\mathbb{C}$. The lifted map $\tilde{f}$: $\mathbb{C} \to \mathbb{C}$ is of the form $z \to az+b$, where $|a|^2 = \deg f$ (see [DH], § 9). The maximal entropy measure for f can be obtained as an image of the Lebesgue measure on $\mathbb{C}$ and is equivalent to the usual Lebesgue measure on $\hat{\mathbb{C}}$.

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